Oscillatory integrals and $L^p$ estimates for Schrödinger equations

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Abstract

This paper is concerned with Schrödinger equations whose principal operators are elliptic. Under certain degenerate condition we show the estimate of an oscillatory integral related to the solution operator in free case, and then employ fractionally integrated groups to obtain the $L^p$ estimate of solutions for the initial data belonging to a dense subspace of $L^p$ in the case of integrable potentials, which improves the corresponding result in [M. Balabane, H.A. Emami-Rad, $L^p$ estimates for Schrödinger evolution equations, Trans. Amer. Math. Soc. 292 (1985) 357–373].

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1. Introduction

In this paper we investigate $L^p$ estimates of solutions for the following Schrödinger equation

$$
\frac{\partial u}{\partial t} = (iP(D) + V)u, \quad u(0, \cdot) = u_0 \in L^p(\mathbb{R}^n),$$

(∗)

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where \( D = -i(\partial/\partial x_1, \ldots, \partial/\partial x_n) \), \( P : \mathbb{R}^n \rightarrow \mathbb{R} \) is an elliptic polynomial, and \( V \) is a suitable potential function.

In order to obtain \( L^p \) estimates of the solution of Eq. (\(*\)), one can first deal with \( L^p - L^q \) estimates of \( e^{itP(D)} (t \neq 0) \), which is the solution operator of Eq. (\(*\)) in free case. It is known that the later depends heavily on the geometric property of level hypersurface, written \( \Sigma \), associated with the principal part of \( P \).

In the case when \( P \) is homogeneous, Miyachi [11] obtained such \( L^p - L^q \) estimates if the Gaussian curvature of \( \Sigma \) is nonzero everywhere. Since the condition of nonvanishing Gaussian curvature plays a key role in estimating many oscillatory integrals [14], dropping such condition has become an important object in harmonic analysis. Based on a powerful result in [4], we recently got the \( L^p - L^q \) estimate of \( e^{itP(D)} (t \neq 0) \), and so the \( L^p \) estimates of solutions of (\(*\)) provided convex hypersurface \( \Sigma \) (see [16]).

In the case when \( P \) is inhomogeneous, Balabane and Emami-Rad [3] also established these estimates under the condition of \( \Sigma \) having nonvanishing Gaussian curvature everywhere. Meanwhile, one may consider the condition relating to Hessian matrix of \( P \), as suggested by stationary phase principle. In [9], they studied regularity of solution of Eq. (\(*\)) in free case provided that the determinant of Hessian matrix is an elliptic polynomial. In fact, both two conditions are equivalent (cf. [10]).

However, there exist many inhomogeneous polynomials whose hypersurfaces \( \Sigma \) allow vanishing Gaussian curvatures at some points, such as \( \xi_1^4 + 6\xi_1^2\xi_2^2 + \xi_3^4 + 2\xi_4^2 + \xi_5^2 \) and \( \xi_6^6 + 2\xi_7^2\xi_8^2 + \xi_9^6 \). Motivated by the above condition of Hessian matrix, in this paper we introduce a larger class of polynomials which satisfy the condition \((H_b)\) (see Section 2). When \( b < 1 \), this means the Gaussian curvature of \( \Sigma \) may vanish at some points, hence \( b \) is an important index that reflects the degeneration of \( P \). By the way, as seen by the above examples, the condition of Hessian matrix as above shall be added in Theorem 3.1(i) in [9].

Since the kernel \( \mathcal{F}^{-1}(e^{i\xi P}) \) of solution operator \( e^{i\xi P(D)} (t \neq 0) \) is an oscillatory integral, the main purpose of Section 2 is to establish an estimate of this oscillatory integral under the condition \((H_b)\) for some \( b \in [\frac{1}{2}, 1] \). Our proof depends on the method in [9, 16]. We remark that the proof of conclusion (i) in [9, p. 52] is incomplete because \( V_2 \) appeared therein is not convex. Some new ideas are thus added in our proof. Moreover, when \( P \) is a polynomial of order 2 we show in this section that \( \mathcal{F}^{-1}(e^{i\xi P}) \in L^1_{loc}(\mathbb{R}^n) (t \neq 0) \) if and only if the corresponding hypersurface \( \Sigma \) has nonvanishing Gaussian curvature everywhere.

On the other hand, it is well known that the semigroup of operators is an abstract tool to treat Cauchy problems. However, Eq. (\(*\)) in \( L^p(\mathbb{R}^n) \) (\( p \neq 2 \)) cannot be treated by classical \( C_0 \)-semigroups (cf. [7]). To this end, several generalizations of \( C_0 \)-semigroups, such as smooth distribution semigroups, integrated semigroups, and regularized semigroups were introduced and applied to general differential operators (cf. [1, 5, 6, 15]). Particularly, it was Balabane and Emami-Rad [3] who first applied smooth distribution semigroups to higher order Schrödinger equations, and showed the \( L^p \) estimate of solutions for Eq. (\(*\)).

In view of the estimate of \( \mathcal{F}^{-1}(e^{i\xi P}) (t \neq 0) \) we first obtain in Section 3 \( L^p - L^{p'} \) estimates of the solution operator \( e^{i\xi P(D)} (t \neq 0) \) and the resolvent operator \((\lambda - i P(D))^{-1} \) (\( \text{Re} \lambda \neq 0 \)). Next, we show that the Schrödinger operator \( i P(D) + V \) with \( V \in L^p(\mathbb{R}^n) \) is the generator of a fractionally integrated group on \( L^p(\mathbb{R}^n) \) (\( 1 \leq p \leq 3 \)), and then give the \( L^p \) estimate of solutions for Eq. (\(*\)) by employing Straub’s fractional powers. Finally, in the case when the hypersurfaces \( \Sigma \) have nonvanishing Gaussian curvature everywhere, we will show how our results present an improvement over Theorems 2 and 6 in [3].
2. The estimate of an oscillatory integral

Throughout this paper, assume that $P : \mathbb{R}^n \to \mathbb{R}$ is always an elliptic polynomial of order $m$ where $n \geq 1$ and $m \geq 2$ ($m$ must be even if $n \geq 2$). Denote by $P_m$ the principal part of $P$, and $HP$ the Hessian matrix of $P$, i.e. $HP(\xi) = (\partial_i \partial_j P(\xi))_{n \times n}$ for $\xi \in \mathbb{R}^n$ where $\partial_j = \partial / \partial \xi_j$ ($1 \leq j \leq n$). Furthermore, we need the following condition:

$$(H_b) \max_{1 \leq k \leq n} |\lambda_k(\xi)|^{-1} = O(|\xi|^{-(m-2)b}) \text{ as } |\xi| \to \infty$$

where $\{|\lambda_k(\xi)|\}^n_{i=1}$ are $n$ eigenvalues of $HP(\xi)$, $\lambda_k(\xi)$ is positive for large $\xi$, and $0 \leq b \leq 1$.

When $n = 1$, the condition $(H_1)$ is automatically satisfied, because every polynomial $P : \mathbb{R} \to \mathbb{R}$ is elliptic. When $m = 2$, clearly $HP$ is a numerical matrix. In this case, the condition $(H_b)$ is independent of $b \in [0, 1]$, and means that $P_2$ is nondegenerate (i.e. det$(HP_2) \neq 0$), which is also equivalent to Proposition 2.1(b) below. When $m > 2$ and $n \geq 2$, the subsequent proposition shows that if $b \in [0, 1)$, then the level hypersurface $\Sigma$ allows to have vanishing Gaussian curvature, where $\Sigma = \{\xi \in \mathbb{R}^n \mid |P_m(\xi)| = 1\}$.

Proposition 2.1. Let $m > 2$ and $n \geq 2$. Then the following statements are equivalent:

(a) $P$ is nondegenerate, i.e. det$(HP)$ is an elliptic polynomial of order $n(m - 2)$;
(b) $\Sigma$ has nonzero Gaussian curvature everywhere;
(c) $P$ satisfies the condition $(H_1)$.

Proof. The implication (c) $\Rightarrow$ (a) is clear because det$(HP(\xi)) = \prod_{k=1}^n \lambda_k(\xi)$ for $\xi \in \mathbb{R}^n$, while the implication (a) $\Rightarrow$ (b) follows by Proposition 1 in [10]. To show (b) $\Rightarrow$ (c), we assume without loss of generality that $P_m(\xi) > 0$ for $\xi \neq 0$, otherwise $P_m$ is replaced by $-P_m$. Then (b) implies that (cf. [10]) $HP_m(\xi)$ is positive definite for every $\xi \neq 0$. Since $P_m$ is homogeneous, there exists a constant $\delta > 0$ such that

$$\min_k |\lambda_k(\xi)| \geq \inf_{|\xi| = 1} \langle HP_m(\xi) x, x \rangle + \inf_{|\xi| = 1} \langle H(P - P_m)(\xi) x, x \rangle \geq \delta |\xi|^{m-2} - o(|\xi|^{m-2})$$

for sufficiently large $|\xi|$ where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$.

Notice that there exist many inhomogeneous polynomials such that the level hypersurfaces $\Sigma$ have vanishing Gaussian curvature. For example, when $P_m(\xi) = \xi_1^m + \xi_2^2 \xi_3^{m-4} + \xi_3^2 \xi_1^{m-4} + \xi_2^m$ for $\xi \in \mathbb{R}^2$,

one has det$(HP_m(\xi)) = m^2 (m-1)^2 \xi_1^{m-2} \xi_2^{m-2}$, and thus $P_m$ is degenerate. By Proposition 2.1 the corresponding hypersurface $\Sigma$ has vanishing Gaussian curvature at some points, more precisely, at $(0, \pm 1)$ and $(\pm 1, 0)$. Moreover, the condition $(H_b)$ with $b = \frac{m-4}{m-2}$ is satisfied, and $b \geq \frac{1}{2}$ if $m \geq 6$.

We now turn to the Cauchy problem (*) with $V = 0$. In this case, for every initial data $u_0 \in S(\mathbb{R}^n)$ (the Schwartz space), the solution is given by

$$u(t, \cdot) = e^{itP(D)}u_0 := F^{-1}(e^{itP}) * u_0,$$
where \( \mathcal{F} \) (or \( \hat{\cdot} \)) denotes the Fourier transform, \( \mathcal{F}^{-1} \) its inverse, and \( \mathcal{F}^{-1}(e^{itP}) \) is understood in the distributional sense, whose estimate is given as follows:

**Theorem 2.2.** Suppose \( P \) satisfies the condition \((H_b)\) for some \( b \in [\frac{1}{2}, 1] \). Then there exists a constant \( C > 0 \) such that

\[
|\left(\mathcal{F}^{-1}(e^{itP})(x)\right)| \leq C(|t|^\rho + |t|^{-\sigma}) \quad \text{for } t \neq 0 \text{ and } x \in \mathbb{R}^n,
\]

where \( \sigma = n[(2b - 1)(m - 2) + 2]^{-1} \) and \( \rho = \max\{\sigma [b(m - 2) + 1] - n, 0\} \).

**Proof.** By our assumptions on \( P \), there exist constants \( L > 0, c_0 > 0 \) such that \( \min_k \lambda_k(\xi) \geq c_0 |\xi|^{(m - 2)b} \), and \( |\nabla P(\xi)| \sim |\xi|^{m - 1} \) for \( |\xi| \geq L \), where and in the rest of this proof we denote by \( \sim \) the equivalent relationship, and \( C (> 1) \) a generic constant independent of \( \xi, x \) and \( t \).

Write \( \mathbb{R}^n = \bigcup_{j=1}^4 \Omega_j \) where

\[
\begin{align*}
\Omega_1 &= \{\xi \in \mathbb{R}^n \mid |\xi| < 2L\}, \\
\Omega_2 &= \{\xi \in \mathbb{R}^n \mid |\xi| > L, \ |\xi| < |t|^{-1/m}\}, \\
\Omega_3 &= \{\xi \in \mathbb{R}^n \mid |\xi| > L, \ |\xi| > \frac{1}{2} |t|^{-1/m} \text{ and } |\nabla P(\xi) + \frac{x}{t}| < \frac{1}{2} |\frac{x}{t}|\}, \\
\Omega_4 &= \{\xi \in \mathbb{R}^n \mid |\xi| > L, \ |\xi| > \frac{1}{2} |t|^{-1/m} \text{ and } |\nabla P(\xi) + \frac{x}{t}| > \frac{1}{4} |\frac{x}{t}|\}.
\end{align*}
\]

Choose the following partition of unity subordinate to this covering:

\[
\begin{align*}
\varphi_1(\xi) &= \varphi(\xi/2L), \\
\varphi_2(\xi) &= (1 - \varphi_1(\xi))\varphi(|t|^{1/m}), \\
\varphi_3(\xi) &= (1 - \varphi_1(\xi) - \varphi_2(\xi))\varphi\left(\left|\nabla P(\xi) + \frac{x}{t}\right|/\frac{1}{2} |\frac{x}{t}|\right), \\
\varphi_4(\xi) &= 1 - \varphi_1(\xi) - \varphi_2(\xi) - \varphi_3(\xi),
\end{align*}
\]

where \( \varphi \in C_0^\infty(\mathbb{R}^n) \) such that

\[
\varphi(\xi) = \begin{cases} 1, & |\xi| \leq \frac{1}{2}, \\
0, & |\xi| \geq 1. \end{cases}
\]

Obviously \( \sum_{j=1}^4 \varphi_j = 1 \), and \( |D^\alpha \varphi_j| \leq C_\alpha |\xi|^{-\alpha} \), \( \forall \alpha \in \mathbb{N}_0^n \) by a direct calculation. Thus \( \mathcal{F}^{-1}(e^{itP}) \) consists of four integrals

\[
I_j = \int_{\mathbb{R}^n} e^{i(tP(\xi) + \langle x, \xi \rangle)} \varphi_j(\xi) \, d\xi \quad \text{for } j = 1, 2, 3, 4.
\]

Clearly \( |I_1| \leq C \) and \( |I_2| \leq C |t|^{-n/m} \).
In order to estimate $I_4$, we write $\Omega_4 = \bigcup_{j=1}^n U_j$ where

$$U_j = \left\{ \xi \in \Omega_4 \left| \partial_j P(\xi) - \frac{x_j}{t} > \frac{1}{\sqrt{2n}} \left| \nabla P(\xi) + \frac{x}{t} \right| \right. \right\},$$

and choose the following partition of unity of $\Omega_4$ subordinate to this covering: $\eta_j = \theta_j / \sum_{l=1}^n \theta_l$ $(j = 1, \ldots, n)$ where

$$\theta_j(\xi) = \psi \left( \frac{\sqrt{2n} \left( \partial_j P(\xi) + \frac{x_j}{t} \right)}{\left| \nabla P(\xi) + \frac{x}{t} \right|} \right) \quad \text{for} \quad \xi \in \Omega_4$$

and $\psi \in C_0^\infty(\mathbb{R})$ with

$$\psi(s) = \begin{cases} 1, & |s| \geq 2, \\ 0, & |s| < 1, \end{cases}$$

again we can get $|D^\alpha \eta_j| \leq C_\alpha |\xi|^{-\alpha}$, $\forall \alpha \in \mathbb{N}_0^n$. Now, to estimate $I_4$ it suffices to consider the integral

$$I_{41} = \int_{\mathbb{R}^n} e^{i(t P(\xi) + \langle x, \xi \rangle)} \varphi_4(\xi) \eta_1(\xi) \, d\xi.$$ 

Set $D_* f = \partial_1 (g f)$ for $f \in C^1(\mathbb{R}^n)$, where $g = (it \partial_1 P + ix_1)^{-1}$. Since by Leibniz’s rule we can check that $|\partial_1^j (\varphi_4 \eta_1)| \leq C |\xi|^{-j}$ and $|\partial_1^j g| \leq C |\xi|^{-1} |\xi|^{1-m-j}$ for $\xi \in U_1$, it follows that $|D_*^n(\varphi_4 \eta_1)| \leq C |\xi|^{-n} |\xi|^{-nm}$ for $\xi \in U_1$ (cf. the proof of (3.2) in [16]). Hence integrating $n$-times by parts leads to

$$|I_{41}| = \int_{\mathbb{R}^n} e^{i(t P(\xi) + \langle x, \xi \rangle)} D_*^n (\varphi_4(\xi) \eta_1(\xi)) \, d\xi$$

$$\leq C |t|^{-n} \int_{|\xi| > \frac{1}{2} |t|^{-1/m}} |\xi|^{-nm} d\xi$$
$$= C |t|^{-n/m}.$$ 

Now turn to $I_3$. Since $|\nabla P(\xi)| \sim |\xi| \sim \frac{2}{|t|}$ for $\xi \in \Omega_3$, there exist constants $c_1, c_2 > 0$ such that $\Omega_3 \subset \{ \xi \in \mathbb{R}^n \left| 2c_1 r \leq |\xi| \leq c_2 r \right. \}$ where $r = |\xi|^{1/(m-1)}$. If $L \geq c_1 r$, then it is easy to get $|I_3| \leq C$. If $L < c_1 r$, we consider further truncated cone decomposition of $\Omega_3$. To this end, choose a finite set $\{ \xi_v \} \subset S^{n-1}$ (the unit sphere in $\mathbb{R}^n$) such that $|\xi_v - \xi_v'| \geq \frac{1}{4} \left( v \neq v' \right)$ and $\min_v |\xi - \xi_v| < \frac{1}{2}$ for every $\xi \in S^{n-1}$. Notice that the set $\{ \xi_v \}$ contains at most $C 4^n$ elements.

Corresponding to $\{ \xi_v \}$, write $\Omega = \bigcup_v \Omega_3^v$ where $\Omega_3^v = \{ \xi \in \Omega_3 \left| \frac{\xi}{|\xi|} - \xi_v \rceil \leq \frac{1}{2} \}$, and choose the following partition of unity subordinate to the covering: $\chi_v = \xi_v (\sum_l \xi_l)^{-1}$ where $\xi_v(\xi) = \varphi(A(\xi/|\xi| - \xi_v))$ for $\xi \in \Omega_3$. Then $I_3 = \sum_v I_3^v$ where

$$I_3^v = \int_{\mathbb{R}^n} e^{i(t P(\xi) + \langle x, \xi \rangle)} \varphi_3(\xi) \chi_v(\xi) \, d\xi.$$
To estimate $I^v_3$, we need the inequality
\[ |\nabla P(\xi) - \nabla P(\xi^{'})| \geq cr^{b(m-2)|\xi - \xi^{'}|} \quad \text{for } \xi, \xi^{'} \in \Omega^v_3, \]
where the constant $c = c_0c_1^{b(m-2)}$. In fact, by the intermediate value theorem one has
\[ |\nabla P(\xi) - \nabla P(\xi^{'})| \geq \min_k |\lambda_k(\bar{\xi})| |\xi - \xi^{'}|. \]
Since $\bar{\xi}$ is located in the convex hull of $\Omega^v_3$, we can show that $|\bar{\xi}| \geq c_1 r > L$. It follows thus by the assumption on $P$ that
\[ \min_k |\lambda_k(\bar{\xi})| \geq c_0|\bar{\xi}|^{b(m-2)} \geq cr^{b(m-2)}, \]
as desired.

We now pick up $\xi_0 \in \Omega^v_3$ such that $|\nabla P(\xi_0) + \frac{x}{t}| \leq \frac{c}{4}r^{b(m-2)|t|^{-\sigma/n}}$, while the case when $\xi_0$ does not exist can be easily treated (cf. [9, p. 53]). Corresponding to the set $V_1 := \{ \xi \in \Omega^v_3 \mid |\xi - \xi_0| < |t|^{-\sigma/n} \}$ and $V_2 = \{ \xi \in \Omega^v_3 \mid |\xi - \xi_0| > \frac{1}{2}|t|^{-\sigma/n} \}$ we split $I^v_3$ into two integrals, $I^v_{31}$ and $I^v_{32}$. It is not hard to obtain $|I^v_{31}| \leq C|t|^{-\sigma}$.

Denote by $W_j$ the set $U_j$ in which $\Omega^v_4$ is replaced by $V_2$, and split $I^v_{32}$ into $n$ new integrals. To estimate $I^v_{32}$, it suffices to estimate one of them, for example,
\[ |I^v_{321}| = \left| \int_{\mathbb{R}^n} e^{i(tP(\xi) + \langle x, \xi \rangle)} D^*_u \phi(\xi) \, d\xi \right|, \]
where supp $\phi \subset W_1$ and $\phi$ is of the form
\[ \phi(\xi) = \phi_3(\xi) \chi_\nu(\xi) \left( 1 - \varphi(|t|^{\sigma/n}(\xi - \xi_0)) \right) \eta_1(\xi). \]

By Leibniz’s formula, it leads to
\[ |\partial_1^k \phi(\xi)| \leq C \sum_{s=0}^k |\xi|^{-s} |\xi - \xi_0|^{-(k-s)} \quad \text{for } \xi \in W_1, \]
since $|\xi| \sim r$ for $\xi \in W_1$, then $|\xi - \xi_0| \leq |\xi| + |\xi_0| \leq C|\xi|$, so it follows that
\[ |\partial_1^k \phi(\xi)| \leq C|\xi - \xi_0|^{-k} \quad \text{for } \xi \in W_1. \]

Using the same method as above, we get
\[ |D^*_u \phi| \leq C|t|^{-n} \sum_{j+k=n} |\xi|^{j(m-2)} |\xi - \xi_0|^{-k} |\partial_1 P + \frac{x_1}{t}|^{-(n+j)} \quad \text{for } \xi \in W_1. \]

Notice that if $\xi \in V_2$, then
\[ \nabla P(\xi) + \frac{x}{t} \geq C|\xi|^{b(m-2)}|\xi - \xi_0|. \]
which can be concluded by the following two inequalities:

\[ |\nabla P(\xi) - \nabla P(\xi_0)| \geq cr^b(m-2)|\xi - \xi_0| \geq \frac{c}{2}r^b(m-2)|t|^{-\sigma/n} \]

and

\[ |\nabla P(\xi_0) + \frac{x}{t}| \leq \frac{c}{4}r^b(m-2)|t|^{-\sigma/n}. \]

Consequently

\[ |D^n_s \phi(\xi)| \leq C|t|^{-n}\sum_{j=0}^n |\xi|^{(|j-b(n+j)|(m-2))}|\xi - \xi_0|^{-2n} \quad \text{for } \xi \in W_1. \]

Since \( j - b(n + j) \leq 0 \) for \( b \in [\frac{1}{2}, 1] \) and \( j = 0, 1, \ldots, n, \)

\[ |D^n_s \phi(\xi)| \leq C|t|^{-n}\sum_{j=0}^n |\xi - \xi_0|^{-2n+|j-b(n+j)|(m-2)} \quad \text{for } \xi \in W_1. \]

It now follows that

\[ |I_{321}^v| \leq \int_{|\xi - \xi_0| \geq \frac{1}{2}|t|^{-\sigma/n}} |D^n_s \phi(\xi)| \, d\xi \]

\[ \leq C|t|^{-n+\sigma(b(m-2)+1)}\sum_{j=0}^n |t|^{-j\sigma(1-b)(m-2)/n} \]

\[ \leq C\left(|t|^\rho + |t|^{-\sigma}\right), \]

and the proof is completed. \( \square \)

We conclude this section with a result related to the case \( m = 2 \), which shows that \( P_2 \) must be nondegenerate even if \( \mathcal{F}^{-1}(e^{it_0 P}) \in L^1_{\text{loc}}(\mathbb{R}^n) \) for some \( t_0 \neq 0. \)

**Theorem 2.3.** Let \( m = 2. \) Then the following statements are equivalent.

(a) \( \mathcal{F}^{-1}(e^{it P}) \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n). \)

(b) \( \mathcal{F}^{-1}(e^{it P}) \in L^1_{\text{loc}}(\mathbb{R}^n). \)

(c) \( P_2 \) is nondegenerate, i.e. \( \det(HP) \neq 0. \)

**Proof.** Let \( A = HP. \) Since there exist \( b \in \mathbb{R}^n \) and \( a \in \mathbb{R} \) such that

\[ \mathcal{F}^{-1}(e^{it P})(x) = e^{ia} \mathcal{F}^{-1}(e^{i(A\cdot \cdot)})(b + x) \quad \text{for } x \in \mathbb{R}^n, \]

we may assume that \( P(\xi) = \langle A\xi, \xi \rangle \) for \( \xi \in \mathbb{R}^n. \)
The implication (a) ⇒ (b) is trivial, while the implication (c) ⇒ (a) is a direct consequence of Theorem 6.7.1 in [8]. It remains to show (b) ⇒ (c). If \( \det A = 0 \), then there exists an orthogonal matrix \( Q \) such that
\[
Q^{-1}AQ = B := \text{diag}(\lambda_1, \ldots, \lambda_l, 0, \ldots, 0)
\]
for some \( l < n \), where \( 0 \neq \lambda_j \in \mathbb{R} \). Let \( K_\delta = \mathcal{F}^{-1}(e^{i(B + i\delta I):\cdot}) \) for \( \delta \geq 0 \). Since \( K_\delta \to K_0 \) and
\[
K_\delta = \mathcal{F}^{-1}(e^{iP - \delta|\cdot|^2})(Q) \to \mathcal{F}^{-1}(e^{iP})(Q) \quad \text{as} \quad \delta \to 0
\]
in the distributional sense, it follows that \( K_0 = \mathcal{F}^{-1}(e^{iP})(Q) \in L^1_{\text{loc}}(\mathbb{R}^n) \).

Choose \( \varphi \in C_0^\infty(\mathbb{R}^l) \) such that \( \langle \mathcal{F}_l^{-1}(e^{iB':\cdot}), \varphi \rangle \neq 0 \), where \( \mathcal{F}_l^{-1} \) denotes the inverse Fourier transform on \( \mathbb{R}^l \) and \( B' = \text{diag}(\lambda_1, \ldots, \lambda_l) \). Also, choose \( \varphi_\varepsilon \in C_0^\infty(\mathbb{R}^{n-l}) \) such that
\[
\varphi_\varepsilon(x'') = \begin{cases} 
1, & |x''| \leq \frac{\varepsilon}{2}, \\
0, & |x''| \geq \varepsilon.
\end{cases}
\]
Let \( \psi_\varepsilon(x)\varphi(x')\varphi_\varepsilon(x'') \) for \( x = (x', x'') \in \mathbb{R}^n \). Then \( \psi_\varepsilon \in C_0^\infty(\mathbb{R}^n) \). In view of \( K_0 \in L^1_{\text{loc}}(\mathbb{R}^n) \), one gets
\[
\langle K_0, \psi_\varepsilon \rangle = \int_{\mathbb{R}^l} \varphi(x') \left( \int_{\mathbb{R}^{n-l}} K_0(x', x'')\varphi_\varepsilon(x'') \, dx'' \right) \, dx' \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

On the other hand, \( \mathcal{F}_l^{-1}(e^{iB':\cdot}) \in L^\infty(\mathbb{R}^l) \) by the implication (c) ⇒ (a), and thus
\[
\langle K_\delta, \psi_\varepsilon \rangle = \langle \mathcal{F}_l^{-1}(e^{i(B' + i\delta I):\cdot}), \varphi \rangle \langle \mathcal{F}_l^{-1}(e^{-i\delta|\cdot|^2}), \varphi \rangle \quad \text{for} \quad \delta > 0.
\]
Since \( \mathcal{F}_l^{-1}(e^{-i\delta|\cdot|^2}) \to \delta_{n-l} \) distributionally, where \( \delta_{n-l} \) is the Dirac measure on \( \mathbb{R}^{n-l} \), letting \( \delta \to 0 \) we obtain
\[
\langle K_0, \psi_\varepsilon \rangle = \langle \mathcal{F}_l^{-1}(e^{iB':\cdot}), \varphi \rangle \varphi_\varepsilon(0) = \langle \mathcal{F}_l^{-1}(e^{iB':\cdot}), \varphi \rangle \neq 0,
\]
which yields a contradiction. \( \square \)

3. \( L^p \) estimates for Schrödinger equations

Let \( \alpha \geq 0 \). A densely defined linear operator \( A \) on a Banach space \( X \) is called the generator of an \( \alpha \)-times integrated semigroup if there exists an exponentially bounded, strongly continuous family \( T(t) \ (t \geq 0) \) of bounded linear operators on \( X \) such that
\[
(\lambda - A)^{-1}x = \lambda^\alpha \int_0^\infty e^{-\lambda t} T(t)x \, dt \quad \text{for large} \ \lambda \ \text{and} \ x \in X.
\]

If \( A \) and \( -A \) both are generators of \( \alpha \)-times integrated semigroups, \( A \) is called the generator of an \( \alpha \)-times integrated group. We refer to [6, p. 30] for a sufficient condition to guarantee
A as the generator, which says that $A$ is the generator of an $\alpha$-times integrated semigroup if $\|(\lambda - A)^{-1}\| \leq C|\lambda|^\beta$ for large $\text{Re} \lambda$, where $-1 \leq \beta < \alpha - 1$.

In the sequel, denote by $p'$ the conjugate index of $p$ ($\geq 1$), and $\| \cdot \|_{L^p-L^q}$ the norm in $L(L^p, L^q)$ (the space of all bounded linear operators from $L^p$ to $L^q$). Assume that the operator $P(D)$ has maximal domain in $L^p(R^n)$ in the distributional sense, and thus it is closed and densely defined in $L^p(R^n)$.

We start with two lemmas, which are concerned with Eq. $(*)$ in case free. The first deals with the $L^p-L^{p'}$ estimate of the integrated group with the generator $iP(D)$, as well as the resolvent operator of $iP(D)$ (cf. [6]).

**Lemma 3.1.** Let $1 \leq p < \infty$ and $\alpha > n_p : n|1/2 - 1/p|$. Then $iP(D)$ is the generator of an $\alpha$-times integrated group $T(t)$ ($t \in R$) on $L^p(R^n)$, and there exists a constant $C > 0$ such that

$$\|T(t)\|_{L^p-L^{p'}} \leq C(1 + |t|^\alpha) \quad \text{for } t \in R$$

and

$$\|(\lambda - iP(D))^{-1}\|_{L^p-L^{p'}} \leq C|\lambda|^\alpha|\text{Re} \lambda|^{-1}(1 + |\text{Re} \lambda|^{-\alpha}) \quad \text{for } \text{Re} \lambda \neq 0.$$ 

The subsequent lemma establishes $L^p-L^{p'}$ estimates of the solution operator and resolvent operator.

**Lemma 3.2.** Suppose $P$ satisfies the condition $(H_b)$ for some $b \in [1/2, 1]$. If $p \in [1, 2]$, then there exists a constant $C > 0$ such that

$$\|e^{itP(D)}\|_{L^p-L^{p'}} \leq C(|t|^\rho(2/p-1) + |t|^{-\sigma(2/p-1)}) \quad \text{for } t \neq 0,$$

where $\sigma$ and $\rho$ are given in Theorem 2.2. If in addition $p > \frac{2\alpha}{1+\alpha}$, then

$$\|(\lambda - iP(D))^{-1}\|_{L^p-L^{p'}} \leq C(|\text{Re} \lambda|^{-\rho(2/p-1)-1} + |\text{Re} \lambda|^{-\sigma(2/p-1)-1}) \quad \text{for } \text{Re} \lambda \neq 0.$$

**Proof.** By Theorem 2.2 and Young’s inequality we have

$$\|e^{itP(D)}\|_{L^1-L^\infty} \leq \|\mathcal{F}^{-1}(e^{itP})\|_{L^\infty} \leq C(|t|^\rho + |t|^{-\sigma}) \quad \text{for } t \neq 0.$$

Since $P(D)$ is self-adjoint in $L^2(R^n)$, $\|e^{itP(D)}\|_{L^2-L^2} = 1$ for $t \geq 0$ by Stone’s theorem. When $1 \leq p < 2$, we deduce from the Riesz–Thorin interpolation theorem that

$$\|e^{itP(D)}\|_{L^p-L^{p'}} \leq \|e^{itP(D)}\|_{L^1-L^\infty}^{1-2/p'} \|e^{itP(D)}\|_{L^2-L^2}^{2/p'} \leq C(|t|^\rho(2/p-1) + |t|^{-\sigma(2/p-1)}) \quad \text{for } t \neq 0.$$

When $\text{Re} \lambda > 0$, one has

$$(\lambda - iP(D))^{-1} f = \int_0^\infty e^{-\lambda t} e^{itP(D)} f \, dt \quad \text{for } f \in \mathcal{S}(R^n).$$
It follows therefore that
\[
\left\| (\lambda - i P(D))^{-1} \right\|_{L^p \to L^{p'}} \leq C \int_0^\infty e^{-(\text{Re}\lambda) t} \left( t^{\frac{n}{2} - 1} + t^{-\sigma(\frac{2}{p} - 1) - 1} \right) dt \\
\leq C \left( |\text{Re}\lambda|^{-\rho(\frac{2}{p} - 1) - 1} + |\text{Re}\lambda|^{\sigma(\frac{2}{p} - 1) - 1} \right).
\]

When \( \text{Re} \lambda < 0 \), we notice
\[
(\lambda - i P(D))^{-1} f = \int_0^\infty e^{\lambda t} e^{-itP(D)} f dt \quad \text{for } f \in S(\mathbb{R}^n),
\]
and thus the desired estimate also holds.

We now turn to Schrödinger operator \( i P(D) + V \), in which \( V \) is a measurable function defined on \( \mathbb{R}^n \). We consider \( V \) as a multiplication operator on \( L^p(\mathbb{R}^n) \) with \( \text{D}(V) := \{ f \in L^p(\mathbb{R}^n) \mid Vf \in L^p(\mathbb{R}^n) \} \). The domain of \( i P(D) + V \) is given by \( \text{D}(P(D)) \cap \text{D}(V) \).

**Theorem 3.3.** Suppose \( P \) satisfies the condition \( (H_b) \) for some \( b \in \left[ \frac{1}{2}, 1 \right] \). Let \( 1 \leq p \leq 3 \), \( n_p < (1 - b/2)(m - 2) + 1 \), and \( V \in L^{p-2} \left( \mathbb{R}^n \right) \). Then \( i P(D) + V \) is the generator of an \( \alpha \)-times integrated group on \( L^p(\mathbb{R}^n) \), where \( \alpha > n \cdot p + 1 \).

**Proof.** Since \( i P(D) + V \) and \( -(i P(D) + V) \) satisfy the same assumptions, it suffices to show that \( i P(D) + V \) is the generator of an \( \alpha \)-times integrated semigroup on \( L^p(\mathbb{R}^n) \). Meanwhile, we notice that for fixed \( p \in [1, 3] \), \( \frac{p}{|p-2|} \geq p \) and thus \( i P(D) + V \) is densely defined in \( L^p(\mathbb{R}^n) \).

We consider first the case \( 1 \leq p \leq 2 \). By Hölder’s inequality and Lemma 3.2 there exists a constant \( \omega \geq 1 \) such that
\[
\left\| V (\lambda - i P(D))^{-1} \right\|_{L^p \to L^p} \leq \| V \|_{L^{p'} \to L^p} \| (\lambda - i P(D))^{-1} \|_{L^p \to L^{p'}} \\
\leq C \| V \|_{L^{p/(2-p)}} \left( |\text{Re}\lambda|^{-\rho(\frac{2}{p} - 1) - 1} + |\text{Re}\lambda|^{\sigma(\frac{2}{p} - 1) - 1} \right) \\
\leq 1/2 \quad \text{for } \text{Re} \lambda > \omega,
\]
where we note that \( \sigma(\frac{2}{p} - 1) - 1 < 0 \) because \( n_p < (1 - b/2)(m - 2) + 1 \). It follows now from Lemma 3.1 that
\[
\left\| (\lambda - i P(D))^{-1} \right\|_{L^p \to L^p} \leq \left\| (\lambda - i P(D))^{-1} \sum_{j=0}^\infty \left( V (\lambda - i P(D))^{-1} \right)^j \right\|_{L^p \to L^p} \\
\leq 2 \left\| (\lambda - i P(D))^{-1} \right\|_{L^p \to L^p} \\
\leq C |\lambda|^{n \cdot p + \varepsilon} \quad \text{for } \text{Re} \lambda > \omega,
\]
where \( \varepsilon \in (0, \alpha - n \cdot p - 1) \), and thus \( i P(D) + V \) is the generator of an \( \alpha \)-times integrated semigroup on \( L^p(\mathbb{R}^n) \).
Next, we consider the case $2 < p \leq 3$. By Theorems 6.1 of Chapter 5 and 10.2 of Chapter 6 in [13] one has that $iP(D) + V = (-iP(D) + \overline{V})^*$ on $L^p(\mathbb{R}^n)$, and thus an adjointness argument yields
\[
\| (\lambda - (iP(D) + V))^{-1} \|_{L^p - L^p} = \| (\lambda - (-iP(D) + V))^{-1} \|_{L^{p'} - L^{p'}}.
\]
Since $n_{p'} = n_p$ and $\frac{p}{p-2} = \frac{p'}{|p'-2|}$, the same estimates as above are also true, and the proof is completed.

In order to give $L^p$ estimates of the solution for Schrödinger equations we need Straub’s fractional powers (cf. [12]). Let $\alpha_0 \geq 0$. If $A$ is the generator of an $\alpha$-times integrated group for every $\alpha > \alpha_0$, then the fractional powers $(\omega \pm A)^{\alpha}$ are well defined for large $\omega \in \mathbb{R}$ and their domains all contain the dense subspace $D(A^{[\alpha]+1})$. The following result is a consequence of Theorem 3.3 and Theorem 1.1 in [12].

**Theorem 3.4.** Suppose $P$, $V$, $p$ and $\alpha$ satisfy the assumptions of Theorem 3.3. Then there exist constants $C, \omega > 0$ such that for every data $u_0 \in D((\omega + iP(D) + V)^{\alpha}) \cap D((\omega - iP(D) - V)^{\alpha})$, Eq. ($\ast$) has a unique solution $u \in C(\mathbb{R}, L^p(\mathbb{R}^n))$ and
\[
\| u(t, \cdot) \|_{L^p} \leq Ce^{\omega|t|} \| (\omega \pm iP(D) \pm V)^{\alpha} u_0 \|_{L^p} \text{ for } t \in \mathbb{R},
\]
where we choose $+$ (respectively $-$) if $t \geq 0$ (respectively $< 0$).

When $P_m$ is nondegenerate, by Proposition 2.1 it follows that $P$ satisfies the condition $(H_1)$. Since $\rho = 0 \iff m - 3 \leq b(m - 2)$, one has in this case that $\rho = 0$ and $\sigma = n/m$ (see Theorem 2.2 for the definition of $\rho$ and $\sigma$). Also, we notice that $p \in [1, 3]$ with $n_p < m/2 \iff p \in I(n, m)$, where
\[
I(n, m) = \begin{cases} 
[1, 3] & \text{if } n < m, \\
\left(\frac{2n}{n+m}, 3\right] & \text{if } m \leq n < 3m, \\
\left(\frac{2n}{n+m}, \frac{2n}{n-m}\right] & \text{if } n \geq 3m.
\end{cases}
\]
Therefore one has the following consequence of Lemma 3.2 and Theorem 3.3, which will be used to compare with the corresponding results in [3].

**Corollary 3.5.** Suppose $P_m$ is nondegenerate.

(a) If $1 \leq p \leq 2$, then there exists a constant $C > 0$ such that
\[
\| e^{itP(D)} \|_{L^p - L^{p'}} \leq C\left(1 + |t|^{-\frac{n}{m}\left(\frac{2}{p}-1\right)}\right) \text{ for } t \neq 0.
\]

(b) If $p \in I(n, m)$ and $V \in L^{\frac{p}{|p-2|}}(\mathbb{R}^n)$, then $iP(D) + V$ is the generator of an $\alpha$-times integrated group on $L^p(\mathbb{R}^n)$, where $\alpha > n_p + 1$.

We note that the hypothesis (H2) in [3] is equivalent to that $P_m$ is nondegenerate, and thus Corollary 3.5(a) is an improvement of Theorem 2 in [3], in which $n \geq 3$ and $\frac{n}{m}$ is replaced by
an integer $> \frac{n}{m-1}$. Indeed, $\frac{n}{m}$ is the best one in Corollary 3.5(a). Corollary 3.5(b) extends Corollary 4.7 in [16] to the case of inhomogeneous polynomials. Now we compare it with Theorem 6 (with $V_2 = 0$) in [3] in two respects.

First, if $p > 3$, by Proposition 4.5(c) in [16] it is possible that the domain of $iP(D) + V$ in $L^p(\mathbb{R}^n)$ only is $\{0\}$ for some $V \in L^{\frac{2c}{c+1}}(\mathbb{R}^n)$. Hence the interval $[1, 3]$ is the largest one that $p$ shall belong to. This means the condition $p \in (\frac{2c}{c+1}, \frac{2c}{c-1})$ in Theorem 6 must be replaced by $p \in (\frac{2c}{c+1}, \frac{2c}{c-1}) \cap [1, 3]$, where $c$ is an integer $> \frac{n}{m-1}$. Since $I(n, m)$ contains properly $(\frac{2c}{c+1}, \frac{2c}{c-1}) \cap [1, 3]$, our condition on $p$ provides an improvement of that in Theorem 6.

Next, the conclusion in Theorem 6 is that $iP(D) + V$ is the generator of a smooth distribution group on $L^p(\mathbb{R}^n)$ of order $\alpha$, which is equivalent to that $iP(D) + V$ is the generator of an $\alpha$-times integrated group on $L^p(\mathbb{R}^n)$ (see [2]), where $\alpha$ is an integer $(n + 3)|\frac{1}{2} - \frac{1}{p}| + 2$. Our conclusion in Corollary 3.5(b) however admits that $\alpha$ is a real number $(n + 3)|\frac{1}{2} - \frac{1}{p}| + 1$.

References