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Two-point Padé-type approximation to the Cauchy transform of certain strong distributions ☆

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Dedicated to Professor Haakon Waadeland on the occasion of his 70th birthday

Abstract

In this paper we are mainly concerned with the Cauchy transform of certain strong distributions satisfying a type of symmetric property introduced by A.S. Ranga. Algebraic properties of the corresponding two-point Padé-type approximants are given along with results about convergence for sequences of such approximants

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1. Introduction

In 1980, Jones et al. [9] introduced and solved the so-called strong Stieltjes moment problem. Namely, for a given sequence $\{c_k\}_{k\in\mathbb{Z}}$, find a distribution function, i.e. a real valued, bounded, nondecreasing function $\phi(t)$ with infinitely many points of increase on $[0,\infty)$ such that

$$c_k = \int_0^\infty t^k \, \mathrm{d}\phi(t), \quad k \in \mathbb{Z}. \tag{1.1}$$

Such functions are usually described as strong distributions and since then, many contributions have been given in connection with continued fractions, orthogonal Laurent polynomials, quadrature formulas, two-point Padé approximants and so on. (For a survey about these topics, see [8] and references therein.)

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Associated with a distribution ϕ, we have its Cauchy transform (Stieltjes function) given by

$$F_{\phi}(z)=\int_0^{\infty}\frac{\mathrm{d}\phi(t)}{z-t} \tag{1.2}$$

which admits the asymptotic expansions (see [9])

$$L_0(z)=-\sum_{j=0}^{\infty}c_{-(j+1)}z^j \quad \text{and} \quad L_{\infty}(z)=\sum_{j=1}^{\infty}c_{j-1}z^{-j} \tag{1.3}$$

around $z=0$ and $z=\infty$, respectively.

When considering rational approximation to $F_{\phi}(z)$, starting from the asymptotic expansions L_0 and L_{∞}, then, two-point Padé approximants immediately arise. Thus, given two nonnegative integers k and m with $0\leqslant k\leqslant 2m$, there exist two polynomials $P_{m-1}(z)$ and $Q_m(z)$ of degrees $m-1$ and m respectively such that,

$$\begin{aligned} L_0(z)-\frac{P_{m-1}(z)}{Q_m(z)}&=\mathrm{O}\left(z^k\right)(z\to 0),\\ L_{\infty}(z)-\frac{P_{m-1}(z)}{Q_m(z)}&=\mathrm{O}\left(\left(\frac{1}{z}\right)^{2m-k+1}\right)(z\to\infty). \end{aligned} \tag{1.4}$$

Furthermore, it is known (see [4,10]) that $Q_m(z)$ coincides, up to a multiplicative factor, with the mth orthogonal polynomial with respect to the distribution $\mathrm{d}\phi(t)/t^k$, so that $Q_m(z)$ has exact degree m and all its zeros lie on $(0,\infty)$. We will refer to the rational function $P_{m-1}(z)/Q_m(z)$ as the two-point Padé approximant (2PA) to $F_{\phi}(z)$ and we will write

$$\frac{P_{m-1}(z)}{Q_m(z)}=[k/m]_{F_{\phi}}(z),\quad 0\leqslant k\leqslant 2m.$$

(For an alternative approach based upon orthogonal Laurent polynomials see the papers [6,7].)

On the other hand, according to the ideas given by Brezinski [1], one could also consider rational approximants with prescribed poles, i.e. the denominator is given in advance. Thus, we have the so-called Padé-type approximation. More precisely, let m and p be nonnegative integers with $0\leqslant p\leqslant m$ and $B_m(z)$ a given polynomial of exact degree m such that $B_m(0)\neq 0$, then there exists a unique polynomial $A_{m-1}(z)$ of degree at most $m-1$, such that,

$$\begin{aligned} L_0(z)-\frac{A_{m-1}(z)}{B_m(z)}&=\mathrm{O}(z^p)(z\to 0),\\ L_{\infty}(z)-\frac{A_{m-1}(z)}{B_m(z)}&=\mathrm{O}\left(\left(\frac{1}{z}\right)^{m-p+1}\right)(z\to\infty). \end{aligned} \tag{1.5}$$

It will be said that $A_{m-1}(z)/B_m(z)$ represents a (p/m) two-point Padé-type approximant (2PTA) to $F_{\phi}(z)$, denoted by

$$(p/m)_{F_{\phi}}(z),\quad 0\leqslant p\leqslant m.$$

In this paper we will be mainly concerned with the study and characterization of these rational approximants to the function $F_{\phi}(z)$ when the distribution ϕ satisfies a certain symmetric property introduced by Ranga in [12].

2. Strong c-inversive Stieltjes distributions

In a series of papers (see e.g. [11–13]), Sri Ranga et al. have dealt with some strong distributions on (a,b) $(0 \leqslant a < b \leqslant +\infty)$ along with properties of sequences of polynomials associated with such distributions satisfying certain orthogonality properties. More exactly, for a given strong distribution $\mathrm{d}\phi(t)$ on (a,b), this author considers the monic polynomials $B_m(z)$ of degree m defined by

$$\int_a^b t^{-m+s} B_m(t)\,\mathrm{d}\phi(t) = 0, \quad s = 0,1,\ldots,m-1 \tag{2.1}$$

and shows that they satisfy a three-term recurrence relation of the type,

$$B_{m+1}(z) = (z-\beta_{m+1})B_m(z) - \alpha_{m+1} z B_{m-1}(z), \quad m \geqslant 0 \tag{2.2}$$

with $B_{-1}=0$ and $B_0=1$.

These polynomials are related to certain continued fractions ($\hat{J}$-fraction) and their zeros provide quadrature formulas exactly integrating certain subspaces of Laurent polynomials (see [7]). Note that these continued fractions must be called $\hat{J}$-fractions. (J-fractions are associated with ordinary orthogonal polynomials, and are not equivalent to $\hat{J}$-fractions.)

On the other hand, when considering strong distributions with a somewhat symmetric behavior with respect to the origin and infinity, the concept of a c-inversive distribution arises. Indeed, a strong distribution $\mathrm{d}\phi(t)$ on (a,b) is said to be c-inversive [12] if there exists a positive number $c>0$ such that for all $t \in (a,b)$ it holds that

$$\frac{c}{t} \in (a,b) \quad \text{and} \quad \frac{\mathrm{d}\phi(t)}{\sqrt{t}} = -\frac{\mathrm{d}\phi(c/t)}{\sqrt{c/t}}. \tag{2.3}$$

If $0<a<b<+\infty$, then $c=ab$. Furthermore, if $a=0$ then $b=+\infty$. In the sequel and for the sake of simplicity we will assume that ϕ is absolutely continuous on (a,b), i.e. a nonnegative function $w(t)$ exists such that

$$\mathrm{d}\phi(t) = w(t)\,\mathrm{d}t, \quad t \in (a,b). \tag{2.4}$$

At the same time, the term '*strong*' could be omitted when confusion does not take place.

Now, Eq. (2.3) can be written as

$$\sqrt{t}w(t) = \sqrt{\frac{c}{t}}\, w\left(\frac{c}{t}\right) \quad \text{or} \quad \frac{\sqrt{c}}{t} w\left(\frac{c}{t}\right) = w(t). \tag{2.5}$$

Let us now consider the Cauchy transform

$$F_w(z) = \int_a^b \frac{w(t)}{z-t}\,\mathrm{d}t, \tag{2.6}$$

of the c-inversive distribution $\mathrm{d}\phi(t) = w(t)\,\mathrm{d}t$. One has $F_w(z) = \int_a^b (\sqrt{c}/t)(w(\frac{c}{t})/z-t)\,\mathrm{d}t$. Setting $c/t = x$, it follows:

$$F_w\left(\frac{c}{z}\right) = -\frac{z}{\sqrt{c}} F_w(z), \quad \forall z \in \mathbb{C}\backslash[a,b]. \tag{2.7}$$

In this case, we will also say that $F_w(z)$ is c-inversive.

Example 2.1.

$$w(t) = \frac{1}{\sqrt{b-t}\sqrt{t-a}}, \quad t \in (a,b), \ 0 < a < b < +\infty.$$

It can be easily verified that $\mathrm{d}\phi(t) = w(t)\,\mathrm{d}t$ is c-inversive with $c = ab$.

Furthermore, $F_w(z)$ is now given by (see [12])

$$F_w(z) = \frac{\pi}{\sqrt{z-b}\sqrt{z-a}},$$

so that Eq. (2.7) holds.

Example 2.2.

$$w(t) = \frac{1}{\sqrt{t}}, \quad t \in (a,b), \ 0 < a < b < +\infty.$$

Again this distribution is c-inversive with $c = ab$. Now, we have

$$F_w(z) = \frac{1}{\sqrt{z}} \ln \frac{(\sqrt{b}+\sqrt{z})(\sqrt{a}-\sqrt{z})}{(\sqrt{b}-\sqrt{z})(\sqrt{a}+\sqrt{z})}.$$

Example 2.3.

$$w(t) = t^{\alpha} \exp\left[\beta\left(t^{\gamma} + \frac{A}{t^{\gamma}}\right)\right], \ t \in (0,\infty), \ \alpha \in \mathbb{R}, \ \beta < 0, \ \gamma > \frac{1}{2}, \ A > 0.$$

The corresponding distributions $\mathrm{d}\phi(t) = w(t)\,\mathrm{d}t$ are included in the class studied recently in [10]. It can be checked that $w(t)$ is c-inversive with $c = A^{1/\gamma}$ if and only if $\alpha = -\frac{1}{2}$. The case $\beta = -\frac{1}{2}$ and $\gamma = 1$ was considered by Ranga in [12,13].

3. Two-point Padé-type approximation

From its definition, one can see that the Cauchy transform of a distribution ϕ supported on $[a,b]$ represents an analytic function on the extended complex plane $\widehat{\mathbb{C}}$ except possibly on $[a,b]$. If we assume that ϕ is c-inversive, then by Eq. (2.7) it is enough to compute $F_w(z)$ for z such that $|z| < r$ with $r > \sqrt{c}\,(z \notin [a,b])$.

Thus, when considering 2PTA to $F_w(z)$, w being c-inversive, it seems natural to study the existence of appropriate denominators such that Eq. (2.7) is preserved. So, let $(p/m)_{F_w}(z)(0 \leqslant p \leqslant m)$ be a 2PTA to $F_w(z)$ and set

$$(p/m)_{F_w}(z) = \frac{A_{m-1}(z)}{B_m(z)}.$$

Is it possible to choose the denominator $B_m(z) \in \Pi_m$ so that

$$(p/m)_{F_w}\left(\frac{c}{z}\right) = -\frac{z}{\sqrt{c}}\,(p/m)_{F_w}(z)?$$

Assume that $B_m(z)$ is a polynomial of degree m with all its zeros in (a,b), satisfying

$$B_m(z)=\lambda_m z^m B_m\left(\frac{c}{z}\right), \quad \lambda_m \neq 0. \tag{3.1}$$

Setting $E_m(z)=F_w(z)-(p/m)_{F_w}(z)$, then one knows [4]

$$E_m(z)=\frac{z^p}{B_m(z)}\int_a^b \frac{B_m(t)}{t^p(z-t)}w(t)\,\mathrm{d}t.$$

Therefore, for the numerator $A_{m-1}(z)$, it follows that

$$A_{m-1}(z)=\int_a^b \frac{t^pB_m(z)-z^pB_m(t)}{t^p(z-t)}w(t)\,\mathrm{d}t.$$

Since, $w(t)=(\sqrt{c}/t)w(c/t)$, one has

$$A_{m-1}(z)=\frac{1}{\sqrt{c}}\int_a^b \frac{t^pB_m(z)-z^pB_m(t)}{t^p(z-t)}\frac{c}{t}w\left(\frac{c}{t}\right)\,\mathrm{d}t.$$

Set, $c/t=x$, then it can be deduced

$$A_{m-1}(z)=\sqrt{c}\lambda_m z^m\int_a^b \frac{x^{m-p}B_m(c/z)-(z^{p-m}c^{-p}/\lambda_m^2)B_m(x)}{x^{m-p}z(x-(c/z))}w(x)\,\mathrm{d}x.$$

Asumme now that $\lambda_m=\sqrt{1/c^m}$. Then,

$$A_{m-1}(z)=-\sqrt{c}\lambda_m z^{m-1}\int_a^b \frac{x^{m-p}B_m(c/z)-(c/z)^{m-p}B_m(x)}{x^{m-p}((c/z)-x)}w(x)\,\mathrm{d}x. \tag{3.2}$$

If we write $A_{m-1}^k(z)$ for the numerator of the $(k/m)_{F_w}(z)$ $(0\leqslant k\leqslant m)$-2PTA with denominator $B_m(z)$, from Eq. (3.2) one has

$$A_{m-1}^p(z)=-\sqrt{c}\lambda_m z^{m-1}A_{m-1}^{m-p}\left(\frac{c}{z}\right),$$

when taking in both approximants the same denominator $B_m(z)$. By choosing p such that $m-p=p$ or equivalently $m=2p$, it follows,

$$(p/m)_{F_w}(z)=\frac{A_{m-1}^p(z)}{B_m(z)}=\frac{-\sqrt{c}\lambda_m z^{m-1}A_{m-1}^p(\frac{c}{z})}{z^m\lambda_m B_m(\frac{c}{z})}=-\frac{\sqrt{c}}{z}(p/m)_{F_w}\left(\frac{c}{z}\right).$$

Thus, we have proved the following,

Theorem 3.1. *Let $\mathrm{d}\phi(t)=w(t)\,\mathrm{d}t$ be a c-inversive distribution on (a,b) and $B_n(z)$ a polynomial of degree n such that*

$$B_n(z)=\lambda_n z^n B_n\left(\frac{c}{z}\right), \quad \lambda_n=\frac{1}{\sqrt{c^n}}. \tag{3.3}$$

Let us consider the $(m/2m)$-2PTA with denominator $B_{2m}(z)$ satisfying Eq. (3.3), *then,*

$$(m/2m)_{F_w}\left(\frac{c}{z}\right)=-\frac{z}{\sqrt{c}}(m/2m)_{F_w}(z). \tag{3.4}$$

In this case, we will say that the $(m/2m)$-2PTA is also c-invesive.

Let us next see how to find polynomials $B_{2m}(z)$ satisfying Eq. (3.3).

Proposition 3.2. *Let c be a real positive number and $\{x_j\}_{j=1}^m$ m points on (a,b). Define,*

$$B_{2m}(z)=\gamma\prod_{j=1}^{m}(z-x_j)\left(z-\frac{c}{x_j}\right),\quad \gamma\neq 0.$$

Then, $B_{2m}(z)$ satisfies Eq. (3.3) *with $c=ab$ when $0<a<b<+\infty$.*

Proof. Since $a<x_j<b$, then $c/x_j=ab/x_j<ab/a=b$, and $c/x_j=ab/x_j>ab/b=a$. Thus, $B_{2m}(z)$ is a polynomial of degree $2m$ with all its zeros in (a,b) at the points $\{x_j,\frac{c}{x_j}\}_{j=1}^m$. Furthermore,

$$B_{2m}\left(\frac{c}{z}\right)=\gamma c^m\prod_{j=1}^{m}\left(\frac{c}{z}-x_j\right)\left(\frac{1}{z}-\frac{1}{x_j}\right)=\frac{c^m}{z^{2m}}\gamma\prod_{j=1}^{m}(c-zx_j)\frac{(x_j-z)}{x_j}$$
$$=\frac{c^m}{z^{2m}}\gamma\prod_{j=1}^{m}(z-x_j)\left(z-\frac{c}{x_j}\right)=\frac{c^m}{z^{2m}}B_{2m}(z).\qquad\square$$

Proposition 3.3. *Let $P_m(z)$ be a polynomial of degree m with all its zeros on (a,b) and $P_m(0)\neq 0$. Take $c>0$, (as before, if $0<a<b<\infty$, $c=ab$). Then $B_{2m}(z)=\frac{z^m}{P_m(0)}P_m(z)P_m(\frac{c}{z})$, satisfies Eq.* (3.3).

Proof. Write $P_m(z)=\gamma\prod_{j=1}^m(z-x_j)(\gamma\neq 0)$. Thus

$$P_m\left(\frac{c}{z}\right)=\gamma z^{-m}(-1)^m\prod_{j=1}^{m}x_j\prod_{j=1}^{m}\left(z-\frac{c}{x_j}\right). \tag{3.5}$$

Then, $B_{2m}(z)=\gamma^2\prod_{j=1}^m(z-x_j)(z-(c/x_j))$ and the proof follows by Proposition 3.2. $\square$

In order to get c-inversive 2PTA with arbitary $p\,(0\leqslant p\leqslant m)$ extra requirements are now needed. Indeed, one has

Theorem 3.4. *Let m and p be nonnegative integers $(m>1)$ such that $0\leqslant p\leqslant m$ and $2p>m$. Let $B_m(z)$ be a polynomial of degree m such that*

(i) $B_m(z)=\lambda_m z^m B_m(\frac{c}{z})$, $\lambda_m=1/\sqrt{c^m}$, $c>0$;

(ii) $\int_a^b t^j B_m(t)[w(t)/t^p]\,dt=0$, $j=0,1,\ldots,E[\frac{2p-m+1}{2}]-1$.

Then, the (p/m) 2PTA with denominator $B_m(z)$ is c-inversive.

Proof. First, we must assure that a polynomial of degree at most m, $B_m(z)$, satisfying (i) and (ii) exists. Set

$$B_m(z)=\sum_{j=0}^{m}b_jz^j.$$

Then, by (i) we deduce for the coefficients $\{b_j\}$ the following linear system:

$$b_j = \lambda_m c^{m-j} b_{m-j}, \quad j = 0, 1, \ldots, E\left[\frac{m+1}{2}\right] - 1.$$

As usual $E[x]$, $x \in \mathbb{R}$, denotes the integer part of x.

Next, we will first consider the case $p < m$. From (i) and (ii) we have an homogeneous linear system of $E[\frac{m+1}{2}] + E[\frac{2p-m+1}{2}]$ equations with $m+1$ unknowns.

Since

$$E\left[\frac{m+1}{2}\right] + E\left[\frac{2p-m+1}{2}\right] = \begin{cases} p & \text{if } m \text{ is even,} \\ p+1 & \text{if } m \text{ is odd,} \end{cases}$$

one sees that such system admits a nontrivial solution.

On the other hand, when $p = m$, by virtue of c-inversivity, (ii) implies that

$$\int_a^b t^j B_m(t) \frac{w(t)}{t^p}\, dt = 0, \quad j = 0, 1, \ldots, m-1.$$

That is, $B_m(z)$ is uniquely determined up to a multiplicative factor and represents the mth orthogonal polynomial with respect to the varying weight function $w(t)/t^m$. In [12] it can be seen that this polynomial satisfies property (i). Furthermore, it should be noted that, in this case, we are actually dealing with the $[m/m]$-2PA to $F_w(z)$. Let us next check that the approximant with denominator $B_m(z)$ as given before is c-inversive.

Indeed, because of c-inversivity (ii) implies that

$$\int_a^b t^j B_m(t) \frac{w(t)}{t^p}\, dt = 0, \quad j = 0, 1, 2, \ldots, 2p-m-1. \tag{3.6}$$

Write

$$E_m(z) = F_w(z) - (p/m)_{F_w}(z) = E_m(z) = \frac{z^p}{B_m(z)} \int_a^b \frac{B_m(t)}{t^p(z-t)} w(t)\, dt. \tag{3.7}$$

Now, for any nonnegative integer k one has,

$$E_m(z) = \frac{z^{p-1}}{B_m(z)} \int_a^b B_m(t) \left[1 + \frac{t}{z} + \cdots + \frac{t^k}{z^k} + \frac{t^{k+1}}{z^k(z-t)}\right] \frac{w(t)}{t^p}\, dt. \tag{3.8}$$

Take $k = 2p - m - 1$. By Eq. (3.6) it follows that

$$E_m(z) = \frac{z^{p-1}}{B_m(z)} \int_a^b \frac{B_m(t)}{t^p} \frac{t^{2p-m}}{z^{2p-m-1}} \frac{w(t)}{z-t}\, dt = \frac{z^{m-p}}{B_m(z)} \int_a^b \frac{B_m(t)}{t^{m-p}} \frac{w(t)}{z-t}\, dt. \tag{3.9}$$

Thus, from Eq. (3.7) see that (3.9) represents the error for the $(m-p/m)$ 2PTA with denominator $B_m(z)$. If we put

$$(p/m)(z) = \frac{A^p_{m-1}(z)}{B_m(z)} \quad \text{and} \quad (m-p/m)(z) = \frac{A^{m-p}_{m-1}(z)}{B_m(z)},$$

then from Eq. (3.9) it follows that both approximants coincide and since they have the same denominator we conclude that

$$A^p_{m-1}(z) = A^{m-p}_{m-1}(z),$$

so that proceeding as in Theorem 3.1, the proof follows. □

Remark 3.5. From Eqs. (3.7)–(3.9) one has actually a higher order 2PTA (see [5]), since,

$$E_m(z) = \mathrm{O}(z^p)\ (z \to 0) \quad \text{and} \quad E_m(z) = \mathrm{O}\left(\frac{1}{z^{p+1}}\right)\ (z \to \infty).$$

So, the total order of correspondence both at $z = 0$ and $z = \infty$ is equal to $2p > m$.

Paralleling the proof of Theorem 3.4, a similar result can be deduced for $m > 2p$. Thus, we have

Theorem 3.6. *Let m and p be nonnegative integers $(m > 1)$ such that $0 \leqslant p \leqslant m$ and $m > 2p$. Let $B_m(z)$ be a polynomial of degree m, such that*

(i) $B_m(z) = \lambda_m z^m B_m(\frac{c}{z}),\ c > 0,\ \lambda_m = \frac{1}{\sqrt{c^m}}$,

(ii) $\int_a^b t^j B_m(t) \frac{w(t)}{t^{m-p}}\, \mathrm{d}t = 0,\ j = 0, 1, \ldots, E[\frac{m-2p+1}{2}] - 1.$

Then, the (p/m)-2PTA with denominator $B_m(z)$ is c-inversive.

Remark 3.7. As before, actually one has again a higher order 2PTA, since now

$$E_m(z) = \mathrm{O}(z^{m-p})\ (z \to 0) \quad \text{and} \quad E_m(z) = \mathrm{O}\left(\frac{1}{z^{m-p+1}}\right)\ (z \to \infty).$$

Thus, the total order of correspondence both at the origin and infinity is equal to $2(m - p) > m$.

Let us next consider a (p/m)-2PTA whose total order of correspondence is exactly equal to m, i.e.

$$\begin{aligned} E_m(z) &= \sum_{j=p}^{\infty} d_j z^j = \mathrm{O}(z^p), \quad d_p \neq 0\ \ (z \to 0), \\ E_m(z) &= \sum_{j=m-p+1}^{\infty} d_j^* z^{-j} = \mathrm{O}\left(\frac{1}{z^{m-p+1}}\right), \quad d_{m-p+1}^* \neq 0\ \ (z \to \infty). \end{aligned} \tag{3.10}$$

Theorem 3.8. *Let $(p/m)_{F_w}(z)$ $(0 \leqslant p \leqslant m)$ be a 2PTA to $F_w(z)$ satisfying Eq.* (3.10). *Set $(p/m)_{F_w}(z) = A_{m-1}(z)/B_m(z)$ with $A_{m-1} \in \Pi_{m-1}$ and $B_m(z)$ a monic polynomial of degree m with $B_m(0) \neq 0$. Assume that $(p/m)_{F_w}(z)$ is c-inversive $(c > 0)$. Then*

(i) $m = 2p$ *and* (ii) $B_m(z) = \lambda_m z^m B_m(c/z),\ \lambda_m = 1/c^p$.

Proof. (i) $F_w(z) - (p/m)_{F_w}(z) = \mathrm{O}(z^p)(z \to 0)$ and $F_w(z) - (p/m)_{F_w}(z) = \mathrm{O}(1/z^{m-p+1})(z \to \infty)$ which gives

$$\frac{\sqrt{c}}{z} F_w\left(\frac{c}{z}\right) - \frac{\sqrt{c}}{z} (p/m)_{F_w}\left(\frac{c}{z}\right) = \frac{\sqrt{c}}{z} \mathrm{O}\left(\frac{1}{z^p}\right) = \mathrm{O}\left(\frac{1}{z^{p+1}}\right)$$

and by c-inversivity, this implies

$$F_w(z) - (p/m)_{F_w}(z) = \mathrm{O}\left(\frac{1}{z^{p+1}}\right) \quad (z \to \infty), \quad F_w(z) - (p/m)_{F_w}(z) = \mathrm{O}\ (z^{m-p})(z \to 0). \tag{3.11}$$

Thus from Eqs. (3.10) and (3.11) it follows: $m - p = p$, i.e. $m = 2p$.

(ii) Let us assume that the approximant is not an irreducible rational fraction, otherwise the proof follows easily. So, we can write

$$A_{2p-1}(z) = P(z)A(z), \qquad deg(P) = l \geqslant 1$$
$$B_{2p}(z) = P(z)B(z).$$

Thus $(p/2p)_{F_w}(z) = A_{2p-1}(z)/B_{2p}(z) = A(z)/B(z)$.

On the other hand, from the error expression for the $(p/2p)$-2PTA it follows:

$$F_w(z) - \frac{A(z)}{B(z)} = \frac{z^p}{P(z)B(z)} \int_a^b \frac{P(t)B(t)}{(z-t)t^p} w(t)\,\mathrm{d}t. \tag{3.12}$$

Since the approximant is c-inversive, one has

$$\frac{A(z)}{B(z)} = -\frac{\sqrt{c}z^{2p-l-1}A(c/z)}{z^{2p-l}B(c/z)} \tag{3.13}$$

and because of $A(z)/B(z)$ is irreducible, this implies that

$$B(z) = \gamma z^{2p-l} B(c/z), \qquad \gamma \neq 0. \tag{3.14}$$

Putting $z = \sqrt{c}$ we deduce that $\gamma = 1/c^{\frac{2p-l}{2}}$. For it, take into account that from Eq. (3.13) $A(\sqrt{c}) = 0$, which gives $B(\sqrt{c}) \neq 0$. Furthermore, $A(z)$ should also satisfy

$$A(z) = -\sqrt{c}\gamma z^{2p-l-1} A(c/z). \tag{3.15}$$

Now, from Eq. (3.12) we have

$$B(z)E(z) = B(z)F_w(z) - A(z) = \frac{z^p}{P(z)} \int_a^b \frac{P(t)}{z-t} \frac{B(t)}{t^p} w(t)\,\mathrm{d}t = M(z).$$

By Eqs. (3.14) and (3.15) and that $F_w(z) = -(c/z)F_w(c/z)$ we see that

$$\sqrt{c}\gamma z^{2p-l-1} E(c/z) = -E(z). \tag{3.16}$$

Now, replacing $E(z)$ by $M(z)$ in Eq. (3.16) and after some elementary calculations (including the usual change of variable $c/t = x$) the following holds:

$$\frac{1}{z^l P(c/z)} \int_a^b \frac{x^l P(c/x)}{z-x} B(x) \frac{w(x)}{x^p}\,\mathrm{d}x = \frac{1}{P(z)} \int_a^b \frac{P(x)}{z-x} B(x) \frac{w(x)}{x^p}\,\mathrm{d}x,$$

or equivalently,

$$\frac{P(z)}{z^l P(c/z)} = \frac{\int_a^b \frac{P(x)}{z-x} B(x) \frac{w(x)}{x^p}\,\mathrm{d}x}{\int_a^b \frac{x^l P(c/x)}{z-x} B(x) \frac{w(x)}{x^p}\,\mathrm{d}x}. \tag{3.17}$$

Now, the functions appearing in the right-hand member of Eq. (3.17) are holomorphic functions outside $[a,b]$. Furthermore, since $w(x) > 0$ a.e. on $[a,b]$, they cannot be rational functions. Thus, there must exist a constant β such that

$$P(z) = \beta z^l P(c/z).$$

Setting, $z = \sqrt{c}$, we have $\beta = 1/\sqrt{c^l}$.

Therefore, $B_{2p}(z) = P(z)B(z) = (1/\sqrt{c^l})P(c/z)(1/\sqrt{c^{2p-l}})B(c/z) = (1/c^p)P(c/z)B(c/z) = (1/c^p)B_{2p}(c/z)$.

This has been done under the assumption that $P(\sqrt{c}) \neq 0$. Suppose that $P(\sqrt{c})=0$. Since $\deg(P)=l$ is even, then $z=\sqrt{c}$ must be a root with multiplicity even. More precisely,

$$P(z)=(z-\sqrt{c})^{2j}R(z), \quad j \geqslant 1.$$

Thus,

$$\begin{aligned} P(z)=(z-\sqrt{c})^{2j}R(z) &= \beta z^l \left(\frac{c}{z}-\sqrt{c}\right)^{2j} R(c/z) \\ &= \beta c^j z^{l-2j} R(z)(z-\sqrt{c})^{2j}, \quad R(\sqrt{c}) \neq 0. \end{aligned}$$

Hence, $\beta c^j(\sqrt{c})^{l-2j}=1$, which gives $\beta = 1/\sqrt{c^l}$. □

Theorems 3.1 and 3.9 provide us with the following

Corollary 3.9. *Let $(p/m)_{F_w}(z)=A_{m-1}/B_m(z)$ be a 2PTA with $B_m(z)$ a polynomial of the degree m where $B_m(0) \neq 0$ and assume that Eq.* (3.10) *holds. Then $(p/m)_{F_w}(z)$ is c-inversive, if and only if,* (i) *$m=2p$ and* (ii) *$B_m(z)=\lambda_m z^m B_m(\frac{c}{z})$, $\lambda_m = 1/c^p$.*

To end this section, let us see the reciprocals of Theorems 3.4 and 3.6. Indeed, one first has,

Proposition 3.10. *Let $(p/m)_{F_w}(z)=A_{m-1}(z)/B_m(z)$ be a 2PTA to $F_w(z)$ where p and m are non-negative integers such that $0 \leqslant p \leqslant m$ and $2p>m$. Assume that $(p/m)_{F_w}(z)$ is c-inversive. Then,* (i) *$\int_a^b t^j B_m(t)(w(t)/t^p)\,\mathrm{d}t=0$, $j=0,1,2,\ldots,2p-m-1$ and* (ii) *$B_m(z)=\lambda_m z^m B_m(c/z)$, $\lambda_m = 1/\sqrt{c^m}$.*

Proof. (a) We have

$$\begin{aligned} F_w(z)-(p/m)_{F_w}(z) &= \mathrm{O}(z^p)\,(z\to 0), \\ F_w(z)-(p/m)_{F_w}(z) &= \mathrm{O}\left(\frac{1}{z^{m-p+1}}\right)(z\to\infty). \end{aligned} \tag{3.18}$$

As the approximant is c-inversive we can also write

$$\begin{aligned} F_w(z)-(p/m)_{F_w}(z) &= \mathrm{O}(z^{m-p})\,(z\to 0), \\ F_w(z)-(p/m)_{F_w}(z) &= \mathrm{O}\left(\frac{1}{z^{p+1}}\right)(z\to\infty). \end{aligned} \tag{3.19}$$

Since $2p>m$ then $p>m-p$. Thus, by Eqs. (3.18) and (3.19), it follows that

$$E_m(z)=\mathrm{O}(z^p)(z\to 0) \quad \text{and} \quad E_m(z)=\mathrm{O}\left(\frac{1}{z^{p+1}}\right)(z\to\infty). \tag{3.20}$$

Since $\deg(B_m)=m$, then one has

$$\frac{1}{B_m(z)}=z^{-m}\sum_{j=0}^{\infty} b_j z^{-j}, \quad b_0 \neq 0.$$

Thus, by Eq. (3.8) one can write:

$$E_m(z)=z^{-(m-p+1)}\left(\sum_{j=0}^{\infty} b_j z^{-j}\right)\left(\sum_{k=0}^{\infty} d_k z^{-k}\right),$$

where

$$d_k = \int_a^b t^k B_m(t)\,\frac{w(t)}{t^p}\,\mathrm{d}t, \quad k = 0, 1, 2, \ldots .$$

Hence, this results in

$$E_m(z) = \left(\frac{1}{z}\right)^{m-p+1} \sum_{j=0}^{\infty} \alpha_j z^{-j} \quad \text{with} \quad \alpha_j = \sum_{i=0}^{j} b_i d_{j-i}, \;\; j = 0, 1, \ldots, \;\; b_0 \neq 0.$$

From Eq. (3.20) it follows $\alpha_j = 0$; $j = 0, 1, \ldots, 2p - m - 1$, and since $b_0 \neq 0$ this implies $d_j = 0$, $j = 0, 1, \ldots, 2p - m - 1$, which yields (i). Now proceeding as in proof of (ii) for Theorem 3.8, (ii) can be achieved. □

In a similar way one can also prove the following.

Proposition 3.11. *Let $(p/m)_{F_w}(z) = A_{m-1}(z)/B_m(z)$ be a 2PTA to $F_w(z)$ where p and m are non-negative integers such that $0 \leqslant p \leqslant m$ and $2p < m$. Assume that $(p/m)_{F_w}(z)$ is c-inversive. Then,* (i) $B_m(z) = \lambda_m z^m B_m(c/z)$, $\lambda_m = 1/\sqrt{c^m}$ *and* (ii) $\int_a^b t^j B_m(t)[w(t)/t^{m-p}]\,\mathrm{d}t = 0$, $j = 0, 1, 2, \ldots, m - 2p - 1$.

4. Convergence

Let us first consider the case $0 < a < b < +\infty$ and $w(t)$ c-inversive ($c = ab$). For a given sequence $\{(n/2n)\}$ of 2PTA, to $F_w(z)$, we will study when it converges to $F_w(z)$. As usual in Padé-type approximation [1], the key is to find an appropriate choice of denominators. Indeed, one has

Theorem 4.1. *Let α be a positive measure on $[a, b]$ such that $\alpha'(x) > 0$ a.e. on $[a, b]$. Let $Q_n(z)$ denote the nth monic orthogonal polynomial with respect to $d\alpha$. Set*

$$B_{2n}(z) = \frac{z^n}{Q_n(0)} Q_n(z) Q_n\left(\frac{c}{z}\right). \tag{4.1}$$

Under these conditions the following holds:

(i) $(n/2n)_{F_w}(z) = A_{2n-1}/B_{2n}(z)$ *is c-inversive*;

(ii) *Let K be a compact in $\mathbb{C} \setminus [a, b]$. Then, there exits a positive constant $\lambda = \lambda(K) < 1$ so that*

$$\limsup_{n\to\infty} \|F_w(z) - (n/2n)_{F_w}(z)\|_K^{1/2n} \leqslant \lambda(K),$$

where $\| - \|_K$ represents the suprem norm.

Proof. (i) It immediately follows from Proposition 3.3.

(ii) Set

$$\begin{aligned} E_{2n}(z) &= F_w(z) - (n/2n)_{F_w}(z) \\ &= \frac{z^n}{B_{2n}(z)} \int_a^b \frac{B_{2n}(t)}{t^n(z-t)} w(t)\,\mathrm{d}t \end{aligned}$$

$$= \frac{z^n Q_n(0)}{z^n Q_n(z) Q_n(\frac{c}{z})} \int_a^b \frac{t^n Q_n(t) Q_n(\frac{c}{t})}{Q_n(0) t^n (z-t)} w(t)\,\mathrm{d}t$$

$$= \frac{1}{Q_n(z) Q_n(\frac{c}{z})} \int_a^b \frac{Q_n(t) Q_n(\frac{c}{t})}{z-t} w(t)\,\mathrm{d}t, \quad \forall z \in \widehat{\mathbb{C}} \backslash [a,b]. \tag{4.2}$$

Define: $\|Q_n\|_\infty = \max_{x\in[a,b]} |Q_n(x)|$. Since $c/t \in [a,b]$ $\forall t \in [a,b]$ then $|Q_n(t)| \leqslant \|Q_n\|_\infty$ and $|Q_n(c/t)| \leqslant \|Q_n\|_\infty \forall t \in [a,b]$. Take $z \in K \subset \mathbb{C}\backslash[a,b]$, K compact. For $z \notin [a,b]$, we have

$$|E_{2n}(z)| \leqslant \frac{\|Q_n\|_\infty^2}{|Q_n(z)||Q_n(\frac{c}{z})|} \int_a^b \frac{w(t)}{|z-t|}\,\mathrm{d}t \leqslant \frac{M(K)\|Q_n\|_\infty^2}{|Q_n(z)||Q_n(\frac{c}{z})|},$$

M being a positive constant dependent on K. Furthermore, if $z \notin [a,b]$, then $c/z \notin [a,b]$, so

$$|E_{2n}(z)|^{1/2n} \leqslant \frac{[M(K)]^{1/2n} ||Q_n||_\infty^{1/n}}{|Q_n(z)|^{1/2n} |Q_n(\frac{c}{z})|^{1/2n}},$$

and consequently

$$\limsup_{n\to\infty} |E_{2n}(z)|^{1/2n} \leqslant \frac{\limsup_{n\to\infty} \|Q_n\|_\infty^{1/n}}{\liminf_{n\to\infty} \{|Q_n(z)|^{1/2n} |Q_n(c/z)|^{1/2n}\}}.$$

On the other hand, one knows (see [14])

$$\limsup_{n\to\infty} ||Q_n||_\infty^{1/n} = \mathrm{Cap}([a,b]),$$

$$\limsup_{n\to\infty} |Q_n(z)|^{1/n} = \mathrm{Cap}([a,b]) \Phi_{[a,b]}(z), \quad \forall z \notin [a,b],$$

where $\mathrm{Cap}([a,b]) = \frac{1}{4}(b-a)$ (see e.g. [14]) and $\Phi_{[a,b]}$ is the conformal transformation mapping $\widehat{\mathbb{C}}\backslash[a,b]$ onto the exterior of the unit circle, preserving the point at infinity. Therefore

$$|\Phi_{[a,b]}(z)| > 1, \quad \forall z \in \widehat{\mathbb{C}}\backslash[a,b].$$

Thus, we finally obtain,

$$\limsup_{n\to\infty} |E_{2m}(z)|^{1/2n} \leqslant \frac{1}{\sqrt{|\Phi_{[a,b]}(z)||\Phi_{[a,b]}(c/z)|}} \leqslant \lambda(K) < 1$$

with

$$\lambda(K) = \frac{1}{\inf_{z\in K} \sqrt{|\Phi_{[a,b]}(z)||\Phi_{[a,b]}(c/z)|}}. \quad \square \tag{4.3}$$

Remark 4.2. Certainly, it should be observed that part (ii) in Theorem 4.1 is valid for a general distribution $\mathrm{d}\phi(t) = w(t)\,\mathrm{d}t$ not necessarily c-inversive. Even more, $w(t)$ can be an L_1-integrable function on $(a,b)(0<a<b<+\infty)$ and an arbitrary sequence of 2PTA $(k/n)_{F_{w(z)}}(z) = A_{n-1}(z)/B_n(z)$ can be considered where $B_n(z) = z^k Q_k(c/z) Q_{n-k}(z)$, with $k = k(n)$ a sequence of nonnegative integers such that $0 \leqslant k(n) \leqslant n$ and $\lim_{n\to\infty} k(n)/n = \theta$ $(0 \leqslant \theta \leqslant 1)$. In this respect see [4]. On the other hand, assume that the '*auxiliary*' measure $\alpha(t)$ is c-inversive too. Here it is assumed that

$d\alpha(t)/t^{2n} = [\alpha'(t)/t^{2n}]\,dt$, i.e., that α is absolutely continuous, with $\alpha'(t) > 0$ a.e. on $[a,b]$. Let $B_{2n}(z)$ be orthogonal with respect to $d\alpha(t)/t^{2n}$ i.e.

$$\int_a^b t^j B_{2n}(t)\frac{\alpha'(t)}{t^{2n}}\,dt = 0, \quad j = 0,1,2,\dots,2n-1.$$

By [12] we have that the 2PTA with denominator $B_{2n}(z)$ is c-inversive. Set

$$(n/2n)_{F_w}(z) = \frac{A_{2n-1}(z)}{B_{2n}(z)}$$

then, making use of [2, Theorem 5.4], we have ($\theta = \frac{1}{2}$)

$$\limsup_{n\to\infty} |E_{2m}(z)|^{1/2n} \leqslant \exp\{-G_{[a,b]}(v,z)\}.$$

Here, $G_{[a,b]}(v,z)$ denotes the Green potencial of the measure $v = \frac{1}{2}[\delta_0 + \delta_\infty]$ and δ_x the Dirac measure corresponding to the point x. Thus, $G_{[a,b]}(v,z) = \frac{1}{2}[g_{[a,b]}(z,0) + g_{[a,b]}(z,\infty)]$.

As usual, $g_{[a,b]}(z,t)$ is the Green function whith singularity in $t \in \widehat{\mathbb{C}}\backslash[a,b]$.

Therefore, one obtains

$$\begin{aligned}\limsup_{n\to\infty} |E_{2n}(z)|^{1/2n} &\leqslant \exp\{-\tfrac{1}{2}[g_{[a,b]}(z,0) + g_{[a,b]}(z,\infty)]\}\\ &= \frac{1}{\sqrt{\exp g_{[a,b]}(z,0)\exp g_{[a,b]}(z,\infty)}}.\end{aligned} \tag{4.4}$$

Compare the estimate deduced from Eq. (4.4) with the one obtained in Eq. (4.3). It can be checked that both estimates coincide.

Let us see next what happens when $[a,b]$ is an infinite interval i.e. $[a,b] = [0,\infty)$. We first have,

Theorem 4.3. *Let $w(t)$ be c-inversive on $[0,\infty)$ ($w(t)$ not necessarily positive) and let $\alpha(t)$ be a positive measure which is also c-inversive on $[0,\infty)$ and the unique solution of a strong Stieltjes moment problem. Assume that $\int_0^\infty(|w(t)|^2/\alpha'(t))\,dt = k_1^2 < +\infty$.*

Let $B_{2n}(z)$ be an orthogonal polynomial of degree $2n$ with respect to $d\alpha(t)/t^{2n}$. Then,

(i) *$(n/2n)_{F_w}(z) = A_{2n-1}(z)/B_{2n}(z)$ is c-inversive,*
(ii) *$\{(n/2n)_{F_w}(z)\}$ converges uniformly to $F_w(z)$ on any compact set of $\mathbb{C}\backslash[0,\infty)$.*

Proof. (i) By Theorem 3.2 of [12] we see that $B_{2n}(z)$ satifies $B_{2n}(z) = \lambda_{2n}z^{2n}B_{2n}(c/z)$ with $\lambda_{2n} = 1/c^n$. Thus, from Theorem 3.1(i) is proved.

(ii) It readily follows from Theorem 5.1 of [3]. □

As we have already seen in Example 1.3, given $c > 0$, the measure

$$d\alpha(t) = t^{-1/2}\exp\left[\beta\left(t^\gamma + \frac{c^\gamma}{t^\gamma}\right)\right]dt, \quad \gamma > \frac{1}{2}, \quad \beta < 0 \tag{4.5}$$

is c-inversive. Thus proceeding as in Theorem 5.7 of [3], we can also give, in this case, an estimate of the rate of convergence. Indeed,

Theorem 4.4. *Let $w(t)$ be c-inversive on $[0,\infty)$. Assume that two constants $\gamma > \frac{1}{2}$ and $\beta < 0$ exist such that*

$$\int_0^\infty \frac{|w(t)|^2\sqrt{t}}{\exp[\beta(t^\gamma + \frac{c^\gamma}{t^\gamma})]}\,\mathrm{d}t = K_1^2 < +\infty.$$

Let $B_{2n}(z)$ be the orthogonal polynomial of degree $2n$ with respect to $\mathrm{d}\alpha(t)/t^{2n}$ with $\mathrm{d}\alpha(t)$ given by Eq. (4.5). *Then, for any compact subset K of $\mathbb{C}\backslash[0,\infty)$ there exists a positive constant $\eta=\eta(K) < 1$ so that*

$$\limsup_{n\to\infty} ||F_w(z) - (n/2n)_{F_w}(z)||^{1/(2n)^r} \leqslant \eta(K)$$

with $0 < r = 1 - \frac{1}{2\gamma} < 1$.

Remark 4.5. Take into account that Eq. (4.5) can be written as

$$\mathrm{d}\alpha(t) = t^{-1/2}\exp(-\tau(t)) \quad \text{where } \tau(t) = |\beta|\left(t^\gamma + \frac{c^\gamma}{t^\gamma}\right), \quad \gamma > \frac{1}{2}.$$

According to [10], $\tau(t)$ should satisfy for some $s > 0$

$$\lim_{t\to 0^+}(st)^\gamma\tau(t) = |\beta| \lim_{t\to+\infty}(st)^{-\gamma}\tau(t) = A > 0. \tag{4.6}$$

Thus,

$$\lim_{t\to 0^+}(st)^\gamma\tau(t) = |\beta| \lim_{t\to 0^+}(st)^\gamma\left[t^\gamma + \frac{c^\gamma}{t^\gamma}\right] = |\beta|s^\gamma c^\gamma.$$

On the other hand,

$$\lim_{t\to+\infty}(st)^{-\gamma}\tau t = |\beta|, \lim_{t\to+\infty}(st)^{-\gamma}\left[t^\gamma + \frac{c^\gamma}{t^\gamma}\right] = |\beta| \text{ and } \lim_{t\to+\infty}\left[s^{-\gamma} + \frac{c^\gamma}{t^{2\gamma}}\right] = |\beta|s^{-\gamma}.$$

In order to fulfill Eq. (4.6), it should hold, $|\beta|s^\gamma c^\gamma = |\beta|s^{-\gamma}$, i.e. $s = 1/\sqrt{c} > 0$. With this choice of the parameter $s > 0$, we have

$$A = |\beta|c^{\gamma/2}. \tag{4.7}$$

In this case, $\eta(K)$ can be expressed as (see [10]) $\eta(K) = \exp(-R)$ with $R = D(\gamma)\inf_{z\in K}\{\delta(z)\} > 0$, where $\delta(z)$ and $D(\gamma)$ are given by

$$\delta(z) = \left(\frac{1}{2}\right)^r \left[Im(sz)^{\frac{1}{2}} + Im(sz)^{-\frac{1}{2}}\right], \quad s = \frac{1}{\sqrt{c}}$$

and

$$D(\gamma) = \frac{2\gamma}{2\gamma-1}\left[\frac{A\Gamma(\gamma+\frac{1}{2})}{\sqrt{\pi}\Gamma(\gamma)}\right]^{\frac{1}{2\gamma}}, \quad A = |\beta|c^{\frac{\gamma}{2}}.$$

Assume $\gamma = 1$ (this measure was considered by Ranga [12]), then $r = 1 - (1/2\gamma) = 1 - \frac{1}{2} = \frac{1}{2}$. Thus,

$$\delta(z) = \frac{1}{\sqrt{2}}\left[Im\left(\frac{z}{\sqrt{c}}\right)^{\frac{1}{2}} + Im\left(\frac{z}{\sqrt{c}}\right)^{-\frac{1}{2}}\right] \quad \text{and} \quad D(1) = \sqrt{2|\beta|\sqrt{\frac{c}{\pi}}}.$$

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