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The Equivalence Problem for Deterministic Two-Tape Automata*

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A decision procedure is described for equivalence of deterministic two-tape (one-way) automata.

1. INTRODUCTION

The notion of an n-tape (one-way, deterministic) automaton was introduced by Rabin and Scott [2]. Although the properties of these devices have been studied extensively, no answer has been forthcoming to the question: for any $n \geqslant 2$, is there an effective procedure for deciding if two n-tape automata are equivalent? The present paper shows that, in the case of two-tape automata, such a procedure exists.

We concentrate on a simplified form of two-tape automaton called a "scheme." A scheme may be thought of as a two-tape automaton with, instead of a set of final states, a single final state (the "exit") from which no transitions are permitted. It is shown that there exists a decision procedure for equivalence of schemes (Sections 2–5), and that the equivalence problem for two-tape automata reduces to the equivalence problem for schemes (Section 6).

The decision procedure is based on the notion of a "closed diagram." Informally, this is a nondeterministic scheme (without an "entry node") with a certain property which guarantees deterministic behavior. It is shown that two schemes are equivalent if and only if they can be mapped in a certain manner into a closed diagram. Given two schemes the procedure attempts to construct a closed diagram related to them in this way.

2. DEFINITIONS

Before giving a formal definition of "schemes," we define two generalizations ("semidiagrams" and "diagrams") which arise during the execution of the decision procedure. We also define a type of mapping ("morphism") between these objects in terms of which many of their properties may be expressed conveniently.

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Let Σ_a and Σ_b denote two disjoint finite alphabets. A *semidiagram* E (over Σ_a, Σ_b) consists of

1. a set (possibly infinite) of *nodes* denoted by $| E |$;
2. a designated *exit node* $\in | E |$, denoted by Λ_E; and
3. a ternary relation $\subseteq | E | \times (\Sigma_a \cup \Sigma_b) \times | E |$. Given $M, N \in | E |$ and $\sigma \in \Sigma_a \cup \Sigma_b$, we write

$$M \xrightarrow{\sigma} N \quad \text{in } E,$$

if (M, σ, N) is in this relation.

In addition, the relation must satisfy two conditions. We will say that the node N is a *successor* of the node M if $M \to^{\sigma} N$ in E for some $\sigma \in \Sigma_a \cup \Sigma_b$; and if $\sigma \in \Sigma_a$ then N is an *a-successor* of M. Similarly for *b-successor*. The conditions follow:

1. If M has an a-successor in E, then for each $\sigma_a \in \Sigma_a$ there is an $N \in | E |$ such that $M \to^{\sigma_a} N$ in E. Similarly for b-successors.
2. Λ_E has no successors.

A semidiagram E is *finite* if $| E |$ is a finite set. A node $N \in | E |$ is *vacant* if it has no successors and is not the exit node.

Given semidiagrams E and E', a mapping $f : | E | \to | E' |$ is a *morphism* from E to E' (written $f : E \to E'$) if

1. $M \to^{\sigma} N$ in E implies $f(M) \to^{\sigma} f(N)$ in E' and
2. $f(\Lambda_E) = \Lambda_{E'}$.

Note that, given morphisms $f : E \to E'$ and $f' : E' \to E''$, the mapping $ff' : | E | \to | E'' |$ defined by $(ff')(N) = f'(f(N))$ is also a morphism. So too is the identity mapping $i_E : | E | \to | E |$. A morphism $f : E \to E'$ is an *isomorphism* if there is a morphism $f' : E' \to E$ such that $ff' = i_E$ and $f'f = i_{E'}$. E and E' are *isomorphic* (written $E \simeq E'$) if there is an isomorphism from one to the other. Clearly $E \simeq E'$ if and only if they differ only by a renaming of their nodes.

Given a morphism $f : E \to E'$, let $\mathrm{Im}(f)$ denote the semidiagram defined as follows:

1. $| \mathrm{Im}(f) | = \{f(M) \mid M \in | E |\}$;
2. $\Lambda_{\mathrm{im}(f)} = f(\Lambda_E)$; and
3. $N_1 \to^{\sigma} N_2$ in $\mathrm{Im}(f)$ if and only if there exist $M_1, M_2 \in | E |$ such that $M_1 \to^{\sigma} M_2$ in E and $f(M_i) = N_i$ $(i = 1, 2)$.

If $\mathrm{Im}(f) = E'$ then f is a morphism of E *onto* E'. Given semidiagrams E_0 and E, E_0 is a *subsemidiagram* of E (written $E_0 \subseteq E$) if $| E_0 | \subseteq | E |$, and the inclusion mapping of $| E_0 |$ into $| E |$ is a morphism.

It will sometimes be convenient to combine several statements such as $N_1 \to^{\sigma} N_2$, $N_1 \to^{\sigma'} N_3$, and $N_2 \to^{\sigma''} N_3$ in E, thus,

$$\begin{tikzcd} & N_2 \arrow[d, "\sigma''"] \\ N_1 \arrow[ur, "\sigma"] \arrow[dr, "\sigma'"'] & N_3 \end{tikzcd} \quad \text{in } E.$$

A semidiagram D is a *diagram* if

$$\begin{tikzcd} & N_1 \\ M \arrow[ur, "\sigma"] \arrow[dr, "\sigma"'] & \\ & N_2 \end{tikzcd} \quad \text{in } D$$

implies $N_1 = N_2$. A node M of a diagram D is *closed* if

$$\begin{tikzcd} M \arrow[r, "\sigma_a"] \arrow[d, "\sigma_b"'] & N \\ N' & \end{tikzcd} \quad \text{in } D,$$

where $\sigma_a \in \Sigma_a$ and $\sigma_b \in \Sigma_b$, implies that for some $N'' \in |D|$

$$\begin{tikzcd} & N \arrow[d, "\sigma_b"] \\ N' \arrow[r, "\sigma_a"'] & N'' \end{tikzcd} \quad \text{in } D.$$

A diagram is *closed* if all its nodes are closed.

A *scheme* is a pair (D, S) where:

1. D is a diagram in which every node other than the exit has either a-successors or b-successors but not both. (Note that D is closed and has no vacant nodes.)
2. S is a node of D called the *entry node* of the scheme.

Let M and N be nodes of a semidiagram E. For any $n \geqslant 0$, a *path from M to N in E of length n* consists of a sequence of $n + 1$ nodes N_0, N_1,..., N_n and a sequence of n letters σ_1, σ_2,..., σ_n such that $N_0 = M$, $N_n = N$ and

$$N_0 \xrightarrow{\sigma_1} N_1 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_n} N_n \quad \text{in } E.$$

(Note that a single node constitutes a path from itself to itself of length zero). With this path we associate the pair of words (u, v) defined as follows. u is the word obtained from the sequence σ_1, σ_2,..., σ_n by selecting, in order all letters which belong to Σ_a. (If the sequence contains no letters belonging to Σ_a, then u is the empty word over Σ_a, denoted e_a.) Similarly v is the word obtained by selecting letters of Σ_b. For any semidiagram E and node $N \in | E |$, let $\tau_E(N)$ denote the set of all word pairs associated

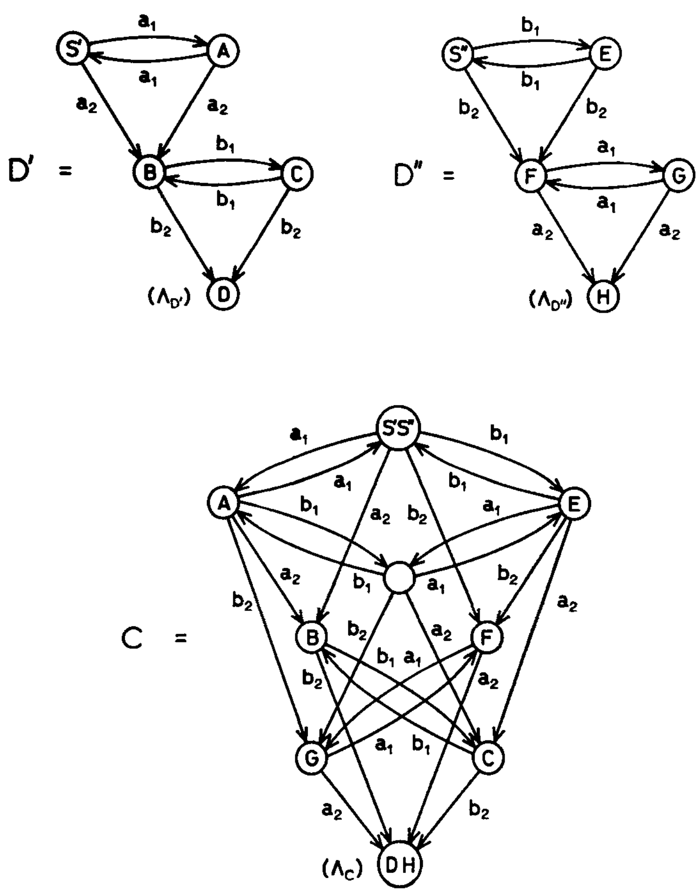


FIG. 1. Examples of diagrams.

with paths in E from N to Λ_E. Two schemes, (D', S') and (D'', S''), are *equivalent* if $\tau_{D'}(S') = \tau_{D''}(S'')$. We aim to establish an effective procedure which, given two finite schemes, will determine whether or not they are equivalent.

EXAMPLE. Let $\Sigma_a = \{a_1, a_2\}$ and $\Sigma_b = \{b_1, b_2\}$, and let D', D'' and C be the diagrams over Σ_a, Σ_b shown in Fig. 1. Clearly (D', S') and (D'', S'') are equivalent schemes, and C is closed. The nodes of the three diagrams have been labelled to indicate mappings from $| D' |$ to $| C |$ and from $| D'' |$ to $| C |$. These mappings are morphisms.

3. The Role of Closed Diagrams

In the example at the end of Section 2 the schemes (D', S') and (D'', S'') have the property that there exists a closed diagram C and morphisms $f' : D' \to C, f'' : D'' \to C$ such that $f'(S') = f''(S'')$. In this section it is shown that this is a necessary and sufficient condition for two schemes to be equivalent.

Define the *universal closed diagram* U as follows:

1. $| U |$ = all sets consisting of word pairs of the form (u, v) where $u \in \Sigma_a{}^*$, $v \in \Sigma_b{}^*$.

2. A set $\theta \in | U |$ has a-successors if $(u, v) \in \theta \Rightarrow u \neq e_a$. If this condition is satisfied then given any $\sigma_a \in \Sigma_a$, $\theta \to^{\sigma_a} \phi$ in U where $\phi = \{(u, v) \mid (\sigma_a u, v) \in \theta\}$. Similarly for b-successors.

3. $\Lambda_U = \{(e_a, e_b)\}$.

PROPOSITION 3.1. *U is a closed diagram.*

Proof. Clearly U is a diagram. Given any $\theta \in | U |$, suppose

$$\begin{tikzcd} \theta \arrow[r, "\sigma_a"] \arrow[d, "\sigma_b"'] & \phi \\ \phi' & \end{tikzcd} \quad \text{in } U,$$

where $\sigma_a \in \Sigma_a$ and $\sigma_b \in \Sigma_b$. Then $(u, v) \in \phi \Rightarrow (\sigma_a u, v) \in \theta \Rightarrow v \neq e_b$ since θ has b-successors. Thus, ϕ has b-successors, and similarly ϕ' has a-successors. Suppose $\phi \to^{\sigma_b} \psi$ and $\phi' \to^{\sigma_a} \psi'$ in U. Then $(u, v) \in \psi \Leftrightarrow (u, \sigma_b v) \in \phi \Leftrightarrow (\sigma_a u, \sigma_b v) \in \theta \Leftrightarrow (\sigma_a u, v) \in \phi' \Leftrightarrow (u, v) \in \psi'$. Thus, $\psi = \psi'$. It follows that U is closed. ∎

Define the *length* of a word pair (u, v) to be the sum of the lengths of u and v. Given any node N of a semidiagram E, and given any $n > 0$, let $\tau_E^{(n)}(N)$ denote the set of all word pairs in $\tau_E(N)$ of length n. Let ω denote the empty set of word pairs.

LEMMA 3.2. *Let C be a closed diagram. For all $n \geqslant 0$, $M \to^{\sigma} N$ in C implies $\tau_C^{(n+1)}(M) \to^{\sigma} \tau_C^{(n)}(N)$ in U.*

Proof. Proof by induction on n. We show first that if $M \to^{\sigma} N$ in C, then $\tau_C^{(1)}(M) \to^{\sigma} \tau_C^{(0)}(N)$ in U. Assume arbitrarily that $\sigma \in \Sigma_a$. If $\tau_C^{(1)}(M)$ does not have a-successors in U then $(e_a, \sigma_b) \in \tau_C^{(1)}(M)$ for some $\sigma_b \in \Sigma_b$. This implies $M \to^{\sigma_b} \Lambda_C$ in C, and, hence, Λ_C has a-successors (since M is closed), which is impossible. Therefore, $\tau_C^{(1)}(M)$ must have a-successors. Suppose $\tau_C^{(1)}(M) \to^{\sigma} \theta$ in U. Then

$$\theta = \begin{cases} \{(e_a, e_b)\} & \text{if } (\sigma, e_b) \in \tau_C^{(1)}(M) \\ \omega & \text{if } (\sigma, e_b) \notin \tau_C^{(1)}(M). \end{cases}$$

But $(\sigma, e_b) \in \tau_C^{(1)}(M) \Leftrightarrow M \to^{\sigma} \Lambda_C$ in $C \Leftrightarrow N = \Lambda_C$. Therefore,

$$\theta = \left.\begin{cases} \{(e_a, e_b)\} & \text{if } N = \Lambda_C \\ \omega & \text{if } N \neq \Lambda_C \end{cases}\right\} = \tau_C^{(0)}(N).$$

Next suppose that the proposition is true for some $n \geqslant 0$, and that $M \to^{\sigma} N$ in C. We will deduce that $\tau_C^{(n+2)}(M) \to^{\sigma} \tau_C^{(n+1)}(N)$ in U. Assume arbitrarily that $\sigma \in \Sigma_a$. Given any $(u, v) \in \tau_C^{(n+2)}(M)$, consider any path from M to Λ_C which defines (u, v). There are two possibilities:

1. The path starts with a b-step i.e. there exist $M_1 \in |C|$, $\sigma_b \in \Sigma_b$ and $v_1 \in \Sigma_b^*$ such that $M \to^{\sigma_b} M_1$ in C, $v = \sigma_b v_1$ and $(u, v_1) \in \tau_C^{(n+1)}(M_1)$. Since M is closed, M_1 must have a-successors. Therefore, by the induction hypothesis $\tau_C^{(n+1)}(M_1)$ has a-successors, and, hence, $u \neq e_a$.

2. The path starts with an a-step. This immediately implies $u \neq e_a$.

It follows that $\tau_C^{(n+2)}(M)$ has a-successors. Suppose $\tau_C^{(n+2)}(M) \to^{\sigma} \phi$ in U. Clearly $(u, v) \in \tau_C^{(n+1)}(N) \Rightarrow (\sigma u, v) \in \tau_C^{(n+2)}(M) \Rightarrow (u, v) \in \phi$. Thus, $\tau_C^{(n+1)}(N) \subseteq \phi$. It remains to show that $\phi \subseteq \tau_C^{(n+1)}(N)$. If $(u, v) \in \phi$, then $(\sigma u, v) \in \tau_C^{(n+2)}(M)$, i.e., there is a path from M to Λ_C which defines $(\sigma u, v)$. Again there are two possibilities to consider:

1. The path starts with a b-step, i.e., there exist $M_1 \in |C|$, $\sigma_b \in \Sigma_b$, and $v_1 \in \Sigma_b^*$ such that $M \to^{\sigma_b} M_1$ in C, $v = \sigma_b v_1$ and $(\sigma u, v_1) \in \tau_C^{(n+1)}(M_1)$. Since M is closed, there exists $N_1 \in |C|$ such that

$$\begin{tikzcd} M \arrow[r, "\sigma"] \arrow[d, "\sigma_b"'] & N \arrow[d, "\sigma_b"] \\ M_1 \arrow[r, "\sigma"] & N_1 \end{tikzcd} \quad \text{in } C.$$

By the induction hypothesis $\tau_C^{(n+1)}(M_1) \to^{\sigma} \tau_C^{(n)}(N_1)$ in U, and, hence, $(u, v_1) \in \tau_C^{(n)}(N_1)$. It follows that $(u, \sigma_b v_1) \in \tau_C^{(n+1)}(N)$, i.e., $(u, v) \in \tau_C^{(n+1)}(N)$.

2. The path starts with the a-step $M \to^{\sigma} N$. Clearly this implies $(u, v) \in \tau_C^{(n+1)}(N)$. In either case $(u, v) \in \tau_C^{(n+1)}(N)$. It follows that $\phi \subseteq \tau_C^{(n+1)}(N)$, and, hence, $\tau_C^{(n+2)}(M) \to^{\sigma} \tau_C^{(n+1)}(N)$ in U. ∎

Given a semidiagram E, there is associated with each $N \in | E |$ the set of word pairs $\tau_E(N) \in | U |$. Thus, we have a mapping from $| E |$ to $| U |$ which, to be consistent with previous notation, we will denote by τ_E.

PROPOSITION 3.3. *If C is a closed diagram, the mapping $\tau_C : | C | \to | U |$ is a morphism.*

Proof. Suppose $M \to^{\sigma} N$ in C. We will deduce that $\tau_C(M) \to^{\sigma} \tau_C(N)$ in U. Assume arbitrarily that $\sigma \in \Sigma_a$. Then $(u, v) \in \tau_C(M) \Rightarrow (u, v) \in \tau_C^{(n+1)}(M)$ for some $n \geqslant 0$ (since $M \neq \Lambda_C$ and, hence, $(u, v) \notin \tau_C^{(0)}(M)) \Rightarrow u \neq e_a$ (since $\tau_C^{(n+1)}(M)$ has a-successors by Lemma 3.2). Hence, $\tau_C(M)$ has a-successors. Further $(u, v) \in \tau_C(N) \Leftrightarrow (u, v) \in \tau_C^{(n)}(N)$ for some $n \geqslant 0 \Leftrightarrow (\sigma u, v) \in \tau_C^{(n+1)}(M)$ for some $n \geqslant 0$ (since $\tau_C^{(n+1)}(M) \to^{\sigma} \tau_C^{(n)}(N)$ in $U) \Leftrightarrow (\sigma u, v) \in \tau_C(N)$. Hence, $\tau_C(M) \to^{\sigma} \tau_C(N)$ in U. Also $\tau_C(\Lambda_C) = \{(e_a, e_b)\} = \Lambda_U$. It follows that τ_C is a morphism. ∎

Thus, given equivalent schemes (D', S') and (D'', S''), we have a closed diagram U and morphisms $\tau_{D'} : D' \to U$, $\tau_{D''} : D'' \to U$ such that $\tau_{D'}(S') = \tau_{D''}(S'')$. So half the required result has been proven.

PROPOSITION 3.4. *Let f be a morphism from a semidiagram E with no vacant nodes to a closed diagram C. Then $f\tau_C = \tau_E$.*

Proof. Given any $M \in | E |$, $\tau_E(M) \subseteq \tau_C(f(M))$ since any path from M to Λ_E in E is mapped by f into a similarly labelled path from $f(M)$ to Λ_C in C. To prove that $\tau_C(f(M)) \subseteq \tau_E(M)$ it is sufficient to prove: given any $n \geqslant 0$, $\tau_C^{(n)}(f(M)) \subseteq \tau_E(M)$ for all $M \in | E |$. We prove this by induction on n.

Note that $f(M) = \Lambda_C \Leftrightarrow M = \Lambda_E$, since $f(M) = \Lambda_C \Rightarrow M$ has no successors $\Rightarrow M = \Lambda_E$. Thus,

$$\tau_C^{(0)}(f(M)) = \begin{cases} \{(e_a, e_b)\} & M = \Lambda_E \\ \omega & \text{if } M \neq \Lambda_E \end{cases}$$

$$\subseteq \tau_E(M).$$

Assume $\tau_C^{(n)}(f(M)) \subseteq \tau_E(M)$ for all $M \in | E |$. We will deduce that

$$\tau_C^{(n+1)}(f(M)) \subseteq \tau_E(M) \qquad \text{for all } M \in | E |.$$

If $M = \Lambda_E$ then $\tau_C^{(n+1)}(f(M)) = \omega \subseteq \tau_E(M)$. If $M \neq \Lambda_E$ then M has successors (since E has no vacant nodes). Assume arbitrarily that M has a-successors. Then $f(M)$ has a-successors and, hence, so has $\tau_C^{(n+1)}(f(M))$. Therefore,

$$(u, v) \in \tau_C^{(n+1)}(f(M)) \Rightarrow u \neq e_a \Rightarrow u = \sigma_a u_1$$

for some $\sigma_a \in \Sigma_a$ and $u_1 \in \Sigma_a{}^*$. Suppose $M \to^{\sigma_a} N$ in E. Then $f(M) \to^{\sigma_a} f(N)$ in C, and, hence, $\tau_C^{(n+1)}(f(M)) \to^{\sigma_a} \tau_C^{(n)}(f(N))$ in U. Therefore, $(u_1, v) \in \tau_C^{(n)}(f(N)) \subseteq \tau_E(N)$ by the induction hypothesis. It follows that $(\sigma_a u_1, v) \in \tau_E(M)$; i.e., $(u, v) \in \tau_E(M)$. Thus, $\tau_C^{(n+1)}(f(M)) \subseteq \tau_E(M)$. ∎

COROLLARY 3.5. *Two schemes, (D', S') and (D'', S''), are equivalent $\Leftrightarrow$ there exists a closed diagram C and morphisms $f' : D' \to C', f'' : D'' \to C$ such that $f'(S') = f''(S'')$.*

Proof. ($\Rightarrow$) Already shown.

($\Leftarrow$) Since D' and D'' have no vacant nodes, $\tau_{D'}(S') = \tau_C(f'(S')) = \tau_C(f''(S'')) = \tau_D(S)$. ∎

4. The Decision Procedure

We now describe four operations on finite semidiagrams and diagrams and a procedure which uses these operations to decide if two finite schemes are equivalent.

The first operation is the identification of two nodes in a semidiagram. Suppose E is a finite semidiagram and $N_1, N_2 \in |E|$. Assume that it is *not* the case that one of N_1, N_2 has successors and the other is Λ_E. Then the following construction yields a finite semidiagram E' together with a morphism $d : E \to E'$.

1. Let $|E'|$ consist of $|E|$ with the nodes N_1 and N_2 replaced by a new node N_0.

2. Let d be the obvious mapping from $|E|$ to $|E'|$ defined by

$$d(M) = \begin{cases} N_0 & \text{if } M = N_1 \text{ or } N_2 \\ M & \text{otherwise.} \end{cases}$$

3. Let $\Lambda_{E'} = d(\Lambda_E)$.

4. Define $N \to^{\sigma} N'$ in E' if and only if there exist $M, M' \in |E|$ such that $M \to^{\sigma} M'$ in E, $d(M) = N$ and $d(M') = N'$.

If one of N_1, N_2 has successors and the other is Λ_E then E' is not a semidiagram, and we will say that the identification of N_1 and N_2 *fails*. Otherwise, the identification *succeeds* and the morphism $d : E \to E'$ is the result. Note that d is onto.

PROPOSITION 4.1. *Suppose N_1 and N_2 are nodes of a finite semidiagram E, and* *suppose that for some semidiagram E'' there exists an $f : E \to E''$ such that $f(N_1) = f(N_2)$. Then the identification of N_1 and N_2 succeeds, and if $d : E \to E'$ is the result, there is a morphism from E' to E''.*

Proof. Suppose there exists $f : E \to E''$ such that $f(N_1) = f(N_2)$. Then the identification will not fail because if N_1 say, has successors, so has $f(N_1)$. Hence, $f(N_2) \neq \Lambda_{E''}$, and, hence, $N_2 \neq \Lambda_E$. g is defined by:

$$g(N) = \begin{cases} f(N) & \text{if } N \neq N_0 \\ f(N_1)\ (=f(N_2)) & \text{if } N = N_0. \end{cases} \quad \blacksquare$$

The next operation, called *packing*, attempts to reduce a semidiagram to a diagram by identifying nodes. We start with a finite semidiagram E_0. If it is a diagram there is no more to do. If not, select any occurrence of the form

$$\begin{tikzcd} & N_1 \\ M \arrow[ur, "\sigma"] \arrow[dr, "\sigma"'] & & \text{in } E_0, \\ & N_2 \end{tikzcd}$$

where $N_1 \neq N_2$, and identify N_1 and N_2. If the identification is successful and the result is $d_1 : E_0 \to E_1$ say, then do the same for E_1 as for E_0. And so on. Clearly the process cannot continue indefinitely since each identification reduces the number of nodes by one. If one of the identifications fails then the operation fails. Otherwise, for some $n \geqslant 0$, successive identifications will produce a sequence of morphisms

$$d_1 : E_0 \to E_1,\ d_2 : E_1 \to E_2, \ldots, d_n : E_{n-1} \to E_n,$$

where E_n is a finite diagram. In this case the operation succeeds and the result is the morphism $p : E_0 \to E_n$ defined by

$$p = \begin{cases} d_1 d_2 \cdots d_n & \text{if } n \geqslant 1 \\ i_{E_0} & \text{if } n = 0. \end{cases}$$

Note that p is onto.

PROPOSITION 4.2. *Suppose there exists a morphism from a finite semidiagram E_0 to a diagram D. Then the packing of E_0 succeeds, and if the result is $p : E_0 \to E_n$, there is a morphism from E_n to D.*

Proof. Consider the packing of E_0. Either E_0 is a diagram, or an occurrence of the form,

$$\begin{tikzcd} & N \\ M \arrow[ur, "\sigma"] \arrow[dr, "\sigma"'] & & \text{in } E_0, \\ & N_2 \end{tikzcd}$$

will be selected and N_1, N_2 identified. In the latter case, if there exists $f_0 : E_0 \to D$ where D is a diagram, we have

$$\begin{tikzcd} & f_0(N_1) \\ f_0(M) \arrow[ur, "\sigma"] \arrow[dr, dash, "\sigma"'] & & \text{in } D. \\ & f_0(N_2) \end{tikzcd}$$

Hence, $f_0(N_1) = f_0(N_2)$. Therefore by proposition 4.1 the identification will succeed, and if $d_1 : E_0 \to E_1$ is the result, there exists a morphism $f_1 : E_1 \to D$. We can now repeat the argument for E_1, and so on. It follows that eventually a diagram E_n will be obtained, together with a morphism $f_n : E_n \to D$. ▌

The third operation, called *augmentation*, simply adds nodes and arrows to a given finite diagram D as follows: for every occurrence of the form

$$\begin{tikzcd} M_1 \arrow[r, "\sigma_a"] \arrow[d, "\sigma_b"'] & M_2 & \text{in } D, \\ M_3 \end{tikzcd}$$

where $\sigma_a \in \Sigma_a$ and $\sigma_b \in \Sigma_b$, add a new node N together with the relations

$$\begin{tikzcd} & M_2 \arrow[d, "\sigma_b"] \\ M_3 \arrow[r, "\sigma_a"'] & N \end{tikzcd} \quad .$$

The operation fails if there is an occurrence of the form

$$\begin{tikzcd} M_1 \arrow[r, "\sigma_a"] \arrow[d, "\sigma_b"'] & M_2 & \text{in } D, \\ M_3 \end{tikzcd}$$

where $\sigma_a \in \Sigma_a$, $\sigma_b \in \Sigma_b$ and M_2 or $M_3 = \Lambda_D$. Otherwise, the resulting figure will be a finite semidiagram $\hat{D}$ such that $D \subseteq \hat{D}$. In this case the operation succeeds and the result is the injection morphism $a : D \to \hat{D}$.

PROPOSITION 4.3. *Suppose there exists a morphism from a finite diagram D to a closed diagram C. Then the augmentation of D succeeds, and if the result is $a : D \to \hat{D}$, there exists a morphism $g : \hat{D} \to C$.*

Proof. The proof is straightforward, g being defined as follows:

1. If $N \in | D |$ then $g(N) = f(N)$.

2. If $N \in | \hat{D} | - | D |$ then there exist unique $M_1, M_2, M_3 \in | D |$ and $\sigma_a \in \Sigma_a$, $\sigma_b \in \Sigma_b$ such that

$$\begin{tikzcd} M_1 \arrow[r, "\sigma_a"] \arrow[d, "\sigma_b"'] & M_2 \\ M_3 & \end{tikzcd} \text{ in } D \quad \text{and} \quad \begin{tikzcd} & M_2 \arrow[d, "\sigma_b"] \\ M_3 \arrow[r, "\sigma_a"'] & N \end{tikzcd} \text{ in } \hat{D}.$$

Since C is closed there exists a unique $P \in | C |$ such that

$$\begin{tikzcd} f(M_1) \arrow[r, "\sigma_a"] \arrow[d, "\sigma_b"'] & f(M_2) \arrow[d, "\sigma_b"] \\ f(M_3) \arrow[r, "\sigma_a"'] & P \end{tikzcd} \text{ in } C.$$

In this case, $g(N) = P$. ∎

The fourth operation, called *closing*, attempts to transform a diagram into a closed diagram by alternately augmenting and packing it. We start with a finite diagram D_0. If it is already closed there is no more to do. Otherwise, augment D_0. If the augmentation is successful, giving $a_1 : D_0 \to E_0$ say, then pack E_0. And if the packing is successful, giving $p_1 : E_0 \to D_1$ say, then repeat the process with D_1 instead of D_0 and so on. If an augmenting or packing operation should fail then the closing operation fails. On the other hand, if for some $n \geqslant 0$, successive augmenting and packing operations produce a sequence

$$a_1 : D_0 \to E_0 \,,\, p_1 : E_0 \to D_1 ,..., a_n : E_{n-1} \to D_{n-1} \,,\, p_n : E_{n-1} \to D_n \,,$$

where D_n is a (finite) closed diagram, then the operation succeeds and the result is $c : D_0 \to D_n$, where

$$c = \begin{cases} a_1 p_1 \cdots a_n p_n & \text{if } n \geqslant 1 \\ i_{D_0} & \text{if } n = 0. \end{cases}$$

Note that for some D_0 this operation does not terminate (see Fig. 2).

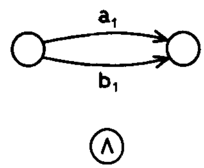


FIG. 2. The closing operation does not terminate when applied to this diagram (over $\{a_1\}$, $\{b_1\}$).

PROPOSITION 4.4. *If there exists a morphism from a finite diagram D_0 to a closed diagram, then the closing of D_0 will not fail.*

Proof. The proof is similar to the proof of proposition 4.2. ▌

We now describe a procedure called the 'equivalence test,' and then show that, provided it halts, it determines whether or not two schemes are equivalent. We start with two finite schemes (D', S') and (D'', S''). First, form a diagram (denoted by $D' + D''$) by taking disjoint copies of D' and D'' and identifying their exists. Let j' denote the obvious morphism : $D' \to D' + D''$, and similarly for j'' : $D'' \to D' + D''$. Next identify $j'(S')$ and $j''(S'')$. If this is successful, giving $d : D' + D'' \to E$ say, then pack E. And if this is successful, giving $p : E \to D$, then close D. If this last operation succeeds then the test succeeds. Should any of these operations fail then the test fails.

PROPOSITION 4.5. *Given two schemes, if the equivalence test succeeds then the schemes are equivalent; if it fails then they are not equivalent.*

Proof. Suppose the test succeeds, given (D', S') and (D'', S''). Then the successive operations of the test will produce morphisms of the form

$$j'\colon D' \to D' + D'',\ j''\colon D'' \to D' + D'',\ d\colon D' + D'' \to E,\ p\colon E \to D,\ c\colon D \to C,$$

where C is closed. Define $f'\colon D' \to C$ by $f' = j'\,dpc$, and $f''\colon D'' \to C$ by $f'' = j''\,dpc$. Then $f'(S') = f''(S'')$, since $d(j'(S')) = d(j''(S''))$. Therefore, the schemes are equivalent by Corollary 3.5.

On the other hand, if (D', S') and (D'', S'') are equivalent, there exist a closed diagram C_1 and morphisms $f_1'\colon D' \to C_1$, $f_1''\colon D'' \to C_1$ such that $f_1'(S') = f_1''(S'')$. If the test is applied to these schemes, the first step will yield morphisms $j'\colon D' \to D' + D''$ and $j''\colon D'' \to D' + D''$. These morphisms have the property that there exists an $f_1 : D' + D'' \to C_1$ such that $j'f_1 = f_1'$ and $j''f_1 = f_1''$. Hence, $f_1(j'(S')) = f_1(j''(S''))$. Applying Propositions 4.1, 4.2, and 4.4, we deduce that none of the operations of the test will fail, and, hence, the test will not fail. The required result follows. ▌

In the next section we show that the equivalence test always halts.

5. The Limit Diagram

In order to show that the test for equivalence always halts we establish a sufficient condition for the closing operation to halt.

Suppose the closing operation does not terminate when applied to a finite diagram D_0. There will be a corresponding sequence of morphisms

$$a_0 : D_0 \to E_0 , p_0 : E_0 \to D_1 , a_1 : D_1 \to E_1 , p_1 : E_1 \to D_2 ,...,$$

where $a_r : D_r \to E_r$ is the result of augmenting D_r and $p_r : E_r \to D_{r+1}$ is the result of packing E_r. Writing f_r for $a_r p_r$ we obtain the sequence

$$f_0 : D_0 \to D_1 , f_1 : D_1 \to D_2 ,...,$$

where all the D_r are diagrams. Given any sequence of this form we can define as follows a diagram L called the *limit* of the sequence, together with morphisms $g_r : D_r \to L$.
For all $0 \leqslant r \leqslant s$ define the morphism $F_{r,s} : D_r \to D_s$ by

1. $F_{r,r} = i_{D_r}$,
2. if $r < s$, $F_{r,s} = f_r f_{r+1} \cdots f_{s-1}$.

Let $\mathscr{N}$ denote the disjoint union of the sets $| D_0 |, | D_1 |,...$. To each $r \geqslant 0$ and $N \in D_r$ there corresponds an element of $\mathscr{N}$ which will be denoted by $\mathscr{G}_r(N)$. Let $\equiv$ denote the equivalence relation on $\mathscr{N}$ defined by: $\mathscr{G}_{r_1}(N_1) \equiv \mathscr{G}_{r_2}(N_2)$ if there exists an $s \geqslant r_1 , r_2$ such that $F_{r_1,s}(N_1) = F_{r_2,s}(N_2)$. Then the nodes of the limit diagram L are the equivalence classes of $\mathscr{N}$ under $\equiv$. For all $r \geqslant 0$, define $g_r : | D_r | \to | L |$ by: $g_r(N)$ is the equivalence class of $\mathscr{G}_r(N)$. (Note that $F_{r,s} g_s = g_r$.) The 'arrows' of L are defined by: $P \to^{\sigma} Q$ in L if and only if for some $r \geqslant 0$ there exist $M, N \in | D_r |$ such that $M \to^{\sigma} N$ in D_r, $g_r(M) = P$ and $g_r(N) = Q$. Λ_L is defined to be $g_0(\Lambda_{D_0})$.

We consider the properties of L, given that it is derived from a nonterminating closing operation as described previously.

Proposition 5.1. *L is closed.*

Proof. Given any $P \in | L |$, suppose

$$\begin{array}{ccc} P & \xrightarrow{\sigma_a} & Q \\ {\scriptstyle \sigma_b}\downarrow & & \\ Q' & & \end{array} \quad \text{in } L,$$

where $\sigma_a \in \Sigma_a$ and $\sigma_b \in \Sigma_b$. Then there exist $r_1 \geqslant 0$ and $M_1 , N_1 \in | D_{r_1} |$ such that $M_1 \to^{\sigma_a} N_1$ in D_{r_1}, $g_{r_1}(M_1) = P$ and $g_{r_1}(N_1) = Q$. And there exist $r_2 \geqslant 0$ and

$M_2, N_2 \in | D_{r_2} |$ such that $M_2 \rightarrow^{\sigma_b} N_2$ in D_{r_2}, $g_{r_2}(M_2) = P$ and $g_{r_2}(N_2) = Q'$. Since $g_{r_1}(M_1) = g_{r_2}(M_2)$, there exists an $s \geqslant r_1, r_2$ such that $F_{r_1,s}(M_1) = F_{r_2,s}(M_2)$. It follows that

$$\begin{array}{ccc} F_{r_1,s}(M_1) & \xrightarrow{\sigma_a} & F_{r_1,s}(N_1) \\ {\scriptstyle\sigma_b}\downarrow & & \\ F_{r_2,s}(N_2) & & \end{array} \qquad \text{in } D_s.$$

Hence, there is an $N \in | E_s |$ such that

$$\begin{array}{ccc} & & a_s(F_{r_1,s}(N_1)) \\ & & \downarrow{\scriptstyle\sigma_b} \\ a_s(F_{r_2,s}(N_2)) & \xrightarrow[\sigma_a]{} & N \end{array} \qquad \text{in } E_s.$$

Applying the morphism $p_s g_{s+1}$ we obtain

$$\begin{array}{ccc} & & Q \\ & & \downarrow{\scriptstyle\sigma_b} \\ Q' & \xrightarrow[\sigma_a]{} & (p_s\, g_{s+1})(N) \end{array} \qquad \text{in } L,$$

since

$$(F_{r_1,s} a_s p_s g_{s+1})(N_1) = (F_{r_1,s+1} g_{s+1})(N_1) = g_{r_1}(N_1) = Q,$$

and similarly $(F_{r_2,s} a_s p_s g_{s+1})(N_2) = Q'$. It follows that L is closed. ▌

Let us say that a path in a semidiagram is an *a-path* if the word pair which it defines is of the form (u, e_b) for some $u \in \Sigma_a{}^*$. Similarly for *b-paths*.

LEMMA 5.2. *Given any $N \in | D_r |$, there is an a-path in D_r from a node of* $\mathrm{Im}(F_{0,r})$ *to N. Similarly for b-paths.*

Proof. Proof by induction on r. If $N \in | D_0 |$, then N itself constitutes an a-path from a node of $\mathrm{Im}(F_{0,0})$ to N. Next assume the result true for some $r \geqslant 0$. Given any $N \in | D_{r+1} |$, there is an $N' \in | E_r |$ such that $p_r(N') = N$. There are two cases to consider.

1. $N' = a_r(N'')$ for some $N'' \in | D_r |$. By hypothesis, there exists an a-path in D_r from a node of $\mathrm{Im}(F_{0,r})$ to N'', and this will be mapped by f_r into an a-path in D_{r+1} from a node of $\mathrm{Im}(F_{0,r+1})$ to N.

2. $N' \notin \mathrm{Im}(a_r)$. In this case there exist $N_1, N_2 \in |D_r|$ such that

$$\begin{array}{ccc} & & a_r(N_1) \\ & & \downarrow{\scriptstyle \sigma_b} \quad \text{in } E_r \\ a_r(N_2) & \xrightarrow[\sigma_a]{} & N' \end{array}$$

for some $\sigma_a \in \Sigma_a$, $\sigma_b \in \Sigma_b$. By hypothesis, there exists $M \in |D_0|$ such that there is an a-path in D_r from $F_{0,r}(M)$ to N_2. This will be mapped by a_r into an a-path in E_r from $a_r(F_{0,r}(M))$ to $a_r(N_2)$. Hence, there is an a-path from $a_r(F_{0,r}(M))$ to N'. And this will be mapped by p_r into an a-path in D_{r+1} from $F_{0,r+1}(M)$ to N. ▌

PROPOSITION 5.3. *L is infinite.*

Proof. Suppose L is finite. We start by defining for all sufficiently large s, morphisms $h_s : L \to D_s$. Clearly any $P \in |L|$ belongs to $|\mathrm{Im}(g_r)|$ for all sufficiently large r. Since $|L|$ is finite, it follows that $|L| = |\mathrm{Im}(g_{r_0})|$ for some $r_0 \geqslant 0$. Define $h_{r_0} : |L| \to |D_{r_0}|$ as follows:

1. $h_{r_0}(\Lambda_L) = \Lambda_{D_{r_0}}$.
2. If $P \neq \Lambda_L$, select any $N \in |D_{r_0}|$ such that $g_{r_0}(N) = P$ and set $h_{r_0}(P) = N$.

(Thus, $h_{r_0} g_{r_0} = i_L$.) For all $r > r_0$, define $h_r : |L| \to |D_r|$ by $h_r = h_{r_0} F_{r_0, r}$. (Note that $h_r g_r = i_L$, and that $g_n(N) = P$ implies $F_{n,s}(N) = h_s(P)$ for all sufficiently large s.) Consider any arrow $P \to^\sigma Q$ in L. There exist $n \geqslant 0$ and $M, N \in |D_n|$ such that $M \to^\sigma N$ in D_n, $g_n(M) = P$ and $g_n(N) = Q$. It follows that for all sufficiently large s, $F_{n,s}(M) \to^\sigma F_{n,s}(N)$ in D_s, $F_{n,s}(M) = h_s(P)$ and $F_{n,s}(N) = h_s(Q)$; i.e., $h_s(P) \to^\sigma h_s(Q)$ in D_s. Since there are only a finite number of arrows in L, it follows that for all sufficiently large s, $P \to^\sigma Q$ in L implies $h_s(P) \to^\sigma h_s(Q)$ in D_s. Also $h_s(\Lambda_L) = \Lambda_{D_s}$ for all $s \geqslant r_0$. Hence, h_s is a morphism for all sufficiently large s.

Given any $N \in |D_0|$, let $P = g_0(N)$. Then for all sufficiently large s, $F_{0,s}(N) = h_s(P) \in |\mathrm{Im}(h_s)|$. Since $|D_0|$ is finite, it follows that $|\mathrm{Im}(F_{0,s})| \subseteq |\mathrm{Im}(h_s)|$ for all sufficiently large s.

Select any $s \geqslant 0$ such that h_s is a morphism and $|\mathrm{Im}(F_{0,s})| \subseteq |\mathrm{Im}(h_s)|$. We show that h_s is an isomorphism. Let K denote the set of fixed points of the morphism $g_s h_s$; i.e., $K = \{N \mid h_s(g_s(N)) = N\}$. For all $P \in |L|$, $h_s(g_s(h_s(P))) = h_s(P)$; hence, $|\mathrm{Im}(h_s)| \subseteq K$, and, hence, $|\mathrm{Im}(F_{0,s})| \subseteq K$. Next suppose that $N_0 \in K$ and $N_0 \to^\sigma N_1$ in D_s. Then $N_0 \to^\sigma h_s(g_s(N_1))$ in D_s, and, hence, $N_1 = h_s(g_s(N_1))$; i.e., $N_1 \in K$. It follows that if there exist a path from a node $N_0 \in K$ to a node N, then $N \in K$. But by Lemma 5.2, given any $N \in |D_s|$, there is a path from a node of $|\mathrm{Im}(F_{0,s})|$ to N. Therefore, since $|\mathrm{Im}(F_{0,s})| \subseteq K$, all $N \in |D_s|$ belong to K. Thus, $g_s h_s = i_{D_s}$. But $h_s g_s = i_L$. Therefore, h_s is an isomorphism.

Thus, for some $s \geqslant 0$, $D_s \simeq L$, and, hence, D_s is closed. But if any of the diagrams D_0, D_1,... is closed, the closing operation halts when applied to D_0. Thus we have a contradiction. Hence, L is infinite. ▌

The final property of L which we require is an immediate consequence of Lemma 5.2. It is that given any $P \in |L|$, there is an a-path from a node of $\mathrm{Im}(g_0)$ to P, and there is a b-path from a node of $\mathrm{Im}(g_0)$ to P.

PROPOSITION 5.4. *The closing operation halts when applied to a finite diagram with no vacant nodes.*

Proof. Suppose D_0 is finite and has no vacant nodes. We have seen that if the closing operation does not halt when applied to D_0, there exists an infinite closed diagram L together with a subdiagram L_0 (namely $L_0 = \mathrm{Im}(g_0)$) such that

1. L_0 is finite (since D_0 is finite);
2. L_0 has no vacant nodes (since $g_0(N)$ vacant in L_0 would imply N vacant in D_0); and
3. given any $P \in |L|$, there is an a-path from a node of L_0 to P and there is a b-path from a node of L_0 to P.

We show that this is an impossibility by proving: if L is a closed diagram containing a subdiagram L_0 with properties 1–3 above, then L is finite.

For all $P \in |L|$ define $d_a(P) = \min\{d \mid$ there exists an a-path in L of length d from a node of L_0 to $P\}$. $d_b(P)$ is defined similarly in terms of b-paths. (Property 3 guarantees that $d_a(P)$ and $d_b(P)$ are well defined.) Note that $P \in |L_0| \Leftrightarrow d_a(P) = 0 \Leftrightarrow d_b(P) = 0$; and that for all $n \geqslant 0$ the sets $\{P \mid d_a(P) \leqslant n\}$ and $\{P \mid d_b(P) \leqslant n\}$ are finite.

We prove that if $P \in |L| - |L_0|$ and $P \to^{\sigma_a} Q$ in L for some $\sigma_a \in \Sigma_a$, then $d_b(P) \geqslant d_b(Q)$. Let

$$P_0 \xrightarrow{\sigma_b^1} P_1 \xrightarrow{\sigma_b^2} \cdots \xrightarrow{\sigma_b^d} P$$

be a b-path in L from a node $P_0 \in |L_0|$ to P of minimal length $d = d_b(P) \geqslant 1$. Clearly $P_1 \notin |L_0|$, and, hence, P_0 does not have b-successors in L_0. But $P_0 \neq \Lambda_{L_0}$ (since $\Lambda_{L_0} = \Lambda_L$ which does not have successors in L) and P_0 is not vacant in L_0; therefore, P_0 has successors in L_0. Hence, P_0 has a-successors in L_0. Suppose $P_0 \to^{\sigma_a} Q_0$ in L_0. Since L is closed, there exist $Q_1, \ldots, Q_{d-1} \in |L|$ such that

$$\begin{array}{ccccccc} P_0 & \xrightarrow{\sigma_b^1} & P_1 & \xrightarrow{\sigma_b^2} & \cdots & \xrightarrow{\sigma_b^d} & P \\ {\scriptstyle\sigma_a}\downarrow & & {\scriptstyle\sigma_a}\downarrow & & & & {\scriptstyle\sigma_a}\downarrow \\ Q_0 & \xrightarrow[\sigma_b^1]{} & Q_1 & \xrightarrow[\sigma_b^2]{} & \cdots & \xrightarrow[\sigma_b^d]{} & Q \end{array} \quad \text{in } L.$$

Thus, there is a b-path of length $d_b(P)$ from a node of L_0 to Q. Hence, $d_b(Q) \leqslant d_b(P)$.

Given any $Q \in |L| - |L_0|$, there is an a-path

$$Q_0 \xrightarrow{\sigma_a^1} Q_1 \xrightarrow{\sigma_a^2} \cdots \xrightarrow{\sigma_a^{n-1}} Q_{n-1} \xrightarrow{\sigma_a^n} Q$$

in L for some $n \geqslant 1$, where $Q_0 \in |L_0|$ and $Q_1, \ldots, Q_{n-1} \in |L| - |L_0|$. Clearly $d_a(Q_1) = 1$ and $d_b(Q) \leqslant d_b(Q_1)$. Let $\bar{d}_b = \max\{d_b(P) |$ all P such that $d_a(P) = 1\}$, which is well defined because the set $\{P \mid d_a(P) = 1\}$ is finite. Then $d_b(Q) \leqslant \bar{d}_b$. It follows that $|L| = \{Q \mid d_b(Q) \leqslant \bar{d}_b\}$, which is finite. ▌

COROLLARY 5.5. *The equivalence test always halts.*

Proof. Suppose the equivalence test is applied to two schemes (D', S') and (D'', S''). If one of the first two operations (identification and packing) of the test fails, then clearly the test halts. If they succeed, then the closing operation will be applied to a diagram D which is related to the original schemes by morphisms of the form

$$j'\colon D' \to D' + D'',\ j''\colon D' \to D' + D'',\ d\colon D' + D'' \to E,\ p\colon E \to D.$$

These morphisms are such that, given any $N \in |D|$, N is either the image of a node of D' under $j'\,dp$, or the image of a node of D'' under $j''\,dp$. Suppose arbitrarily that $N = (j'\,dp)(M')$, where $M' \in |D'|$. Then N cannot be vacant, otherwise M' would be vacant. It follows that D has no vacant nodes. Hence, the closing operation halts, and, hence, the equivalence test halts. ▌

6. Application to Two-Tape Automata

In order to prove that equivalence of two-tape automata is decidable, we now show that that equivalence problem reduces to the equivalence problem for schemes. The definition of two-tape automata used here is based on that of Rabin and Scott [2]. A similar reduction can be made for two-tape automata in the sense of Mirkin [1].

For a given alphabet Θ and end-marker $\epsilon \notin \Theta$, a *two-tape* (*one-way, deterministic*) *automaton* is a quintuple (S, M, s_0, F, k) where

1. S is a finite set (of *states*);
2. M is a function: $S \times (\Theta \cup \{\epsilon\}) \to S$ (the *state transition* function);
3. $s_0 \in S$ (the *initial* state);
4. $F \subseteq S$ (the set of *final* states); and
5. k is a function: $S \to \{0, 1\}$ (the *tape selector* function).

We say that a pair (u_0, u_1) of words over Θ takes this automaton to the state s if, for some $m \geqslant 0$, there exists a sequence of states

$$s_1, s_2, \ldots, s_m \quad (s_r \in S),$$

together with a word $\sigma_1 \sigma_2 \cdots \sigma_m$ over Θ such that

1. $M(s_{r-1}, \sigma_r) = s_r$ $(r = 1, \ldots, m)$;
2. $s_m = s$; and
3. for $i = 0, 1$, u_i is the word obtained by selecting, in order, all letters σ_r from $\sigma_1 \cdots \sigma_m$ such that $k(s_r) = i$.

A pair (t_0, t_1) of words over Θ is *accepted* by the automaton if either of the following conditions is satisfied:

1. There exists a word u_1 such that
   (i) $t_1 = u_1 u_1'$ for some word u_1' and
   (ii) (t_0, u_1) takes the automaton to a state s such that $k(s) = 0$ and $M(s, \epsilon) \in F$.
2. There exists a word u_0 such that
   (i) $t_0 = u_0 u_0'$ for some word u_0' and
   (ii) (u_0, t_1) takes the automaton to a state s such that $k(s) = 1$ and $M(s, \epsilon) \in F$.

Let $T(\mathscr{A})$ denote the set of all pairs accepted by the automaton $\mathscr{A}$. Then two automata, $\mathscr{A}_1$ and $\mathscr{A}_2$, are *equivalent* if $T(\mathscr{A}_1) = T(\mathscr{A}_2)$.

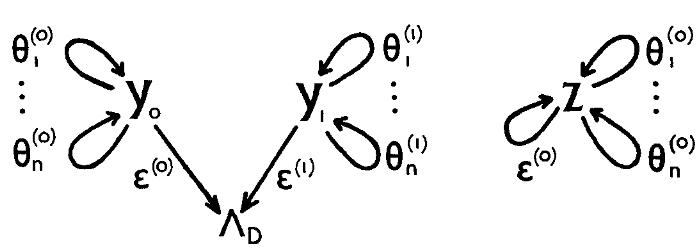
Suppose $\Theta = \{\theta_1, \ldots, \theta_n\}$. We select two disjoint copies of $\Theta \cup \{\epsilon\}$, denoted $\Sigma_0 = \{\theta_1^{(0)}, \ldots, \theta_n^{(0)}, \epsilon^{(0)}\}$ and $\Sigma_1 = \{\theta_1^{(1)}, \ldots, \theta_n^{(1)}, \epsilon^{(1)}\}$. Then with the automaton $\mathscr{A} = (S, M, s_0, F, k)$ we associate the scheme $\mathrm{sch}(\mathscr{A})$ over Σ_0, Σ_1, defined as follows: $\mathrm{sch}(\mathscr{A}) = (D, s_0)$ where $|D| = S \cup \{y_0, y_1, z, \Lambda_D\}$ for some $y_0, y_1, z, \Lambda_D \notin S$, and where the arrows of D are given by:

1. $s \to^{\theta_r^{(i)}} s'$ in D (where $i = 0$ or 1, $\theta_r \in \Theta$ and $s, s' \in S$) if $M(s, \theta_r) = s'$ and $k(s) = i$.

2. $s \to^{\epsilon^{(0)}} y_1$ in D (where $s \in S$) if $M(s, \epsilon) \in F$ and $k(s) = 0$. Similarly, $s \to^{\epsilon^{(1)}} y_0$ in D if $M(s, \epsilon) \in F$ and $k(s) = 1$.

3. $s \to^{\epsilon^{(i)}} z$ in D (where $i = 0$ or 1, and $s \in S$) if $M(s, \epsilon) \notin F$ and $k(s) = i$.

4. The remaining arrows are such that



in D

It will be seen that all pairs in $\tau_D(s_0)$ are of the form

$$(\theta_{i_1}^{(0)} \dots \theta_{i_p}^{(0)}\epsilon^{(0)}, \theta_{j_1}^{(1)} \dots \theta_{j_q}^{(1)}\epsilon^{(1)}),$$

for some $\theta_{i_1}, \dots, \theta_{j_q} \in \Theta$, and that $(\theta_{i_1}^{(0)} \cdots \theta_{i_p}^{(0)}\epsilon^{(0)}, \theta_{j_1}^{(1)} \cdots \theta_{j_q}^{(1)}\epsilon^{(1)}) \in \tau_D(s_0)$ if and only if $(\theta_{i_1} \cdots \theta_{i_p}, \theta_{j_1} \cdots \theta_{j_q}) \in T(\mathscr{A})$. It follows that $T(\mathscr{A})$ uniquely determines $\tau_D(s_0)$ and vice versa. Thus two automata, $\mathscr{A}_1$ and $\mathscr{A}_2$, are equivalent if and only if sch($\mathscr{A}_1$) and sch($\mathscr{A}_2$) are equivalent. Hence, the equivalence test for schemes can be used to decide if two automata are equivalent.

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References

1. B. G. Mirkin, On the theory of multitape automata (Russian, English summary), *Kibernetica (Kiev)* (5), (1966), 12–18.
2. M. O. Rabin and D. Scott, Finite automata and their decision problems, *IBM J. Res. Develop.* 3 (1959), 114–125.

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