Homology and cohomology with coefficients, of an algebra over a quadratic operad

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Abstract

The aim of this paper is to give a definition of groups of homology and cohomology with coefficients, for an algebra over a quadratic operad in characteristic zero. Our work completes the works of Ginzburg and Kapranov (1994) of Kimura and Voronov (1995) and of Fox and Markl (1997). In particular, we emphasize that coefficients for homology and cohomology are radically different. If $U(A)$ denotes the enveloping algebra of a $P$-algebra $A$, then one can define the homology of $A$ with coefficients in a right module over $U(A)$ and the cohomology of $A$ with coefficients in a left module over $U(A)$. For classical operads such as those encoding associative, commutative, Lie or Poisson algebras, there is no difference between left module and right module; it is not the case for the operads encoding Leibniz algebras and dual Leibniz algebras. This phenomenon has already been observed by Loday (1995) for Leibniz algebras. At the end of this paper, we study the new case of dual Leibniz algebras and the relations between our theory and Barr–Beck’s theory of homology. © 1998 Elsevier Science B.V. All rights reserved.

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Introduction

The aim of this paper is to give a definition of groups of homology and cohomology with coefficients, for an algebra over a quadratic operad in characteristic zero. This problem has already been undertaken by Ginzburg and Kapranov in [11], where they defined homology with coefficients in the ground field $k$. However, their definition is not very explicit (it is based on the language of trees), in particular for the determination
of the differential of the complex calculating these groups. In this paper we propose an effective method for such a determination based on the knowledge, for an operad, of its \textit{comp}-operations. Furthermore, although in [11] the concept of a module over an algebra is defined, there is no approach to the definition of homology or cohomology with coefficients. This gap is partly filled by Kimura and Voronov in [13], where they defined cohomology with coefficients in a module by using the machinery of trees as in [11].

Cohomology with coefficients has also been studied by Fox and Markl in [6], but again their approach is voluntarily theoretical and the practical determination of the differential remains a problem. To illustrate our writings let us cite the following sentence of [6]: “The problem is that we do not know an easy way to describe the differential... the only way we know is the one based on the somewhat explicit description of the Koszul dual operad using the language of trees as in [10,11]”. Furthermore, Fox and Markl do not define a theory of homology with coefficients.

In their nature, coefficients for homology and cohomology are radically different. More precisely, if $M$ is a module over a $\mathcal{P}$-algebra $A$ (in our paper we prefer to use the expression “representation of $A$”) although cohomology of $A$ with coefficients in $M$ can be defined, in general one cannot give a sense to a theory of homology of $A$ with coefficients in $M$. The fundamental reason for this fact is the following: for every vector space $A$ the collection $\text{End}(A) = \{\text{Hom}_k(A^\otimes n,A); n \in \mathbb{N}^*\}$ is naturally an operad; this supplementary structure can be used to define the cohomology of $A$ with coefficients in itself, and then the cohomology of $A$ with coefficients in $M$ (since $A \oplus M$ is a particular $\mathcal{P}$-algebra). If we now want to construct a theory of homology such that $A$ is an admissible coefficient, $\text{End}(A)$ has to be a cooperad (the dual notion) which is not the case in general. On the contrary, for $A$ finite dimensional, the collection $\{\text{Hom}_k(A^\otimes n,A^*); n \in \mathbb{N}^*\}$ is a cooperad because $\text{Hom}_k(A^\otimes n,A^*)$ coincides with $(\text{Hom}_k(A^*,(A^*)^\otimes n))^*$ and because, for every vector space $B$, the collection $\text{CoEnd}(B) = \{\text{Hom}_k(B,B^\otimes n); n \in \mathbb{N}^*\}$ is naturally an operad (so its dual is a cooperad). Therefore, we can define the homology of $A$ with coefficients in $A^*$; if we examine the new structure inherited by $A^*$, we see that it is the dual notion of representation of $A$ called corepresentation in this paper. More precisely, the category of representations of $A$ is equivalent to the category of left module over the enveloping algebra $U_\mathcal{P}(A)$ (in the case where $\mathcal{P}$ is quadratic, we give an explicit description of $U_\mathcal{P}(A)$); by definition, a corepresentation of $A$ is a right module over $U_\mathcal{P}(A)$.

In this paper we define the homology of $\mathcal{P}$ algebras with coefficients in corepresentations; for classical operads encoding associative, commutative, Lie and Poisson algebras (whose operads are denoted by $\text{Ass}$, $\text{Com}$, $\text{Lie}$ and $\text{Poiss}$), the category of representations is equivalent to the category of corepresentations, hence theories of homology and cohomology use the same coefficients. It is not the case for other operads such as the operads of Leibniz and dual Leibniz algebras denoted by $\text{Leib}$ and $\text{Leib}^!$ (for more precisions about these operads see [15–17]). For Leibniz algebras, this phenomenon has been already observed by Loday in [15]; as an appli-
cation of our theory and in order to show that it is really effective, at the end of this paper we define homology and cohomology with coefficients for dual Leibniz algebras.

This paper is divided into seven sections: in Section 1 and 2 we define the category of representation and corepresentation of a \( \mathcal{P} \)-algebra \( A \). In Section 3 we define the cohomology of \( A \), with coefficients in a representation \( M \), denoted by \( H^*_\mathcal{P}(A,M) (\ast \geq 1) \); Section 4 is devoted to the homology of \( A \), with coefficients in a corepresentation \( N \), denoted by \( H^*_\mathcal{P}(A,N) (\ast \geq 1) \); in Section 5 we introduce the complex \( W^*_\mathcal{P}(A) \) which plays the role of the bar construction, and we establish the following isomorphisms:

\[
H^*_\mathcal{P}(A,M) \simeq H^*_\mathcal{P}(\text{Hom}_{\mathcal{P}(A)}(W^*_\mathcal{P}(A),M)),
\]

\[
H^*_\mathcal{P}(A,N) \simeq H^*_\mathcal{P}(N \otimes_{\mathcal{P}(A)} W^*_\mathcal{P}(A)).
\]

In general, the theories of homology and cohomology we define do not come from derived functors; however, at the end of this part we construct two spectral sequences \( E^p_{r,q} \) and \( F^p_{r,q} \) such that: \( E^r_{2,p} = E^r_{\mathcal{P}(A)}(H^r_{p+1}(W^*_\mathcal{P}(A)), M) \Rightarrow H^r(\mathcal{P}(A),M) \) and \( F^p_{r,q} = \text{Tor}^r_{\mathcal{P}(A)}(N, H^q_{p+1}(W^*_\mathcal{P}(A))) \Rightarrow H^p_{r,q}(A,N) \). In Section 6, as an application of our theory, we define homology and cohomology with coefficients for dual Leibniz algebras. Finally, in Section 7 we study the relations between our theory and Barr–Beck’s theory of homology [3].

0. Preliminaries

All objects are assumed to be defined over a fixed ground field \( k \) of characteristic zero. The tensor product over \( k \) will be noted \( \otimes \). For any natural \( n \in \mathbb{N}^* \), \( \Sigma_n \) will denote the group of permutations of a finite set having \( n \) elements, and \( 1_n \) will denote the identity permutation; if \( \alpha_1, \ldots, \alpha_k \) are \( k \) numbers belonging to \( \{1, \ldots, n\} \), \( (\alpha_1 \alpha_2 \cdots \alpha_k) \) will stand for the cyclic permutation on \( \{\alpha_1, \ldots, \alpha_k\} \); for \( \sigma \in \Sigma_n \), \( \varepsilon(\sigma) \in \{-1,1\} \) will be the sign of \( \sigma \) and \( \text{sgn}_n \) will denote the sign representation of \( \Sigma_n \). The following convention will often be used in all the paper: if \( f: \otimes_{i=1}^n E_i \rightarrow F \) is a map, then \( f(x_1, \ldots, x_n) \) will stand for \( f(x_1 \otimes \cdots \otimes x_n) \).

In the sequel, we briefly recall what will be needed about operads; for more details the reader should refer to [10, 11, 16, 21].

**Definition 0.1.** A \( k \)-linear operad is a sequence \( \mathcal{P} = \{\mathcal{P}(n), n \in \mathbb{N}^*\} \) of vector spaces over \( k \) such that:

**OPER 1:** Each \( \mathcal{P}(n) \) is a right module over \( k[\Sigma_n] \).

**OPER 2:** For all \( n \geq 1 \) and \( 1 \leq i \leq n \) there exists maps

\[
\sigma_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \rightarrow \mathcal{P}(n + m - 1),
\]
such that the following identities holds: for \( l, m, n \in \mathbb{N}^* \), \( \lambda \in \mathcal{P}(l) \), \( \mu \in \mathcal{P}(m) \) and \( v \in \mathcal{P}(n) \),

\[
(\lambda \circ_i \mu) \circ_j v = \begin{cases} 
(\lambda \circ_j v) \circ_{i+n} \mu, & 1 \leq j \leq i - 1, \\
(\lambda \circ_i (\mu \circ_{j+i} v), & i \leq j \leq m + i - 1, \\
(\lambda \circ_{j-m} v) \circ_i \mu, & i + m \leq j.
\end{cases}
\]  

(0.1)

Furthermore, we have the following equivariant conditions: for all \( \mu, v \in \mathcal{P}(m) \times \mathcal{P}(n) \), \( \sigma, \tau \in \Sigma_m \times \Sigma_n \) and \( i \in \{1, \ldots, m\} \),

\[
(\mu \circ_i v)(\sigma \circ_i \tau) = \mu \sigma \circ_{\sigma^{-1}(i)} v \tau;
\]

where \( \sigma \circ_i \tau \) stands for the permutation of \( \Sigma_{m+n-1} \) defined by

\[
\sigma \circ_i \tau = \hat{\sigma} \circ (1 \times \cdots \times \tau \times \cdots \times 1)
\]

(\( \tau \) is at the \( i \)th place), where \( \hat{\sigma} \) permutes the blocks in which \( 1 \times \cdots \times \tau \times \cdots \times 1 \) acts in the same way as \( \sigma \) permutes \( \{1, \ldots, n\} \). We called \( \circ_i \) the comp\(_i\)-operation between \( \mathcal{P}(n) \) and \( \mathcal{P}(m) \): as pointed out to us by Martin Markl, the comp\(_i\)-operations more or less coincide to the operations which define a comp-algebra in the sense of Gerstenhaber [7, 9].

**OPER 3:** There exists a unit element \( 1 \in \mathcal{P}(1) \) such that for all \( n \in \mathbb{N}^* \) and \( 1 \leq i \leq n \), \( \mu \circ_i 1 = \mu \) and \( 1 \circ_i \mu = \mu \).

**Definition 0.2.** Let \( \mathcal{P} \) and \( \mathcal{Q} \) be two operads. A morphism of operads from \( \mathcal{P} \) to \( \mathcal{Q} \) is a sequence \( a = \{a(n), n \in \mathbb{N}^*\} \) of \( k[\Sigma_n] \)-linear maps \( a(n) : \mathcal{P}(n) \to \mathcal{Q}(n) \) such that:

(i) \( a(1)(1) = 1; \)

(ii) for all \( n, m \in \mathbb{N}^* \), \( 1 \leq i \leq n \), \( \mu \in \mathcal{P}(n) \), \( v \in \mathcal{P}(m) \), \( a(n+m-1)(\mu \circ_i v) = a(n)(\mu) \circ_i a(m)(v) \).

**Example 0.3.** Let \( V \) be a vector space over \( k \) and for every \( n \in \mathbb{N}^* \) let \( \text{End}(V)(n) \) denote the vector space \( \text{Hom}_k(V^\otimes n, V) \). Then \( \text{End}(V) = \{\text{End}(V)(n), n \in \mathbb{N}^*\} \) is naturally an operad: the right action of \( \Sigma_n \) is given, for \( \sigma \in \Sigma_n \), \( f \in \text{End}(V)(n) \) and \( v_1, \ldots, v_n \in V \), by

\[
(f \sigma)(v_1, \ldots, v_n) = f(v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)}).
\]

The comp\(_i\)-operation between \( \text{End}(V)(n) \) and \( \text{End}(V)(m) \) is given, for \( 1 \leq i \leq n \), \( f \in \text{End}(V)(n) \) and \( g \in \text{End}(V)(m) \), by

\[
f \circ_i g(v_1, \ldots, v_{n+m-1}) = f(v_1, \ldots, g(v_i, \ldots, v_{i+m-1}), v_{i+m}, \ldots, v_{n+m-1}).
\]

The unit is given by \( \text{id} : V \to V \). We called \( \text{End}(V) \) the endomorphism operad.
The key notion which will interest us in all the paper is the notion of an algebra over an operad.

**Definition 0.4.** Let \( V \) be a vector space over \( k \). We say that \( V \) is a \( \mathcal{P} \)-algebra or an algebra over \( \mathcal{P} \), if there is a morphism of operads \( a \) from \( \mathcal{P} \) to \( \text{End}(V) \). In the sequel, for \( \mu \in \mathcal{P}(n) \), \( v_1, \ldots, v_n \in V \), we will set \( a(\mu)(v_1 \otimes \cdots \otimes v_n) = \mu(v_1, \ldots, v_n) \).

A morphism of \( \mathcal{P} \)-algebras \( \phi : (V, a) \to (W, b) \) is a \( k \)-linear map \( \phi : V \to W \) such that, for any \( v_1, \ldots, v_n \in V \) and \( \mu \in \mathcal{P}(n) \), \( \phi(\mu(v_1, \ldots, v_n)) = \mu(\phi(v_1), \ldots, \phi(v_n)) \).

**0.5. Quadratic operads.** We recall without proof the following proposition [10, 11]:

**Proposition 0.5.1.** Let \( E \) be a \( k[\Sigma] \)-module, then there exists an operad \( \mathcal{F}(E) \) such that \( \mathcal{F}(E)(1) = k \) and \( \mathcal{F}(E)(2) = E \) (the free operad generated by \( E \)) such that the following property holds: for any operad \( \mathcal{J} \) and for any morphism of \( k[\Sigma] \)-modules \( a : \mathcal{J} \to \mathcal{J}(2) \), there is a unique morphism of operads, \( \tilde{a} : \mathcal{F}(E) \to \mathcal{J} \), such that \( \tilde{a}(2) = a \).

**Definition 0.5.2.** Let \( \mathcal{P} \) be an operad; an ideal of \( \mathcal{P} \) is a sequence \( I = \{ I(n), n \in \mathbb{N}^* \} \) of \( k[\Sigma] \)-submodules of \( \mathcal{P}(n) \) such that, for \( \mu \in \mathcal{P}(n) \), \( v \in \mathcal{P}(m) \), \( x \in I(m) \), \( y \in I(n) \), \( 1 \leq i, j \leq n \), we have

\[
\mu \circ_i x \in I(n + m - 1), \quad y \circ_j v \in I(n + m - 1).
\]

We recall the definition of a quadratic operad [11]:

**Definition 0.5.3.** Let \( E \) be a \( k[\Sigma] \)-module and \( R \) be a \( k[\Sigma] \)-submodule of \( \mathcal{F}(E)(3) \). Let \( (R) \) denote the ideal generated by \( R \), we call \( \mathcal{F}(E)/(R) \) the quadratic operad generated by \( E \) with relations \( R \) denoted \( \mathcal{P}(k, E, R) \) in the sequel.

**Remark 0.5.4.** (1) Let \( \mathcal{P} = \mathcal{P}(k, E, R) \) be a quadratic operad then we have \( \mathcal{P}(2) = E \) and \( \mathcal{P}(3) = \mathcal{F}(E)(3)/R \); furthermore, \( \mathcal{F}(E)(3) = (E \otimes E) \otimes \Sigma_3 k[\Sigma] \) where \( \Sigma_2 \) acts trivially on the first factor \( E \).

(2) The operads \( \text{Ass}, \text{Com}, \text{Lie}, \text{Poiss} \) and \( \text{Leib} \) are quadratic.

We have the useful following proposition (see [1] for a proof):

**Proposition 0.5.5.** Let \( \mathcal{P} = \mathcal{P}(k, E, R) \) be a quadratic operad and \( V \) be a \( k \)-vector space. A \( \mathcal{P} \)-algebra structure on \( V \) is entirely determined by a morphism of \( k[\Sigma] \)-modules \( a : E \to \text{End}(V)(2) \), such that for any \( r \in R \), \( \tilde{a}(3)(r) = 0 \), where \( \tilde{a} \) is the unique morphism of operads associated to \( a \).

**0.6. Quadratic duality.** We end these remarks by the definition of quadratic duality. Let \( F \) be a right module over \( k[\Sigma] \). We denote by

\[
F^\vee = \text{Hom}_k(F, k) \otimes \text{sgn}_n,
\]
the right module equipped with the action of \( \Sigma_3 \) given, for \( \varphi \in \text{Hom}_k(F, k) \), \( x \in F \), by:

\[
(\varphi \cdot \sigma)(x) = \varepsilon(\sigma) \varphi(x \cdot \sigma^{-1}).
\]

Let \( E \) be a right module over \( k[\Sigma_2] \), then as a module over \( k[\Sigma_3] \)

\[
\mathcal{F}(E)(3) \simeq (\mathcal{F}(E)(3))^\vee,
\]

the isomorphism being given by the evaluation map [19],

\[
(\cdot, \cdot) : \mathcal{F}(E)(3) \otimes \mathcal{F}(E)(3) \to k,
\]

characterized by the following conditions: for \( \mu \in \mathcal{F}(E)(3) \), \( f \in \mathcal{F}(E)(3) \) and \( \sigma \in \Sigma_3 \),

\[
\langle \mu \sigma, f \sigma \rangle = \varepsilon(\sigma) \langle \mu, f \rangle;
\]

for \( \mu, \nu \in E \), \( f, g \in E^\vee \), \( i \in \{1, 2\} \),

\[
\langle \mu \circ_i \nu, f \circ_j g \rangle = \varepsilon_{ij} f(\mu)g(\nu)
\]

where \( \varepsilon_{ij} = 0 \) if \( i \neq j \), \( \varepsilon_{11} = 1 \) and \( \varepsilon_{22} = -1 \). Furthermore for \( h \in \mathcal{F}(E)(3) \) and \( \sigma \in \Sigma_3 \) we have \((\mu \circ_i \nu)\sigma, h) = 0 \) if \( \sigma \) is not of the form \( \sigma_1 \circ_2 \sigma_2 \) with \( \sigma_1, \sigma_2 \in \Sigma_2 \).

Let \( R \subset \mathcal{F}(E)(3) \) be a \( k[\Sigma_3] \)-submodule, and \( R^\perp \subset \mathcal{F}(E)(3) \) be the annihilator of \( R \) in \( (\mathcal{F}(E)(3))^\vee \simeq \mathcal{F}(E)(3) \).

**Definition 0.6.1.** The Koszul dual of the quadratic operad \( \mathcal{P} = \mathcal{P}(k, E, R) \) is the quadratic operad \( \mathcal{P}^! = \mathcal{P}(k, E^\vee, R^\perp) \).

**Remark 0.6.2.** We have \( \mathcal{P}^!(2) = E^\vee \) and \( (\mathcal{P}^!)(3) \simeq R \).

**Standing assumption.** In the remainder of this paper, we fix a quadratic operad \( \mathcal{P} = \mathcal{P}(k, E, R) \) such that \( E \) is finite dimensional and such that \( \mathcal{P}(1) = k \) (the unit corresponding to the unit of \( k \)); we also fix an algebra over \( \mathcal{P} \) we denote by \( A \), and we note \( \pi : \mathcal{P}(2) \otimes_\mathcal{A} A \otimes_\mathcal{A} A \to A \) the structural map of \( A \) in the sense of Proposition 0.5.5.

For the convenience of the reader, we recall that a Leibniz algebra is a vector space \( L \) equipped with a bilinear bracket \([, , ] : L \times L \to L\), such that the following identity holds (see [15]): for \( x, y, z \in L \),

\[
[x, [y, z]] = [[x, y], z] - [[x, z], y].
\]

A dual Leibniz algebra is a vector space \( L \) equipped with a bilinear product \( \times : L \times L \to L \), such that the following identity holds (see [17]): for \( x, y, z \in L \),

\[
(x \times y) \times z = x \times (y \times z) + x \times (z \times y).
\]

1. **Representations of an algebra over a quadratic operad**

Let \( M \) be a vector space over \( k \) and \( \psi : \mathcal{P}(2) \otimes A \otimes M \to M \) be a linear map and set \( B = A \oplus M \).
**Lemma 1.1.** There is a unique linear map \( \tilde{\psi} : \mathcal{P}(2) \otimes_{\Sigma_3} B^{\otimes 2} \to B \) such that

(i) \( \tilde{\psi}|_{\mathcal{P}(2) \otimes_{\Sigma_3} B^{\otimes 2}} = \pi \);
(ii) \( \tilde{\psi}(\mu, a, m) = \psi(\mu, a, m) \), for all \( \mu \in \mathcal{P}(2) \), \( a, m \in A \times M \);
(iii) \( \tilde{\psi}(\mu, m_1, m_2) = 0 \), for all \( \mu \in \mathcal{P}(2) \), \( m_1, m_2 \in M \).

**Proof.** It suffices to set:
\[
\tilde{\psi}(\mu, a_1 + m_1, a_2 + m_2) = \mu(a_1, a_2) + \psi(\mu, a_1, m_2) + \psi(\mu(12), a_2, m_1). \]

The preceding lemma and the general paper [2] justify the following definition:

**Definition 1.2.** A representation of \( A \) is a vector space \( M \) equipped with a linear map, \( \psi : \mathcal{P}(2) \otimes A \otimes M \to M \), such that the extended map \( \tilde{\psi} \) defined by Lemma 1.1 makes \( A \oplus M \) a \( \mathcal{P} \)-algebra. A morphism of representations \( f : (M, \psi) \to (M', \psi') \) is a linear map \( f : M \to M' \) such that, for \( \mu \in \mathcal{P}(2) \), \( a \in A \) and \( m \in M \), \( f \circ \psi(\mu, a, m) = \psi'(\mu, a, f(m)) \).

Let \( \psi(3) \) and \( \pi(3) \) denote the extended maps of \( \tilde{\psi} \) and \( \pi \) to \( \mathcal{F}(E)(3) \otimes_{\Sigma_3} B^{\otimes 3} \) and to \( \mathcal{F}(E)(3) \otimes_{\Sigma_3} A^{\otimes 3} \) (see Proposition 0.5.1). For all \( \mu \in \mathcal{F}(E)(3) \), \( a_1, a_2, a_3 \in A \) and \( m_1, m_2, m_3 \in M \), we have:
\[
\psi(3)(\mu, a_1 + m_1, a_2 + m_2, a_3 + m_3) = \pi(3)(\mu, a_1, a_2, a_3) + \psi(3)(\mu, a_1, m_2, a_3) + \psi(3)(\mu, m_1, a_2, a_3) + \psi(3)(\mu, a_1, a_2, m_3);
\]
this equality is due to the fact that \( \mathcal{F}(E)(3) \) is generated as a module over \( k[\Sigma_3] \) by the \( \mu \circ_i v's \) \( (\mu, v \in \mathcal{P}(2), i \in \{1, 2\}) \) and to Lemma 1.1. We then deduce the following proposition:

**Proposition 1.3.** With the same notations, \( M \) is a representation of \( A \) if and only if we have, for all \( a_1, a_2 \in A \), \( m \in M \) and \( r \in R \),
\[
\psi(3)(r, a_1, a_2, m) = 0. \tag{1.1}
\]

**Proof.** Simple application of Proposition 0.5.5. \( \square \)

1.4. As a module over \( k[\Sigma_3] \), \( R \) is generated by the \( \mu \circ_i v's \) \( (\mu, v \in \mathcal{P}(2), i \in \{1, 2\}) \); this means that for all \( r \in R \), there exists \( (\mu_\alpha, v_\alpha)_{\alpha \in \mathcal{A}} \in \mathcal{P}(2)^{\mathcal{A}}, (\mu'_\beta, v'_\beta)_{\beta \in \mathcal{B}} \in \mathcal{P}(2)^{\mathcal{B}} \) and \( (\lambda_\alpha, \lambda'_\beta) \in (\Sigma_2)^{\mathcal{A}} \times (\Sigma_2)^{\mathcal{B}} \), such that:
\[
r = \sum_{\alpha \in \mathcal{A}} (\mu_\alpha \circ_1 v_\alpha) \lambda_\alpha + \sum_{\beta \in \mathcal{B}} (\mu'_\beta \circ_2 v'_\beta) \lambda'_\beta.
\]
Hence, to express condition (1.1) it suffices to know $\psi(3)$ on the $(\mu \circ_i v, a, b, m)$’s and on all their permutations; this is given by the following formulas:

\[
\begin{align*}
\psi(3)(\mu \circ_1 v, a, b, m) &= \psi(\mu, v(a, b), m), \\
\psi(3)(\mu \circ_2 v, a, b, m) &= \psi(\mu, a, \psi(v, b, m)), \\
\psi(3)(\mu \circ_1 v, a, m, b) &= \psi(\mu v, b, \psi(v, a, m)), \\
\psi(3)(\mu \circ_2 v, a, m, b) &= \psi(\mu, a, \psi(v, b, m)), \\
\psi(3)(\mu \circ_1 v, m, a, b) &= \psi(\mu v, a, \psi(v, b, m)), \\
\psi(3)(\mu \circ_2 v, m, a, b) &= \psi(\mu, v(a, b), m),
\end{align*}
\]

where $\tau$ denotes the permutation $(12)$ of $\Sigma_2$.

**Example 1.5.**

1. **Associative operad:** If $\mathcal{P} = \text{Ass}$, a representation of the associative algebra $A$ is a $A$-bimodule.

2. **Commutative operad:** If $\mathcal{P} = \text{Com}$, a representation of the commutative algebra is a symmetric $A$-bimodule.

3. **Lie operad:** If $\mathcal{P} = \text{Lie}$, a representation of the Lie algebra $A$ is a classical representation of $A$ (see [4] for instance).

4. **Leibniz operad:** if $\mathcal{P} = \text{Leib}$, a representation of the Leibniz algebra $A$ is a representation of $A$ in the sense of Loday [15].

5. **Leibniz dual operad:** if $\mathcal{P} = \text{Leib}'$, a representation of the dual Leibniz algebra $(A, \times)$ is a vector space $M$ equipped with two actions of $A$, $\alpha : A \otimes M \to M$ et $\beta : M \otimes A \to M$ such that, with the notations $\alpha(a, m) = a \times m$ and $\beta(m, a) = m \times a$, we have the following relations:

\[
\begin{align*}
(m \times x) \times y &= m \times (x \times y) + m \times (y \times x), \\
(x \times m) \times y &= x \times (m \times y) + x \times (y \times m),
\end{align*}
\]

1.6. **Notations.** In the rest of the paper, we often use the following notation: $\mu(x, y, z) = \psi(3)(\mu, x, y, z)$, where $(x, y, z) \in \{(a, b, m), (a, m, b), (m, a, b)\}$; in the same way, we will note $\mu(x, y) = \bar{\psi}(\mu, x, y)$, where $(x, y) \in \{(a, m), (m, a), (a, b)\}$.

1.7. **Enveloping algebra.** Let us consider the free associative algebra generated by $\mathcal{P}(2) \otimes A$, i.e. $T(\mathcal{P}(2) \otimes A)$ and let $\chi(\mu, a)$ denote its generators. Let $\chi : \mathcal{F}(E)(3) \otimes A^{\otimes 2} \to T(\mathcal{P}(2) \otimes A)$ be the map defined, for $\mu, v \in \mathcal{P}(2)$ and $a, b \in A$, by

\[
\begin{align*}
\chi(\mu \circ_1 v, a, b) &= \chi(\mu, v(a, b)), \\
\chi((\mu \circ_1 v)(12), a, b) &= \chi(\mu, v(b, a)), \\
\chi((\mu \circ_1 v)(13), a, b) &= \chi(\mu v, a) \chi(v, b),
\end{align*}
\]
\[ x((\mu \circ_1 v)(23), a, b) = \chi(\mu \tau, b)\chi(v, a), \]
\[ x((\mu \circ_1 v)(231), a, b) = \chi(\mu \tau, b)\chi(v\tau, a), \]
\[ x((\mu \circ_1 v)(321), a, b) = \chi(\mu \tau, a)\chi(v, b). \quad (1.3) \]
and
\[ x((\mu \circ_2 v, a, b) = x(\mu, a)\chi(v, b), \]
\[ x((\mu \circ_2 v)(12), a, b) = x(\mu, b)\chi(v, a), \]
\[ x((\mu \circ_2 v)(13), a, b) = x(\mu \tau, v(b, a)), \]
\[ x((\mu \circ_2 v)(23), a, b) = x(\mu, a)\chi(v\tau, b), \]
\[ x((\mu \circ_2 v)(231), a, b) = x(\mu \tau, v(a, b)), \]
\[ x((\mu \circ_2 v)(321), a, b) = x(\mu, b)\chi(v\tau, a). \quad (1.4) \]

where \( \tau \) still denotes the permutation \((12)\) of \( \Sigma_2 \).

**Definition 1.7.1.** Let \( I_R \) be the two-sided ideal generated by the set \( \{ \chi(r, a, b)/r \in R, a, b \in A \} \). The universal enveloping algebra of the \( \mathcal{P} \)-algebra \( A \) is the associative and unital algebra \( \mathcal{U}_\mathcal{P}(A) = T(\mathcal{P}(2) \otimes A)/I_R \).

**Theorem 1.7.2.** The category of representations of the \( \mathcal{P} \)-algebra \( A \) is equivalent to the category of left modules over \( \mathcal{U}_\mathcal{P}(A) \).

**Proof.** Let \( M \) be a left module over \( \mathcal{U}_\mathcal{P}(A) \). For \( \mu \in \mathcal{P}(2), a \in A \) and \( m \in M \) set \( \mu(a, m) = \chi(\mu, a)m \).

**Lemma.** For \( i \in \{1, 2\} \) and \( \sigma \in \Sigma_3 \), with obvious notations, we have:
\[ ((\mu \circ_i v)\sigma)(a, b, m) = \bar{\chi}((\mu \circ_i v)\sigma, a, b)m. \]

**Proof of the Lemma.** It is a direct application of formulas (1.2)–(1.4); for instance, for \( i = 1 \) and \( \sigma = (23) \):
\[
((\mu \circ_1 v)(23))(a, b, m) = \mu \circ_1 v(a, m, b) = \mu \tau(b, v(a, m)) \\
= x(\mu \tau, b)\chi(v, a)m = x(\mu \tau, b)\chi(v, a))m \\
\equiv \bar{\chi}((\mu \circ_1 v)(23), a, b)m. \quad \square
\]

Consequently, for \( r \in R \) we have \( r(a, b, m) = \bar{\chi}(r, a, b)m = 0 \). Hence, by Proposition 1.3 \( M \) is a representation of \( A \). We leave the converse to the reader. \( \square \)
Example 1.7.3

- If \( P = \text{Ass} \), then \( \mathcal{U}_P(A) = k \oplus A^e \) where \( A^e = A \otimes A^{op} \) is the classical enveloping algebra.

- If \( P = \text{Com} \), then \( \mathcal{U}_P(A) = k \oplus A \).

- If \( P = \text{Lie} \), then \( \mathcal{U}_P(A) = T(A^l \oplus A^r)/I \) where \( A^l \) and \( A^r \) are two copies of the Leibniz algebra \((A, [\cdot, \cdot])\); if we denote by \( l_a = \chi(1_2, a) \) and \( r_b = \chi((12), b) \) the elements of \( A^l \) and \( A^r \) \((\text{Leib}(2) = k\{1_2, (12)\})\), then \( I \) is the two-sided ideal generated by the elements \((a, b \in A)\):

\[
I_{[a, b]} - r_b l_a + l_a r_b, \quad r_{[a, b]} - r_b r_a + r_a r_b, \quad l_a(l_b + r_b).
\]

This description differs from those given by Loday in [15]; it is simply because in [15] a representation is a right module over the enveloping algebra.

- If \( P = \text{Leib}^l \), then \( \mathcal{U}_P(A) = T(A^l \oplus A^r)/I \) where \( A^l \) and \( A^r \) are two copies of the dual Leibniz algebra \((A, \chi)\); if we denote by \( l_a = \chi(1_2, a) \) and \( r_b = \chi((12), b) \) the elements of \( A^l \) and \( A^r \) \((\text{Leib}^l(2) = k\{1_2, (12)\})\), then \( I \) is the two-sided ideal generated by the elements \((a, b \in A)\):

\[
I_{[a, b]} - l_a l_b - l_a r_b, \quad l_{[a, b]} - r_b l_a, \quad r_b r_a - r_{a \times b} - r_b a.
\]

2. Corepresentations of an algebra over a quadratic operad

Theorem 1.7.2 justifies the following definition:

Definition 2.1. Let \( A \) be a \( P \)-algebra; a vector space \( N \) is a corepresentation of \( A \) if it is a right module over \( \mathcal{U}_P(A) \).

Let \( N \) be a corepresentation of \( A \) and let \( \theta : F(2) \otimes A \otimes N \to N \) be the morphism defined, with obvious notations, by \( \theta(\mu, a, n) = n\chi(\mu, a) \). We extend the definition of \( \theta \) to \( F(2) \otimes N \otimes A \) by setting \( \theta(\mu, n, a) = \theta(\mu(12), a, n) \). In the same way, we define \( \theta(3) : F(E)(3) \otimes A^{\otimes 3} \otimes N \to N \) by setting \( \theta(3)(\mu, a, b, n) = n\chi(\mu, a, b) \) and we extend the definition to \( F(E)(3) \otimes A \otimes N \otimes A \oplus F(E)(3) \otimes N \otimes A^{\otimes 2} \) by setting \( \theta(3)(\mu, n, a, b) = \theta(3)(\mu(23), a, b, n) \) and \( \theta(3)(\mu, n, a, b, c) = \theta(3)(\mu(312), a, b, c) \); by construction, for all \( \sigma \in S_3 \) and \( x_1, x_2, x_3 \) we have:

\[
\theta(3)(\mu\sigma, x_1, x_2, x_3) = \theta(3)(\mu, x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}).
\]

Proposition 2.2. The morphism \( \theta(3) \) satisfies the following identities (for \( \mu, \nu \in P(2), a_1, a_2 \in A \) and \( n \in N \)):

(i) \( \theta(3)(\mu \circ_1 \nu, a_1, a_2, n) = \theta(\mu, \nu(a_1, a_2), n) \);

(i') \( \theta(3)(\mu \circ_2 \nu, a_1, a_2, n) = \theta(\nu, a_2, \theta(\mu, a_1, n)) \);
(ii) \( \theta(3)(\mu \circ_1 v, a_1, n, a_2) = \theta(v, a_1, \theta(\mu, n, a_2)) \);

(ii') \( \theta(3)(\mu \circ_2 v, a_1, n, a_2) = \theta(v, \theta(\mu, a_1, n), a_2) \);

(iii) \( \theta(3)(\mu \circ_1 v, n, a_1, a_2) = \theta(v, \theta(\mu, n, a_2), a_1) \);

(iii') \( \theta(3)(\mu \circ_2 v, n, a_1, a_2) = \theta(\mu, n, \nu(a_1, a_2)) \).

**Proof.** Each identity can be obtained from formulas (1.3) and (1.4); for instance

\[
\theta(3)(\mu \circ_1 v, a_1, n, a_2) = \theta(3)(\mu \circ_1 v(23), a_1, a_2, n)
\]

\[
= \eta(\mu \circ_1 v(23), a_1, a_2) = (\eta(\mu, a_2))\chi(v, a_1)
\]

\[
= \theta(v, a_1, \theta(\mu, n, a_2, n)) = \theta(v, a_1, \theta(\mu, n, a_2)).
\]

Conversely, let \( \theta_1 : \mathcal{P}(2) \otimes A \otimes N \to N \) be a linear map; define

\[
\theta_1(3) : \mathcal{F}(E)(3) \otimes A^{\otimes 2} \otimes N \bigoplus \mathcal{F}(E)(3) \otimes A \otimes N \otimes A
\]

\[
\bigoplus \mathcal{F}(E)(3) \otimes N \otimes A^{\otimes 2} \to N
\]

by using the formulas of Proposition 2.2 (with the convention that \( \theta_1(\mu, n, a) = \theta_1(\mu(12), a, n) \)). Then we have the analogue of Proposition 1.3.

**Proposition 2.3.** A corepresentation of the \( \mathcal{P} \)-algebra \( A \) is a vector space \( N \) equipped with a linear map \( \theta_1 : \mathcal{P}(2) \otimes A \otimes N \to N \) such that, for all \( a_1, a_2 \in A, n \in N \) and \( r \in R \), the extended map \( \theta_1(3) \) satisfies \( \theta_1(3)(r, a_1, a_2, n) = 0 \).

**Proof.** Left to the reader (use the structure of \( \mathcal{H}_\mathcal{P}(A) \)).  

**Remark 2.4.** If \( M \) is a representation of \( A \), then \( M^* = \text{Hom}_k(M, k) \) is a corepresentation. In the same way, if \( N \) is a corepresentation of \( A \) then \( N^* \) is a representation. The two morphisms \( \tilde{\psi} \) and \( \theta \) are linked by the following formulas: for \( \mu \in \mathcal{P}(2), a \in A, m, m^* \in M \times M^* \) and \( n, n^* \in N \times N^* \),

\[
\langle m, \theta(\mu, a, m^*) \rangle = \langle \tilde{\psi}(\mu, a, m), m^* \rangle;
\]

\[
\langle n, \tilde{\psi}(\mu, a, n^*) \rangle = \langle \theta(\mu, a, n), n^* \rangle.
\]

**Example 2.5.** (1) \( \mathcal{P} = \text{Ass} \): Let \( N \) be a corepresentation of the associative algebra \( A \) and let

\[ \theta : \mathcal{P}(2) \otimes A \otimes N \to N \]
be the structural morphism. For \(a, n \in A \times N\), we set \(a \cdot n = \theta((12), a, n)\) and \(n \cdot a = \theta(1, a, n)\) \((\text{Ass}(2) = k[\Sigma_2])\). Then \(N\) equipped with these two actions of \(A\) is a representation of \(A\) (i.e. a bimodule over \(A\)). Conversely, any representation \(M\) of \(A\) is given the natural structure of corepresentation of \(A\) by setting \(\theta((12), a, m) = ma\) and \(\theta((12), a, m) = am\). In fact we have an equivalence between the category of representations of \(A\) and the category of corepresentations of \(A\).

(2) \(\mathcal{P} = \text{Com}\): Similarly, the category of representations of the commutative algebra \(A\) is equivalent to the category of corepresentations of \(A\).

(3) \(\mathcal{P} = \text{Lie}\): A corepresentation of the Leibniz algebra \(A\) is a corepresentation in the sense of Loday [15].

(5) \(\mathcal{P} = \text{Lie}'\): A corepresentation of the dual Leibniz algebra \((A, x)\) is a vector space \(N\) equipped with two actions of \(A\), \(\alpha: A \otimes N \to N\) and \(\beta: N \otimes A \to N\) such that, if we set \(\alpha(n, a) = a \star n\) and \(\beta(a, n) = n \star a\), we have the following relations:

\[
x \star (y \star n) = (x \times y) \star n + (y \times x) \star n,
\]

\[
n \star (x \times y) = (n \star x) \star y + y \star (n \star x)
\]

\[
= (y \star n) \star x.
\]

3. Cohomology of a \(\mathcal{P}\)-algebra with coefficients in a representation

For the rest of the paper, \((\mathcal{P}^1)^\vee(n)\) will stand for \((\mathcal{P}^1(n))^\vee\) (see also 0.6).

3.1. Cohomology \(H^n_{\mathcal{P}}(A, A)\)

3.1.1. The graded Lie algebra \(L_{\mathcal{P}}(V)\)

Let \(V\) be a vector space, and for \(n \in \mathbb{N}^*\) let \(L_{\mathcal{P}}^{-n}(V)\) denote the set

\[
L_{\mathcal{P}}^{-n}(V) = \mathcal{P}^1(n) \otimes_{\Sigma_n} \text{End}(V)(n),
\]

where \(\text{End}(V)(n) = \text{End}(V)(n) \otimes \text{sgn}_n\) (with its natural structure of a left module over \(k[\Sigma_n]\)). Set \(L_{\mathcal{P}}(V) = \bigoplus_{n \geq 0} L_{\mathcal{P}}^{-n}(V)\) and for \(\mu^* \otimes f \in L_{\mathcal{P}}^n(V)\) and \(v^* \otimes g \in L_{\mathcal{P}}^m(V)\) define \((\mu^* \otimes f) \circ (v^* \otimes g) \in L_{\mathcal{P}}^{n+m}(V)\) by the formula (see paragraph 2.4.4 of [1] for the validity of this definition):

\[
\sum_{i=1}^{n+1} (-1)^{m(i-1)} (\mu^* \circ_i v^*) \otimes (f \circ_i g).
\]

We have the following proposition [1]:

**Proposition 3.1.2.** There exists on \(L_{\mathcal{P}}(V)\) a natural structure of graded Lie algebra; for \(\mu^* \otimes f \in L_{\mathcal{P}}^n(V)\) and \(v^* \otimes g \in L_{\mathcal{P}}^m(V)\) the bracket, denoted by \([, ,]\), is defined by

\[
[\mu^* \otimes f, v^* \otimes g] = (\mu^* \otimes f) \circ (v^* \otimes g) + (-1)^{nm+1}(v^* \otimes g) \circ (\mu^* \otimes f).
\]
Remark 3.1.3. The expression (3.1) has a sense, i.e. it is independent of the choice of elements representing $\mu^* \otimes f$ and $v^* \otimes g$, but in general, it is not the case for each term of the sum (3.1).

Proposition 3.1.4. With the above notations, for any vector space $V$ there is an isomorphism of vector spaces: $L_{\mathscr{P}}^{-1}(V) \cong \Hom_k((\mathcal{P}^!)^\vee(n) \otimes \Sigma_n V^\otimes n, V)$, where for $\mu^* \otimes f \in L_{\mathscr{P}}^{-1}(V)$ and $\mu \otimes x \in (\mathcal{P}^!)^\vee(n) \otimes nV^\otimes n$,

$$\Upsilon(\mu^* \otimes f)(\mu \otimes x) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \epsilon(\sigma) \langle \mu, \mu^* \sigma \rangle f(\sigma x)$$

$$= \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \langle \mu \sigma^{-1}, \mu^* \rangle f(\sigma x).$$

Proof. We recall that $\mathscr{P} = \mathscr{P}(k, \mathcal{E}, R)$ where $\mathcal{E}$ is finite dimensional; therefore, $\mathcal{P}^!(n)$ is also finite dimensional. So we have an isomorphism of right modules over $k[\Sigma_n]$

$$\mathcal{P}^!(n) \otimes \text{sgn}_n \cong \Hom_k((\mathcal{P}^!)^\vee(n), k).$$

Set $W = \text{End}(V)(n)$, then $W$ is a left projective module over $k[\Sigma_n]$ and hence we have the isomorphism of vector spaces:

$$\mathcal{P}^!(n) \otimes \text{sgn}_n \otimes \Sigma_n \text{End}(V)(n) \cong \Hom_k((\mathcal{P}^!)^\vee(n), k) \otimes \Sigma_n \text{End}(V)(n).$$

But, $\Hom_k((\mathcal{P}^!)^\vee(n), k) \otimes \Sigma_n \text{End}(V)(n) = (\Hom_k((\mathcal{P}^!)^\vee(n), k) \otimes \text{End}(V)(n))_{\Sigma_n}$ and $\Hom_k((\mathcal{P}^!)^\vee(n), k) \otimes \text{End}(V)(n) \cong \Hom_k((\mathcal{P}^!)^\vee(n) \otimes V^\otimes n, V)$ since $\mathcal{P}^!(n)$ is finite dimensional. Therefore,

$$L_{\mathscr{P}}^{-1}(V) \cong \Hom_k((\mathcal{P}^!)^\vee(n) \otimes V^\otimes n, V)_{\Sigma_n}$$

$$\Upsilon \cong \Hom_k((\mathcal{P}^!)^\vee(n) \otimes V^\otimes n)^{\Sigma_n}_V.$$

The group $\Sigma_n$ has its cardinal invertible in $k$, so we have:

$$(\mathcal{P}^!)^\vee(n) \otimes V^\otimes n)^{\Sigma_n}_V \cong (\mathcal{P}^!)^\vee(n) \otimes \Sigma_n V^\otimes n.$$  

This ends the proof; furthermore, if we recover $\Upsilon$ from the $\Upsilon_i$'s, we find the formula of the proposition.

We recall now the following theorem of [1]; we will give another proof of this theorem at the end of this section (see Remark 3.2.10).

Theorem 3.1.5. Let $V$ be any vector space. There is a one to one correspondence between elements $\varphi \in L_{\mathcal{P}}^1(V)$ which satisfy $[\varphi, \varphi] = 0$ and $k[\Sigma_2]$-morphisms $\phi: \mathcal{P}(2) \to \text{End}(V)(2)$ defining $\mathcal{P}$-algebra structures on $V$. 


In the sequel, we will identify $\varphi$ and $\phi$; for instance, the structural morphism of $A$, $\pi : \mathcal{P}(2) \otimes A^{\otimes 2} \to A$ satisfies $[\pi, \pi] = 0$.

**Definition 3.1.6.** Let $C^n_\mathcal{P}(A) = \text{Hom}_k((\mathcal{P}^1)^{(n)} \otimes \Sigma_n A^{\otimes n}, A)$, and for $n \geq 1$, let $\delta^n_\pi : C^n_\mathcal{P}(A) \to C^{n+1}_\mathcal{P}(A)$ be the map defined by the formula:

$$\delta^n_\pi(f) = -\frac{1}{2}(n + 1) \mathcal{T}((\mathcal{T}^{-1}(f), \mathcal{T}^{-1}(\pi))).$$

The next proposition is an easy consequence of the fact that $[\pi, \pi] = 0$ and that $L_\mathcal{P}(A)$ is a graded Lie algebra:

**Proposition 3.1.7.** The map $\delta^*_\pi$ is a differential. The homology of $(C^n_\mathcal{P}(A), \delta^*_\pi)$ is called the cohomology of the $\mathcal{P}$-algebra $A$ with coefficients in itself and is denoted by $H^n_\mathcal{P}(A, A)$ or $H^n_\mathcal{P}(A)$.

**Remark 3.1.8.** (1) The coefficient $\frac{1}{2}(n + 1)$ is a technical coefficient enabling us to recover the differentials in the classical cases.

(2) Each $C^n_\mathcal{P}(A)$ is put on degree $n$; hence the groups of cohomology $H^n_\mathcal{P}(A)$ are only defined for $n \geq 1$. In general, the group $H^0_\mathcal{P}(A)$ cannot be defined (see the example of dual Leibniz algebras at the end of this paper).

**Example 3.1.9.** Here are some classical examples: the reader will find all the proofs in Section 3 of [1] (here $\text{Der}(A)$ stands for the set of derivations in each category).

(1) $\mathcal{P} = \text{Ass}$:

$$H^n_\mathcal{P}(A) = \begin{cases} \text{Der}(A) & \text{if } n = 1, \\ \text{Hoch}^n(A, A) & \text{if } n > 1, \end{cases}$$

where $\text{Hoch}^*$ stands for Hochschild cohomology (see [4] for instance).

(2) $\mathcal{P} = \text{Com}$:

$$H^n_\mathcal{P}(A) = \begin{cases} \text{Der}(A) & \text{if } n = 1, \\ \text{Harr}^n(A, A) & \text{if } n > 1, \end{cases}$$

where $\text{Harr}^*$ stands for Harrison cohomology [12].

(3) $\mathcal{P} = \text{Lie}$:

$$H^n_\mathcal{P}(A) = \begin{cases} \text{Der}(A) & \text{if } n = 1, \\ H^n(A, A) & \text{if } n > 1, \end{cases}$$

where $H^*$ stands for Chevalley-Eilenberg cohomology (see [4] for instance).

(4) $\mathcal{P} = \text{Leib}$:

$$H^n_\mathcal{P}(A) = \begin{cases} \text{Der}(A) & \text{if } n = 1, \\ \text{HL}^n(A, A) & \text{if } n > 1, \end{cases}$$
where $HL^*$ stands for Leibniz cohomology [15].

(5) $\mathcal{P} = \text{Leib}^t$: see Section 6.

Remark 3.1.10. In a general manner, we have $H^1_{\mathcal{P}}(A) = \text{Der}_{\mathcal{P}}(A,A)$, i.e $H^1_{\mathcal{P}}(A)$ is the set of derivations from $A$ to $A$ (see Proposition 3.2.7).

3.2. The differential $\delta_{\pi}$

The aim of this paragraph is to give an explicit description of the differential $\delta_{\pi}$. For this we need to define some new operations.

3.2.1. The operations $\circ_i$

Let $\mathcal{P}$ be an operad and for $1 \leq i \leq n$, let $\circ_i^{*,*}$ denote its $\text{comp}_i$-operations (see Definition 0.1, OPER 2):

$$\circ_i^{n,m} : \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(m + n - 1).$$

For $m,n \in \mathbb{N}^+$ such that $n - m + 1 > 0$ and $1 \leq j \leq n$, this enables us to define the maps $\circ_i^{j,m} : \mathcal{P}^j(n) \otimes \mathcal{P}(m) \to \mathcal{P}^j(n - m + 1)$ by the formula: for $\mu^* \in \mathcal{P}^j(n)$, $\nu \in \mathcal{P}(m)$ and $\lambda \in \mathcal{P}(n - m + 1)$,

$$(\mu^* \circ_i^{j,m} \nu, \lambda) = (\mu^* \circ \circ_i^{m,n-m+1} \nu, \lambda). \quad (3.2)$$

In the same way, for $m - n + 1 > 0$ and $1 \leq i < m - n + 1$, we define the maps $\circ_i^{i,m} : \mathcal{P}(n) \otimes \mathcal{P}^i(m) \to \mathcal{P}^i(m - n + 1)$, by the formula: for $\mu \in \mathcal{P}(n)$, $\nu^* \in \mathcal{P}^i(m)$ and $\lambda \in \mathcal{P}(m - n + 1)$,

$$(\nu^* \circ_i^{i,m} \mu, \lambda) = (\nu^* \circ \circ_i^{i,m-n+1,n} \mu, \lambda). \quad (3.3)$$

Let $(A, \pi)$ be a $\mathcal{P}$-algebra and let us consider $\mathcal{T}^{-1}(\pi) \in \mathcal{P}(2) \otimes \Sigma_2 \text{End}(A)(2)$ (see Proposition 3.1.4). We set

$$\mathcal{T}^{-1}(\pi) = \sum_{\alpha} \mu_{\alpha}^* \otimes \Pi_{\alpha}. \quad (3.4)$$

In the same way, for $f \in C^n_{\mathcal{P}}(A)$ we set

$$\mathcal{T}^{-1}(f) = \sum_{\beta} \nu_{\beta}^* \otimes F_{\beta}. \quad (3.5)$$

By definition, $\delta_{\pi}^n(f) = -\frac{n+1}{2} \mathcal{Y}([\mathcal{T}^{-1}(f), \mathcal{T}^{-1}(\pi)]) = \delta_{\pi}^{1,n}(f) + \delta_{\pi}^{2,n}(f)$, where

$$\delta_{\pi}^{1,n}(f) = \frac{n + 1}{2} \mathcal{Y} \left( \sum_{i=1}^{n} (-1)^{i} \sum_{\alpha, \beta} (\nu_{\beta}^* \circ_i \mu_{\alpha}^*) \otimes (F_{\beta} \circ_i \Pi_{\alpha}) \right) \quad (3.6)$$

and

$$\delta_{\pi}^{2,n}(f) = \frac{n + 1}{2} \mathcal{Y} \left( \sum_{i=1}^{n} (-1)^{i+1} \sum_{\alpha, \beta} (\mu_{\alpha}^* \circ_i \nu_{\beta}^*) \otimes (\Pi_{\alpha} \circ_i F_{\beta}) \right). \quad (3.7)$$
3.2.2. Calculus of $\delta_{n}^{1,n}(f)$

Let $\mu \otimes X \in (\mathcal{P}^1)^{(n+1)} \otimes \Sigma_{n+1} A^{\otimes n+1}$, according to (3.6), we have

$$\frac{2}{n+1} \delta_{n}^{1,n}(f)(\mu \otimes X)$$

$$= \frac{1}{(n+1)!} \sum_{i=1}^{n} (-1)^{i} \sum_{\tau_{i} \in \Sigma_{n+1}} \varepsilon(\tau_{i}) \sum_{\alpha, \beta} \langle \mu_{\alpha} \circ \mu_{\beta} \tau_{i}, (F_{\beta} \circ \Pi_{\alpha}) \tau_{i}(X) \rangle.$$

Let us fix an element $\tau \in \Sigma_{n}$ and set $\tau_{i} = (\tau \circ 1_2) \tau_{i}'$ with $\tau_{i}' \in \Sigma_{n+1}$; according to formula 7 of [1], we have $\varepsilon(\tau_{i}) = (-1)^{i-1}\varepsilon(\tau') \varepsilon(\tau'_{i})$, and so (see also (0.2)) the equalities:

$$(-1)^{i} \varepsilon(\tau_{i}) \langle \mu_{\alpha} \circ \mu_{\beta} \tau_{i}, (F_{\beta} \circ \Pi_{\alpha}) (\tau_{i}(X) \rangle = (-1)^{i-1(i)} \varepsilon(\tau) \langle \mu_{\alpha} \circ \mu_{\beta} \tau_{i}, (F_{\beta} \circ \Pi_{\alpha}) (\tau_{i}(X) \rangle$$

$$(F_{\beta} \circ \Pi_{\alpha}) \tau_{i} = (F_{\beta} \circ \tau_{i-1(i)} \Pi_{\alpha}) \tau_{i}'.$$

Hence, if we set $j = \tau^{-1}(i)$, $\sigma_{j^{-1}(i)} = (\tau_{i}')^{-1}$, we have

$$\frac{2}{n+1} \delta_{n}^{1,n}(f)(\mu \otimes X)$$

$$= \frac{1}{(n+1)!} \sum_{j=1}^{n} (-1)^{j} \sum_{\tau_{j} \in \Sigma_{n+1}} \varepsilon(\tau_{j}) \sum_{\alpha, \beta} \langle \mu_{\alpha} \circ \mu_{\beta} \tau_{j}, (F_{\beta} \circ \Pi_{\alpha}) \sigma_{j}^{-1} \rangle.$$

But, if we use the new operations defined in Section 3.2.1 (with $\mathcal{Q} = \mathcal{P}^1$) and particularly formula (3.3), for any $\tau \in \Sigma_{n}$ we find that

$$\frac{2}{n+1} \delta_{n}^{1,n}(f)(\mu \otimes X)$$

$$= \frac{1}{(n+1)!} \sum_{j=1}^{n} (-1)^{j} \sum_{\tau_{j} \in \Sigma_{n+1}} \varepsilon(\tau_{j}) \sum_{\alpha, \beta} \langle \mu_{\alpha} \circ \mu_{\beta} \tau_{j}, (F_{\beta} \circ \Pi_{\alpha}) \sigma_{j}^{-1} \rangle.$$

Since this last equality is valid for any $\tau \in \Sigma_{n}$ we have

$$2n! \delta_{n}^{1,n}(f)(\mu \otimes X)$$

$$= \sum_{j=1}^{n} (-1)^{j} \sum_{\tau_{j} \in \Sigma_{n+1}} \sum_{\alpha} \frac{1}{n!} \sum_{\tau \in \Sigma_{n}} \varepsilon(\tau) \sum_{\beta} \langle \mu_{\alpha} \circ \mu_{\beta} \tau_{j}, (F_{\beta} \circ \Pi_{\alpha}) \sigma_{j}^{-1} \rangle.$$

Using formula (3.5) and Proposition 3.1.4 we finally have

**Theorem 3.2.3.** With the above notations, for all $\mu \in (\mathcal{P}^1)^{(n+1)}$ and $a_{1}, \ldots, a_{n+1} \in A$ the map $\delta_{n}^{1,n}(f)$ is given by

$$2n! \delta_{n}^{1,n}(f)(\mu, a_{1}, \ldots, a_{n+1})$$

$$= \sum_{j=1}^{n} (-1)^{j} \sum_{\tau_{j} \in \Sigma_{n+1}} \sum_{\alpha} f(\mu_{\alpha} \tau_{j}, a_{\sigma_{j}(1)}, \ldots, a_{\sigma_{j}(j+1)}), \ldots, a_{\sigma_{j}(n+1)}).$$
With the same kind of techniques we have the next theorem:

**Theorem 3.2.4.** With the above notations, for any \( \mu \in (\mathcal{P})^\vee(n + 1) \) and any \( a_1, \ldots, a_{n+1} \in A \) the map \( \delta^2_n(f) \) is given by

\[
2n! \delta^2_n(f)(\mu, a_1, \ldots, a_{n+1}) = \sum_{\sigma \in \Sigma_{n+1}} \sum_{\alpha} \Pi_\alpha(a_{\sigma(1)}, f(\mu \sigma \circ_{n+1,2}^2 \mu^*_\alpha, a_{\sigma(1)}, \ldots, a_{\sigma(n+1)})) \\
+ (-1)^{n+1} \sum_{\sigma \in \Sigma_{n+1}} \sum_{\alpha} \Pi_\alpha(f(\mu \sigma \circ_{n+1,2}^1 \mu^*_\alpha, a_{\sigma(1)}, \ldots, a_{\sigma(n)}), a_{\sigma(n+1)}).
\]

3.2.5. **The case \( n = 1 \)**

We have

\[
2\delta^1_n(f)(\mu, a_1, a_2) = - \sum_{\sigma \in \Sigma_2} \sum_{\alpha} f(\mu^*_\alpha \circ_{2,1}^1 \mu \sigma, \Pi_\alpha(a_{\sigma(1)}, a_{\sigma(2)}));
\]

but \( \mu^*_\alpha \circ_{2,1}^1 \mu \sigma \in k \) and according to (3.3) (with \( \mathcal{P} = \mathcal{P}^1 \)) \( \langle \mu^*_\alpha \circ_{2,1}^1 \mu \sigma, 1 \rangle = \langle \mu \sigma, \mu^*_\alpha \rangle \); hence,

\[
2\delta^1_n(f)(\mu, a_1, a_2) = - \sum_{\sigma \in \Sigma_2} \sum_{\alpha} \langle \mu \sigma, \mu^*_\alpha \rangle f(\Pi_\alpha(a_{\sigma(1)}, a_{\sigma(2)}));
\]

using Proposition 3.1.4 and (3.4), we finally have

\[
\delta^1_n(f)(\mu, a_1, a_2) = - f(\pi(\mu, a_1, a_2)).
\]

In the same way, we can prove that

\[
\delta^2_n(f)(\mu, a_1, a_2) = \pi(\mu, f(a_1), a_2) + \pi(\mu, a_1, f(a_2)).
\]

Therefore, the differential \( \delta^1_n \) is given, for \( \mu \in \mathcal{P}(2) \), \( a_1, a_2 \in A \), by

\[
\delta^1_n(f)(\mu, a_1, a_2) = \pi(\mu, f(a_1), a_2) + \pi(\mu, a_1, f(a_2)) - f(\pi(\mu, a_1, a_2)). \tag{3.8}
\]

**Definition 3.2.6.** Let \( M \) be a representation of the \( \mathcal{P} \)-algebra \( A \). A derivation from \( A \) to \( M \) is a linear map \( d : A \to M \) such that, for all \( \mu \in \mathcal{P}(2) \) and \( a_1, a_2 \in A \):

\[
d(\mu(a_1, a_2)) = \mu(d(a_1), a_2) + \mu(a_1, d(a_2)).
\]

In the sequel, we denote by \( \text{Der}_{\mathcal{P}}(A, M) \) the vector space of all derivations from \( A \) to \( M \).

Since we have (3.8), we have proved the following proposition:

**Proposition 3.2.7.** With the above notations: \( H^1_{\mathcal{P}}(A, A) = \text{Der}_{\mathcal{P}}(A, A) \).
3.2.8. The case \( n = 2 \)

This case is more complex and more technical than the previous one. We have

\[
4\delta_n^{1,2}(f)(\mu, a_1, a_2, a_3) = \sum_{\sigma \in \Sigma_n, 2} \left\{ -f\left(\mu_2^* \mathcal{O}_{2,3}^1 \sigma, \Pi_4(a_{\sigma(1)}, a_{\sigma(2)}), a_{\sigma(3)}\right) + f\left(\mu_2^* \mathcal{O}_{2,3}^2 \sigma, a_{\sigma(1)}, \Pi_4(a_{\sigma(2)}, a_{\sigma(3)})\right) \right\}.
\]

The operations \( \mathcal{O}_{2,3}^i \) are maps from \( \mathcal{F}(E^i)(2) \otimes R \to \mathcal{F}(E)(2) \); we can also define the same kind of maps for the operad \( \mathcal{F}(E^i) \):

\[
\mathcal{O}_{2,3}^i : \mathcal{F}(E^i)(2) \otimes \mathcal{F}(E)(3) \to \mathcal{F}(E)(2),
\]

by \( \langle \mu^* \mathcal{O}_{2,3}^i, v^* \rangle = \langle \mu^* \mathcal{O}, \mu^* \rangle \), where here \( \mathcal{O} \) stands for the \( \text{comp} \)-operation for the operad \( \mathcal{F}(E^i) \).

**Lemma.** For any \( \mu \in R, \mu^*, v^* \in \mathcal{P}(2) \) and \( i \in \{1, 2\} \), \( \mu^* \mathcal{O}_{2,3}^i \mu = \mu^* \mathcal{O}_{2,3}^i \mu \).

**Proof.** The lemma simply results from the fact that \( v^* \mathcal{O} \mu^* - v^* \circ_i \mu^* + f^* \) with \( f^* \in R^\perp \).

Let \( F : \mathcal{F}(E)(3) \otimes \Lambda^3 A \to A \) be the linear map defined by

\[
4F(\mu, a_1, a_2, a_3) = \sum_{\sigma \in \Sigma_3, 2} \left\{ -f\left(\mu_2^* \mathcal{O}_{2,3}^1 \sigma, \Pi_4(a_{\sigma(1)}, a_{\sigma(2)}), a_{\sigma(3)}\right) + f\left(\mu_2^* \mathcal{O}_{2,3}^2 \sigma, a_{\sigma(1)}, \Pi_4(a_{\sigma(2)}, a_{\sigma(3)})\right) \right\}.
\]

According to the preceding lemma, for any \( r \in R, \) we have the equality:

\[
\delta_n^{1,2}(f)(r, a_1, a_2, a_3) = F(r, a_1, a_2, a_3).
\]

So we have to calculate \( F \). The advantage of this method is the following: since \( \mathcal{F}(E)(3) \) is generated by the \( \mu \circ_i v \)'s \( (\mu, v \in \mathcal{P}(2)) \), it suffices to calculate \( F(\mu \circ_i v, a_1, a_2, a_3) \). This is done in the next proposition.

**Proposition.** With the above notations, we have

\[
F(\mu \circ_1 v, a_1, a_2, a_3) = -f(\mu, v(a_1, a_2), a_3),
\]

\[
F(\mu \circ_2 v, a_1, a_2, a_3) = -f(\mu, a_1, v(a_2, a_3)).
\]

**Proof.** By definition, we have \( \langle \mu^* \mathcal{O}_{2,3}^1(\mu \circ_1 v) \sigma, v^* \rangle = \langle (\mu \circ_1 v) \sigma, v^* \circ_1 \mu^* \rangle \); according to Section 0.6, \( \mu^* \mathcal{O}_{2,3}^1(\mu \circ_1 v) \sigma \) is null except if \( \sigma = \tau \circ_2 \omega \) with \( \tau \in \Sigma_2 \), i.e. if \( \sigma \in \{1_3, (12)\} \). In this case, we have

\[
\mu^* \mathcal{O}_{2,3}^1(\mu \circ_1 v)(1_2 \circ_1 \tau) = \langle v \tau, \mu^* \rangle \mu
\]
because
\[ \langle \mu^* \Omega_{2,3}^1(\mu \circ_1 \nu)(1_2 \circ_1 \tau), v^* \rangle = \langle \mu \circ_1 \nu \tau, v^* \circ_1 \mu^* \rangle = \langle \mu, v^* \rangle \langle \nu \tau, \mu^* \rangle. \]

In the same way, \( \mu^* \Omega_{2,3}^2(\mu \circ_1 \nu) \sigma \) is null except if \( \sigma = (12) \circ_1 \tau \) with \( \tau \in \Sigma_2 \), i.e. if \( \sigma \in \{(321), (13)\} \). In this case, we have
\[ \mu^* \Omega_{2,3}^2(\mu \circ_1 \nu)((12) \circ_1 \tau) = - \langle \nu \tau, \mu^* \rangle \mu(12). \]

Therefore, the following identity holds:
\[ 4F(\mu \circ_1 \nu, a_1, a_2, a_3) = - \sum_{\tau \in \Sigma_2, \sigma} \langle \nu \tau, \mu^*_\sigma \rangle \{ f(\mu, \Pi_\sigma(a\tau(1), a\tau(2)), a_3) \}
+ f(\mu(12), a_3, \Pi_\sigma(a\tau(1), a\tau(2))). \]

Hence,
\[ F(\mu \circ_1 \nu, a_1, a_2, a_3) = - \frac{1}{2} \sum_{\tau \in \Sigma_2, \sigma} \langle \nu \tau, \mu^*_\sigma \rangle \{ f(\mu, \Pi_\sigma(a\tau(1), a\tau(2)), a_3) \}; \]

i.e. using (3.4) and Proposition 3.1.4,
\[ F(\mu \circ_1 \nu, a_1, a_2, a_3) = - f(\mu, v(a_1, a_2), a_3). \]

The second formula is proved in the same manner. \( \square \)

We have the same kind of formula for \( \delta^2_2 \); this is included in the following proposition:

**Proposition 3.2.9.** Let \( f \in C^2_\bullet(A) \) and \( \tilde{f} : \mathcal{F}(E)(3) \otimes \Sigma_3 A^{\otimes 3} \rightarrow A \) be the map defined, with obvious notations, by
\[ f(\mu \circ_1 \nu, a_1, a_2, a_3) = f(\mu, v(a_1, a_2), a_3) + \pi(\mu, f(\nu, v(a_1, a_2)), a_3), \]
\[ \tilde{f}(\mu \circ_2 \nu, a_1, a_2, a_3) = f(\mu, a_1, v(\nu, a_2, a_3)) + \pi(\mu, a_1, f(\nu, v(a_2, a_3))). \]

Then, the differential \( \delta^2_\bullet(f) \in C^3_\bullet(A) \) is given by: \( \delta^2_\bullet(f) = - \tilde{f} \mid_{R \otimes \Sigma_3 A^{\otimes 3}}. \)

**Remark 3.2.10.** We can now give a proof of Theorem 3.1.5: the condition \([\pi, \pi] = 0\) is equivalent to the condition \( \delta^1_\bullet(\pi) = 0 \); but \( \delta^2_\bullet(\pi) = - \tilde{\pi} \mid_{R \otimes \Sigma_3 A^{\otimes 3}}, \) where \( \tilde{\pi} \) is defined in the above proposition. Since obviously \( \tilde{\pi} \) coincide with \( \hat{\pi}(3) \), it follows that Theorem 3.1.5 comes from Proposition 0.5.5.

### 3.3. Cohomology \( H^n_\bullet(A, M) \)

In this paragraph, we fix a representation of \( A \) denoted by \( (M, \psi) \). We still denote by \( \tilde{\psi} \) the associated map (see Lemma 1.1). For \( n > 1 \) we set
\[ C^n_\bullet(A, M) = \text{Hom}_k((\mathcal{S}^1)^{\vee}(n) \otimes \Sigma_n A^{\otimes n}, M). \]
Since $A \oplus M$ is a $\mathcal{P}$-algebra, we can consider the vector space $C^n_\mathcal{P}(A \oplus M)$; we have the injection:
\[ C^n_\mathcal{P}(A, M) \xrightarrow{\xi} C^n_\mathcal{P}(A \oplus M) \]
\[ f \mapsto \xi(f) \]
given by $\xi(f)(\mu, x_1, \ldots, x_n) = 0$ if there is an $i \in \{1, \ldots, n\}$ such that $x_i \in M$ and $\xi(f)(\mu, a_1, \ldots, a_n) = f(\mu, a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n \in A$. Noticing that $\Psi \in C^2_\mathcal{P}(A \oplus M)$ satisfies Theorem 3.1.5, we can consider $\delta^*_\Psi$ we more simply denote by $\delta^*_\Psi : C^n_\mathcal{P}(A \oplus M) \to C^{n+1}_\mathcal{P}(A \oplus M)$. Let $d^n_\Psi : C^n_\mathcal{P}(A, M) \to C^n_\mathcal{P}(A \oplus M)$ be the map defined, for any $f \in C^n_\mathcal{P}(A, M)$, by $d^n_\Psi(f) = \delta^*_\Psi(\xi(f))$.

**Proposition 3.3.1.** With the above notations, $\text{Im}(d^n_\Psi) \subset C^{n+1}_\mathcal{P}(A, M)$.

So we have a subcomplex and we make the following definition:

**Definition 3.3.2.** The homology of $(C^*_\mathcal{P}(A, M), d^*_\mathcal{P})$, denoted by $H^*_\mathcal{P}(A, M)$, is called the cohomology of the $\mathcal{P}$-algebra $A$ with coefficients in the representation $M$.

**Proof of Proposition 3.3.1.** Let us consider $T^{-1}(\Psi) - \sum_{\alpha} e^*_\alpha \otimes \Psi_\alpha$, where $\Psi_\alpha \in \text{Hom}_k((A \oplus M)^{\otimes 2}, A \oplus M)$. We have the equality (see also the proof of Lemma 1.1):
\[ \Psi_\alpha(a_1 + m_1, a_2 + m_2) = \Pi_\alpha(a_1, a_2) + \Psi_\alpha(a_1, m_2) + \Psi_\alpha(m_1, a_2). \]
Using Theorems 3.2.3 and 3.2.4 we find that $\delta^*_\Psi(\xi(f)) = \xi(d^1_\Psi(f) + d^2_\Psi(f))$, where
\[ 2n! d^1_\Psi(f)(\mu, a_1, \ldots, a_{n+1}) = \sum_{j=1}^n (-1)^j \sum_{\sigma, j \in \Sigma_{n+1}} \sum_{x} f(\epsilon^*_\alpha \circ_{n+1}^1 \mu \sigma_j, a_{\sigma_j(1)}, \ldots, \Pi_\alpha(a_{\sigma_j(j)}, a_{\sigma_j(j+1)}), \ldots, a_{\sigma_j(n+1)}); \]
\[ 2n! d^2_\Psi(f)(\mu, a_1, \ldots, a_{n+1}) = \sum_{\sigma \in \Sigma_{n+1}} \sum_{x} \Psi_\alpha(a_{\sigma(1)}, f(\mu \sigma \circ_{n+1, 2}^1 e^*_\alpha, a_{\sigma(2)}, \ldots, a_{\sigma(n+1)})) \]
\[ + (-1)^{n+1} \sum_{\sigma \in \Sigma_{n+1}} \sum_{x} \Psi_\alpha(f(\mu \sigma \circ_{n+1, 2}^1 e^*_\alpha, a_{\sigma(1)}, \ldots, a_{\sigma(n)}), a_{\sigma(n+1)}). \]

Therefore, directly from these expressions, we see that the differential of the complex $C^*_\mathcal{P}(A, M)$ can be deduced from the differential of the complex $C^*_\mathcal{P}(A, A)$, by keeping for $d^1_\Psi$ the same expression as those of $\delta^1_\Psi$ and by taking for $d^2_\Psi$ the expression of $\delta^2_\Psi$ where the action of $A$ onto itself has been replaced by the action of $A$ onto $M$.

**Remark 3.3.3.** (1) The Propositions 3.2.7 and 3.2.9 can be transposed for cohomology with coefficients. It suffices to replace, in Proposition 3.2.7, $\text{Der}_\mathcal{P}(A, A)$ by $\text{Der}_\mathcal{P}(A, M)$, and in Proposition 3.2.9, the action of $A$ onto itself by the action of $A$ onto $M$ via $\Psi$. 

(2) If we consider the Examples 3.1.9 again, we rediscover the same classical cohomologies with coefficients.

3.4. Abelian extensions of $\mathcal{P}$-algebras and $H^2_{\mathcal{P}}$

An abelian extension of $A$ by $M$

$$0 \rightarrow M \xrightarrow{i} \mathcal{E} \xrightarrow{\rho} A \rightarrow 0$$

is an extension of vector spaces (so the sequence is split over $k$, and we denote the splitting map by $s$) such that

1. $\mathcal{E}$ is a $\mathcal{P}$-algebra;
2. for any $\mu \in \mathcal{P}(2)$ and $m, m' \in M$, $\mu(i(m), i(m')) = 0$ (in $\mathcal{E}$).

Since $\mathcal{E} = A \oplus M$ condition (2) enables $M$ to inherit a natural structure of representation of $A$ given by

$$\mu(a, m) = \mu(s(a), m), \quad \mu(m, a) = \mu(m, s(a)).$$

**Definition 3.4.1.** Two extensions $(\mathcal{E})$ and $(\mathcal{E}')$ with $A$ and $M$ fixed are said to be equivalent if there exists a morphism of $\mathcal{P}$-algebras, $\phi: \mathcal{E} \rightarrow \mathcal{E}'$ which commutes with $id_M$ and $id_A$:

$$\begin{array}{ccc}
0 & \rightarrow & M & \xrightarrow{i} & \mathcal{E} & \xrightarrow{\rho} & A & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \phi & & \downarrow & & \downarrow \\
0 & \rightarrow & M & \xrightarrow{i'} & \mathcal{E}' & \xrightarrow{\rho'} & A & \rightarrow & 0.
\end{array}$$

For a fixed representation $M$ of $A$, let us consider the set of equivalence classes of extensions of $A$ by $M$, for which the structure of representation of $A$ induced on $A'$ is the prescribed one. We denote by $\mathcal{E}_{xt_{\mathcal{P}}}(A, M)$ this set.

Any two cocycle $f: (\mathcal{P}^1)^{\vee}(2) \otimes \mathcal{E} \rightarrow M$ gives rise to such an extension $(\mathcal{E})$ by the following procedure: as a vector space $\mathcal{E} = A \oplus M$; the structure of $\mathcal{P}$-algebra is given, with obvious notations, by

$$\mu((a_1, m_1), (a_2, m_2)) = (\mu(a_1, a_2), \mu(m_1, a_2) + \mu(a_1, m_2) + f(\mu(a_1, a_2))).$$

This defines a structure of $\mathcal{P}$-algebra on $\mathcal{E}$, since for any $r \in R$ we have

$$r((a_1, m_1), (a_2, m_2), (a_3, m_3))$$

$$= \psi(3)(r, a_1 \mid m_1, a_2 \mid m_2, a_3 \mid m_3) + (0, f(r, a_1, a_2, a_3))$$

$$= 0 + 0,$$
(use Proposition 0.5.5) where $\bar{f}$ is the associated map in the sense of Proposition 3.2.9 (see also Remark 3.3.3). Furthermore, the induced structure of representation of $A$ onto $M$ is obviously the former one.

**Theorem 3.4.2.** Let $A$ be a $\mathcal{P}$-algebra and $M$ be a representation of $A$. The construction described above induces a canonical bijection $\text{Ext}_P(A, M) \simeq H^2_P(A, M)$.

**Proof.** It is exactly the same proof as those for classical algebras (associative, Lie, etc.) which can be found in many textbooks. $\square$

4. **Homology of a $\mathcal{P}$-algebra with coefficients in a corepresentation**

In this section, we fix a corepresentation of $A$ denoted by $(N, 0)$.

4.1. **The chains**

For $n \geq 1$ we set

$$C_n^\mathcal{P}(A, N) = \mathcal{P}(\mathcal{P}^! \otimes (n) \otimes_{S_n} A^\otimes_n).$$

Any vector space can be embedded in its bidual, so we have the inclusion

$$C_n^\mathcal{P}(A, N) \subset \text{Hom}_k(\text{Hom}_k(C_n^\mathcal{P}(A, N), k), k).$$

By the adjunction property of the tensor product we have the isomorphism of vector spaces:

$$\text{Hom}_k(\text{Hom}_k(C_n^\mathcal{P}(A, N), k), k) \cong \text{Hom}_k(C_n^\mathcal{P}(A, N^*), k);$$

hence we have an injection

$$C_n^\mathcal{P}(A, N) \rightarrow \text{Hom}_k(C_n^\mathcal{P}(A, N^*), k)$$

$$x \otimes \mu \otimes a_1 \otimes \cdots \otimes a_n \rightarrow f \mapsto \langle x, f(\mu, a_1, \ldots, a_n) \rangle.$$  

The vector space $N^*$ is a representation of $A$ (see Remark 2.4); let

$$\bar{\psi} \in \text{Hom}_k(\mathcal{P}(2) \otimes_{S_2} (A \oplus N^*)^\otimes_2, A \otimes N^*)$$

be its structural map. Now let us consider the complex $(C_n^\mathcal{P}(A, N^*), d^\mathcal{P}_n)$ (see Section 3.3).

**Lemma 4.2.** Let $b_n^\mathcal{P}: C_n^\mathcal{P}(A, N) \rightarrow \text{Hom}_k(C_{n-1}^\mathcal{P}(A, N^*), k)$ be the map defined, with obvious notations, by

$$\langle b_n^\mathcal{P}(x \otimes \mu \otimes a_1 \otimes \cdots \otimes a_n), f \rangle = \langle x, d_n^\mathcal{P}(f)(\mu, a_1, \ldots, a_n) \rangle.$$  

Then $\text{Im}(b_n^\mathcal{P}) \subset C_{n-1}^\mathcal{P}(A, N)$. 

Therefore, we have a subcomplex. This justifies the next definition:

**Definition 4.3.** The homology of the complex \((C^\mathcal{P}_*(A,N), b^0_n, n)\), denoted in the sequel by \(H^\mathcal{P}_*(A,N)\), is called the homology of the \(\mathcal{P}\)-algebra \(A\) with coefficients in the corepresentation \(N\).

**Proof of Lemma 4.2.** We have \(b^0_n = b_{1,n}^0 + b_{2,n}^0\), where

\[
\begin{align*}
(b^0_{1,n}(x \otimes \mu \otimes a_1 \otimes \cdots \otimes a_n), f) &= (x, d^1_{\psi} n^{-1}(f)(\mu, a_1, \ldots, a_n)), \\
(b^0_{2,n}(x \otimes \mu \otimes a_1 \otimes \cdots \otimes a_n), f) &= (x, d^2_{\psi} n^{-1}(f)(\mu, a_1, \ldots, a_n)).
\end{align*}
\]

Let us prove, separately, that each \(h^0_n(x \otimes \mu \otimes a_1 \otimes \cdots \otimes a_n)\) belongs to \(C^\mathcal{P}_{n-1}(A,N)\). Let us consider again the structural map \(\psi \in \text{Hom}_k(\mathcal{P}(2) \otimes \Sigma_2(A \otimes N^*) \otimes^2, A \otimes N^*)\) and set

\[
\gamma^{-1}(\psi) = \sum \xi^* \otimes \Psi,
\]

where \(\Psi \in \text{Hom}_k((A \otimes N^*) \otimes^2, A \otimes N^*)\). Let \(\Theta_n \in \text{Hom}_k(A \otimes N \otimes A, N)\) be the map defined by the equalities (for \(n, n^* \in N \times N^*, a \in A\)):

\[
\begin{align*}
(n, \Theta_n(a, n^*)) &= (\Theta_n(a, n^*)), \\
(n, \Theta_n(n^*, a)) &= (\Theta_n(n, a^*), n^*).
\end{align*}
\]

Then, for all \(\mu \in \mathcal{P}(2), a \in A\) and \(x \in N\)

\[
\theta(\mu, a, x) = \frac{1}{2} \sum \{ (\mu, \xi^*_n) \Theta_n(a, x) + (\mu(12), \xi^*_n) \Theta_n(a, x) \}.
\]  

(4.1)

Furthermore, we have the following formulas which can be easily computed from the proof of Proposition 3.3.1:

\[
\begin{align*}
2(n - 1)! b^0_n(x \otimes \mu \otimes a_1 \otimes \cdots \otimes a_n) \\
= \sum_{j=1}^{n-1} (-1)^j \sum_{\sigma_j \in \Sigma_n} \sum_{x} x \otimes \xi^*_n \otimes_{2,n}^j \mu \sigma_j \otimes a_{\sigma_j(1)} \otimes \cdots \otimes a_{\sigma_j(n)}; \\
2(n - 1)! b^0_n(x \otimes \mu \otimes a_1 \otimes \cdots \otimes a_n) \\
= \sum_{\sigma \in \Sigma_n} \sum_{x} \Theta_n(1, x) \otimes \mu \sigma \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)} + (-1)^n \sum_{\sigma \in \Sigma_n} \sum_{x} \Theta_n(x, a_{\sigma(n)}) \otimes \mu \sigma \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}.
\end{align*}
\]

This proves our proposition. ☐

**Remark 4.4.** The proof of the last proposition gives us a means to deduce the differential of the complex \(C^\mathcal{P}_*(A,N)\) from the differential of the complex \(C^\mathcal{P}_*(A,N^*)\): the
term $b^0_{1,n}$ is simply the dual of $d^{1,n-1}_\psi$; the term $b^0_{2,n}$ is obtained from the dual of $d^{2,n-1}_\psi$ taking into account the duality between $\theta$ and $\psi$.

4.5. Calculus of $b^0_n$ for $n = 1$ and $n = 2$

From Propositions 3.2.7 and 3.2.9 we deduce the following propositions:

**Proposition 4.5.1.** The differential $b^0_2 : C^\varphi_2(A, N) \to C^\varphi_1(A, N)$ is given, for $x \in N$, $\mu \in (\mathcal{P})' \otimes (\mathcal{P})$, and $a_1, a_2 \in A$, by

$$b^0_2(x \otimes \mu \otimes a_1 \otimes a_2) = \theta(\mu, a_1, x) \otimes a_2 + \theta(\mu, x, a_2) \otimes a_1 - x \otimes \mu(a_1, a_2).$$

**Definition 4.5.2.** Let $A$ be a $\mathcal{P}$-algebra. The module of Kähler differentials, denoted by $\Omega^1_{\mathcal{P}}(A/k)$, is the left module over $\mathcal{P}(A)$ generated by the $k$-linear symbols $da$ for $a \in A$ (so we have $d(\lambda a \mid \lambda' a') = \lambda da + \lambda' da'$ for $a, a' \in A$ and $\lambda, \lambda' \in k$) with the relation

$$d(\mu(a, b)) = \chi(\mu) da + \chi(\mu(12), b) da, \quad a, b \in A.$$

**Proposition 4.5.3.** We have the isomorphism of vector spaces over $k$:

$$H^1_{\mathcal{P}}(A, N) \cong N \otimes_{\mathcal{P}(A)} \Omega^1_{\mathcal{P}}(A/k).$$

**Proof.** Left to the reader. 0

**Remark 4.5.4.** If $M$ is a representation of $A$ we also have

$$H^1_{\mathcal{P}}(A, M) = \text{Der}_{\mathcal{P}}(A, M) \overset{\xi}{\cong} \text{Hom}_{\mathcal{P}(A)}(\Omega^1_{\mathcal{P}}(A/k), M),$$

where $\xi(D)(da) = Da \quad (D \in \text{Der}_{\mathcal{P}}(A, M), a \in A)$.

**Proposition 4.5.5.** The differential $b^0_3 : C^\varphi_3(A, N) \to C^\varphi_2(A, N)$ is given by: $b^0_3 = -\tilde{b}_3|_{N \otimes R \otimes \Lambda^3 A^\otimes 3}$, where $\tilde{b}_3 : N \otimes \mathcal{F}(E)(3) \otimes \Lambda^3 A^\otimes 3 \to N \otimes \mathcal{P}(2) \otimes \Lambda^2 A^\otimes 2$ is defined, for all $x \in N$, $\mu, \nu \in \mathcal{P}(2)$, $a_1, a_2, a_3 \in A$, by

$$\tilde{b}_3(x \otimes \mu \otimes a_1 \otimes a_2 \otimes a_3) = \theta(\mu, x, a_3) \otimes \nu \otimes a_1 \otimes a_2 + x \otimes \mu \otimes v(a_1, a_2) \otimes a_3,$$

$$\tilde{b}_3(x \otimes \mu \otimes a_2 \otimes a_1 \otimes a_3) = \theta(\mu, a_1, x) \otimes \nu \otimes a_2 \otimes a_3 + x \otimes \mu \otimes a_1 \otimes v(a_2, a_3).$$

5. The complex $W^\varphi_*(A)$

In this short paragraph we are interested in the following question: do our theories of homology and cohomology come from derived functors? In general we cannot answer this question; nevertheless there is always a spectral sequence for which the first terms are related to the $\text{Ext}$-functor (respectively to the $\text{Tor}$-functor) and which abuts to cohomology (respectively to homology). For a few operads this spectral sequence collapses and so enables us to answer positively the above question.
5.1. Definition of $W^n(A)$

For any natural $n \geq 1$ we denote by $W^n(A)$ the left module over $U(A)$,

$$U(A) \otimes (\mathcal{P}^1)^\vee(n) \otimes_{\Sigma_n} A^\otimes n.$$ 

Since $U(A)$ is a right module over $U(A)$, it is a corepresentation with morphism

$$\theta^1: \mathcal{P}(2) \otimes A \otimes U(A) \to U(A),$$

defined, for $\mu \in \mathcal{P}(2)$, $a \in A$ and $\rho \in U(A)$ by $\theta^1(\mu, a, \rho) = \rho \chi(\mu, a)$. Therefore, $(W^n(A), b^n_\theta)$ is defined as the complex calculating the homology $H^n_\theta(A, U(A))$ (see Section 4).

**Proposition 5.2.** Let $M$ be a representation of $A$ and $N$ be a corepresentation. Then there are isomorphisms of vector spaces:

$$H^n_\theta(A, M) \simeq H^*_\theta(H(A, U(A), \theta(A), M)),$$

$$H^n_\theta(A, N) \simeq H^*_\theta(N \otimes U(A), W^n(A)).$$

**Proof.** We give a proof for homology, leaving cohomology to the reader. Let $(N, \theta)$ be a corepresentation of $A$; we have the isomorphism of vector spaces:

$$N \otimes U(A) W^n(A) \simeq N \otimes (\mathcal{P}^1)^\vee(n) \otimes_{\Sigma_n} A^\otimes n,$$

where $\zeta$ is defined, for $x \in N$, $\rho \in U(A)$, $v \in (\mathcal{P}^1)^\vee(n)$ and $a_1, \ldots, a_n \in A$, by:

$$\zeta(x \otimes \rho \otimes v \otimes a_1 \otimes \cdots \otimes a_n) = x \rho \otimes v \otimes a_1 \otimes \cdots \otimes a_n.$$ 

Therefore, the only point to establish is the fact that $\zeta(1 \otimes b^n_\theta) = b^n_\theta \circ \zeta$; this results from the following lemma:

**Lemma.** Let $(\Theta_\beta^0, \eta^0_\beta)_{\beta \in \mathcal{A}}$ and $(\Theta_\alpha^*, \epsilon^*_\alpha)_{\alpha \in \mathcal{A}}$ be the elements associated to $\theta^1$ and $\theta$ (see (4.1)), so that the following holds:

$$\Theta_\beta^0(\mu, a, x) = \frac{1}{2} \sum_{A} \{ \langle \mu, \eta^0_\beta \rangle \Theta_\beta^0(a, x) + \langle \mu(12), \eta^0_\beta \rangle \Theta_\beta^0(p, a) \},$$

$$\Theta_\alpha^*(\mu, a, x) = \frac{1}{2} \sum_{A} \{ \langle \mu, \epsilon^*_\alpha \rangle \Theta_\alpha(a, x) + \langle \mu(12), \epsilon^*_\alpha \rangle \Theta_\alpha(x, a) \}. $$

Then $\mathcal{A} = \mathcal{A}$; for any $\alpha \in \mathcal{A}$, $\eta^*_\alpha = \epsilon^*_\alpha$ and for $a \in A$, $x \in N$, $\rho \in U(A)$,

$$\Theta_\alpha(a, x \rho) = x \Theta_\alpha^0(a, \rho), \quad \Theta_\alpha(x \rho, a) = x \Theta_\alpha(a, \rho).$$

**Proof of the Lemma.** The lemma results firstly, from the fact that the $(\Theta_\alpha, \epsilon^*_\alpha)_{\alpha \in \mathcal{A}}$ are entirely determined by $\theta$ and secondly, from equalities:

$$\theta(a, x \rho) = x \theta^1(a, \rho), \quad \theta(x \rho, a) = x \theta^1(\rho, a). \quad \square$$
Using the above lemma and the formulas given in the proof of Lemma 4.2, we easily show that $\zeta$ is a map of complexes.

5.3. The spectral sequences

5.3.1. Hypercohomology of a bifunctor

In this section we briefly recall some results concerning hypercohomology; we refer to [4] for more details.

Let $\mathcal{C}$ be the category of left modules over $\mathcal{A}$ and let $T$ be the bifunctor

$$T: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

defined by $T(M_1, M_2) = \text{Hom}_{\mathcal{A}}(M_1, M_2)$. Let $C = (C_n, d_n)$ and $D = (D^n, \delta_n)$ be two complexes in the category $\mathcal{C}$, and let $X = (X_{p, q})$ be a projective resolution of $C$ and $Y = (Y_{p, q})$ be an injective resolution of $D$: this means that $X$ and $Y$ are double complexes (such that for all $p, q$, $X_{p, q}$ is a projective module over $\mathcal{A}$ and $Y_{p, q}$ is an injective module) satisfying conditions expressed in [4]. Then $T(X, Y)$ is a quadruple complex which can be reduced into a double complex by gathering the first and third indexes and the second and fourth indexes:

$$T^{p, q}(X, Y) = \bigoplus_{p_1 + p_2 = p, q_1 + q_2 = q} T(X_{p_1, q_1}, Y_{p_2, q_2}).$$

The main results of [4] are:

1. The cohomology $H^*(T(X, Y))$ is independent of the resolutions $X$ and $Y$. It is called the hypercohomology of $T(C, D)$ and is denoted by $\mathcal{R}^* T(C, D)$.

2. The two spectral sequences associated to the double complex $T(X, Y)$ are independent of $X$ and $Y$ and abut both to $\mathcal{R}^* T(C, D)$.

3. Let $E_2^{p, q}$ and $E_2^{p, q}$ be the two spectral sequences abutting to hypercohomology; then we have

$$E_2^{p, q} = H^q(R^p T(C, D)) \Rightarrow \mathcal{R}^{p+q} T(C, D),$$

$$E_2^{p, q} = \bigoplus_{q_1 + q_2 = q} R^p T(H_{q_1}(C), H_{q_2}(D)) \Rightarrow \mathcal{R}^{p+q} T(C, D),$$

where $R^p T$ is the $p$th derived functor of $T$, i.e. $\text{Ext}_{\mathcal{A}}^p$ and where $R^p T(C, D)$ is the complex $Z_p^*$ such that

$$Z_p^* = \bigoplus_{q_1 + q_2 = q} R^p T(C_{q_1}, D_{q_2}).$$

In addition, if we suppose that for all $q_1$, $C_{q_1}$ is a free module over $\mathcal{A}$ then, for $p > 0$, $R^p T(C_{q_1}, D^{q_1}) = 0$. Therefore, the spectral sequence $E_2^{p, q}$ collapses and we have

$$\mathcal{R}^n T(C, D) = H^n(R^0 T(C, D)).$$
Now, let us apply this machinery to the following situation: let $M$ be a representation of $A$, $C$ be the complex defined by $C^* = W^*_{W}(A)$ and $D$ be the complex defined by $D^0 = M$ and $D^q = 0$ if $q > 0$. Then for any $q$, $C_q$ is a free module over $W(A)$ and hence

$$\mathcal{A}^n T(C, D) = H^n(R^0 T(C, D)) = H^n(\text{Hom}_{W(A)}(C^*, M)).$$

Therefore, according to Proposition 5.2, we have: $\mathcal{A}^n T(C, D) \simeq H^{n+1}_P(A, M)$. The other spectral sequence is given by: $E_2^{p,q} = \bigoplus_{q_1 + q_2 = q} \text{Ext}_{W}(H_q(C), H^q(D))$; but $H^q(D) = M$, if $q_2 = 0$ and $H^q(D) = 0$ if $q_2 > 0$; furthermore $H_q(C) = H_{q+1}(W^*(A))$. Therefore, we have $E_2^{p,q} = \text{Ext}_{W}(H_q(W^*(A)), M)$. Hence, we have proved the following proposition:

**Proposition 5.3.2.** Let $\mathcal{P}$ be a quadratic operad, $A$ be a $\mathcal{P}$-algebra and $M$ be a representation of $A$. Then there exists a spectral sequence $E_{p,q}$ such that

$$E_2^{p,q} = \text{Ext}_{W}(H_{q+1}(W^*(A)), M) \Rightarrow H^{p+q+1}_{\mathcal{P}}(A, M).$$

We have the analogue for homology:

**Proposition 5.3.3.** Let $\mathcal{P}$ be a quadratic operad, $A$ be a $\mathcal{P}$-algebra and $N$ be a corepresentation of $A$. Then there exists a spectral sequence $E_{p,q}$ such that

$$E_2^{p,q} = \text{Tor}_{W}(H_{q+1}(W^*(A)), N) \Rightarrow H_{P+q+1}_{\mathcal{P}}(A, N).$$

5.3.4. Particular cases

If for all $p > 0$, $H_{p+1}(W^*(A)) = 0$ (or equivalently if $H_{p+1}(W^*(A))$ is a projective module over $W(A)$), then we have:

$$H^*_{\mathcal{P}}(A, M) \simeq \text{Ext}^*_W(A/k, M),$$

$$H^*_{\mathcal{P}}(A, N) \simeq \text{Tor}^*_W(N, A/k).$$

As a matter of fact $W^*(A)$ is a projective resolution of $A$.

**Example 5.3.5.** The complex $W^*(A)$ is acyclic in the following cases:

1. $\mathcal{P} = \text{Ass}$ and $A$ is an unital algebra ($W^*(A)$ is then the bar resolution of $A$);
2. $\mathcal{P} = \text{Lie}$ (see [4]);
3. $\mathcal{P} = \text{Leib}$ (see [18]).
6. Application to dual Leibniz algebras

In all this section $\mathcal{P}$ will stand for the operad $\text{Leib}^!$ and $(A, \times)$ will denote a dual Leibniz algebra (see the end of Section 0).

6.1. According to Section 3, we first have to study the dual operad $\mathcal{P}^! = \text{Leib}$ and particularly its \textit{comp}-operations:

$$o_i : \text{Leib}(n) \otimes \text{Leib}(m) \to \text{Leib}(n + m - 1), \quad 1 \leq i \leq n.$$ 

Let $V = k \{x_1, \ldots, x_n\}$ be the vector space generated by $n$ elements and let $\tilde{T}(V) = \bigoplus_{m \geq 0} V^\otimes m$ be the free Leibniz algebra generated by $V$ (see [15]). With these notations, as a module over $k[\Sigma_n]$, $\text{Leib}(n)$ is the $n$th multi-linear part of $\tilde{T}(V)$. Hence, a generator of $\text{Leib}(n)$ is of the form $x_\sigma(1) \otimes \cdots \otimes x_\sigma(n)$; this means that $\text{Leib}(n) = k[\Sigma_n]$; nevertheless, the structure of right module over $k[\Sigma_n]$ is not the usual one: for $\sigma \in \text{Leib}(n)$ and $\tau \in \Sigma_n$, we have $\sigma \cdot \tau = \tau^{-1} \sigma$; if we take the "naive" right action, then formula (0.2) is not satisfied. The next section is now devoted to the study of $\tilde{T}(V)$.

6.2. Let $[\cdot \cdot \cdot]$ be the Leibniz bracket of $\tilde{T}(V)$. According to [15] this bracket is entirely determined by the Leibniz identity (see the end of Section 0) and by the following statement: for all $x \in \tilde{T}(V)$ and $v \in V$, $[x, v] = x \otimes v$.

Let $I_n$ be the set of $\sigma \in \Sigma_n$ for which there is $n - 1$ permutations $x_i \in \Sigma_2$ such that

$$\sigma = x_1 \circ_1 (x_2 \circ_1 (\cdots \circ_1 (x_{n-2} \circ_1 x_{n-1})\cdots),$$

(for $n = 1$ we set $I_1 = \{1\}$). The set $I_n$ is characterized by the following condition: $\sigma \in I_n$ if and only if there is $x \in \Sigma_2$ and $\tau \in I_{n-1}$ such that $\sigma = x \circ_1 \tau$ (x and $\tau$ are then unique); therefore the cardinal of $I_n$ is $2^{n-1}$.

Let $\omega : I_n \to \{-1, 1\}$ be the map defined by $\omega(\sigma) = (-1)^{\tau(\sigma)}$ where $\tau(\sigma)$ is the cardinal of the set of $i \in \{1, \ldots, n - 1\}$ such that $x_i = (12)$ in the writing of $\sigma$.

\textbf{Proposition 6.3.} \textit{Let $V$ be a vector space and $(\tilde{T}(V), [\cdot \cdot \cdot])$ be the free Leibniz algebra generated by $V$. Then the Leibniz bracket is given, for $X \in \tilde{T}(V)$ and $v_1, \ldots, v_n \in V$, by: $[X, v_1 \otimes \cdots \otimes v_n] = \sum_{x \in I_n} \omega(x) X \otimes v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$.}

\textbf{Proof.} By induction. $\square$

Now we are going to describe the \textit{comp}-operations on $\text{Leib}$: in fact we only need to calculate $1_n \circ_i 1_m$.

\textbf{Lemma 6.4.} \textit{In the operad $\text{Leib}$, for $1 \leq i \leq n$, we have}

$$1_n \circ_i 1_m = \begin{cases} 1_{m+n-1} & \text{if } i = 1, \\ \sum_{x \in I_m} \omega(x) a_i^x & \text{if } i > 1, \end{cases}$$
where $\sigma_i^2$ stands for the permutation of $\Sigma_{m+n-1}$ defined by

$$
\sigma_i^2(j) = \begin{cases} 
  j & \text{if } 1 \leq j \leq i - 1, \\
  a(j) & \text{if } i \leq j \leq i + m - 1, \\
  j & \text{if } i + m \leq j.
\end{cases}
$$

**Proof.** By definition, $1_n \circ_i 1_m$ is the element $X_i \in \bar{F}(k\{x_1, \ldots, x_{m+n-1}\})$ such that:

- if $i = 1$, $X_1 = [x_1 \otimes \cdots \otimes x_n] \otimes x_{n+1} \otimes \cdots \otimes x_{m+n-1}$;
- if $i > 1$, $X_i = [X, Y] \otimes Z$, where $X = x_1 \otimes \cdots \otimes x_{i-1}$, $Y = x_i \otimes \cdots \otimes x_{m+i-1}$ and $Z = x_{m+i} \otimes \cdots \otimes x_{m+n-1}$. According to the last proposition we have:

$$
[X, Y] = \sum_{a \in I_n} \omega(a) X \otimes x_{a(i)} \otimes \cdots \otimes x_{a(i+m-1)},
$$

whence the result. $\square$

In order to define homology and cohomology of $A$, we now have to describe the Lie algebra structure on $L_{\varphi}(V)$. According to 3.1.1, $L_{\varphi}(V)$ is isomorphic to $\text{Hom}_k(V \otimes^{n+1}, V)$, the isomorphism being given by

$$
\zeta : \text{Leib}(n + 1) \otimes_{\Sigma_{n+1}} \overline{\text{End}(V)(n + 1)} \rightarrow \text{Hom}_k(V \otimes^{n+1}, V),
$$
such that $\zeta(\sigma \otimes f) = \varepsilon(\sigma)^{-1} f = \varepsilon(\sigma)f \sigma$. Let $\mathcal{L} = \bigoplus_{n \in \mathbb{N}} \text{Hom}_k(V \otimes^n, V)$.

**Proposition 6.5.** There exists on $\mathcal{L}$ a graded Lie algebra structure with bracket $[,]$ given, for $f, g \in \text{Hom}_k(V \otimes^{n+1}, V) \times \text{Hom}_k(V \otimes^{n+1}, V)$, by

$$
[f, g] = I_f(g) + (-1)^{mn+1} I_g(f),
$$

where $I_f(g) \in \text{Hom}_k(V \otimes^{n+1}, V)$ is defined by

$$
I_f(g) = f \circ_1 g + \sum_{x \in I_{n+1}} \sum_{i=2}^{n+1} (-1)^{n(i-1)} \varepsilon(x) \omega(x)f \circ_i^2 g
$$

with

$$
\begin{align*}
\circ_1 g(v_1, \ldots, v_{m+n+1}) &= f(g(v_1, \ldots, v_{m+1}), \ldots, v_{m+n+1}), \\
\circ_i^2 g(v_1, \ldots, v_{m+n+1}) &= f(v_1, \ldots, v_{i-1}, g(v_{a-1(i)}, \ldots, v_{a-1(i+m)}), \ldots, v_{m+n+1}).
\end{align*}
$$

**Proof.** Simple application of Lemma 6.4 and Proposition 3.1.2. $\square$

**6.6. The operator $\mu_n$**

In this paragraph we consider the left action of $\Sigma_n$ on $V \otimes^n$ given by

$$
\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)};
$$
we extend this action by k-linearity to $k[\Sigma_n]$. Let $\mu_n \in k[\Sigma_n]$ be the operator defined by $\mu_n = \sum_{\alpha \in \Lambda} \varepsilon(\alpha) \omega(\alpha) \alpha$.

**Lemma 6.6.1.** Let $\tau_n$ be the cycle $(12\ldots n)$. For every $n \geq 1$ we have:

$$\mu_{n+1} = \mu_n + (-1)^{n+1} \mu_n \circ \tau_{n+1}^{-1},$$

where $\mu_n$ is considered as an element of $k[\Sigma_{n+1}]$ via the injection of $\Sigma_n$ into $\Sigma_{n+1}$.

**Proof.** By induction. □

We recall that a $(p, q)$-shuffle is an element $\sigma \in \Sigma_{p+q}$ such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$. In the sequel, we will denote by $\mathcal{S}_{p,q}$ the set of $(p, q)$-shuffles. Let $s_{p,q}$ be the element of $k[\Sigma_n]$ $(n = p + q)$ defined by the equality

$$s_{p,q} = \sum_{\sigma \in \mathcal{S}_{p,q}} \varepsilon(\sigma) \sigma,$$

we also make the convention that if $p$ or $q$ is strictly negative then $s_{p,q} = 0$.

**Lemma 6.6.2.** The element $s_{p,q}$ is defined inductively by

$$s_{p,q}(v_1, \ldots, v_n) = s_{p-1,q}(v_1, \ldots, v_{p-1}) \otimes v_p + (-1)^{q-p} s_{p-1,q}(v_1, \ldots, v_{p-1}, v_p, \ldots, v_n) \otimes v_p.$$

**Proof.** Left to the reader. □

We can now give an expression of $\mu_n$, originally found by Muriel Livernet in another context [14].

**Proposition 6.6.3.** Let $v_1, \ldots, v_n \in V$, then

$$\mu_n(v_1, \ldots, v_n) = \sum_{k=1}^{n} (-1)^{\frac{k(k+1)}{2} - 1}(id \otimes s_{k-1,n-k})(v_k, \ldots, v_k, v_{k+1}, \ldots, v_n).$$

**Proof.** By induction (use Lemmas 6.6.1 and 6.6.2). □

In the following two short paragraphs, we give the definition of the complexes calculating homology and cohomology of dual Leibniz algebras. The formulas for the differentials can be computed directly from Proposition 6.5, and from the general theory exposed in Sections 3 and 4.

6.7. Cohomology of dual Leibniz algebras

Let $(A, \times)$ be a dual Leibniz algebra and $M$ be a representation of $A$ (see Example 1.5 (5)). The cohomology $H^*_d(A, M)$ denoted by $HD^*(A, M)$ is the homology of the
complex \((C^*(A,M), d^k)\), where for \(n \geq 1\), \(C^n(A,M) = \text{Hom}_k(A^\otimes n, M)\). The differential \(d^n : C^n(A,M) \to C^{n+1}(A,M)\) is given by

\[
d^n(f) (a_1, \ldots, a_{n+1}) = a_1 \times f(\mu_n(a_2, \ldots, a_{n+1})) - f(a_1 \times a_2, a_3, \ldots, a_{n+1}) + \sum_{i=2}^{n} (-1)^i f(a_1, \ldots, a_{i-1}, a_i * a_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1} f(a_1, \ldots, a_n) \times a_{n+1}
\]

where for \(a, b \in A\) we have set \(a * b = a \times b + b \times a\).

### 6.8. Homology of dual Leibniz algebras

Let \((A, \times)\) be a dual Leibniz algebra and \((N, \star)\) be a corepresentation of \(A\) (see 2.5, Example 5). The homology \(H^q_\star(A,N)\) denoted by \(HD_\star(A,N)\) is the homology of the complex \((C_\star(A,N), b_\star)\), where for \(n \geq 1\), \(C_n(A,N) = N \otimes A^\otimes n\). The differential \(b_n : C_n(A,N) \to C_{n-1}(A,N)\) is given by

\[
b_n(x \otimes a_1 \otimes \cdots \otimes a_n) = x \star a_1 \otimes \mu_{n-1}(a_2, \ldots, a_n) - x \otimes a_1 \times a_2 \otimes a_2 \otimes \cdots \otimes a_n + \sum_{i=2}^{n-1} (-1)^i x \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i * a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n * x \otimes a_1 \otimes \cdots \otimes a_{n-1}.
\]

**Remark 6.9.** Our theory does not give a sense to the groups \(H^q(A,N)\) and \(HD^0(A,M)\); furthermore, no differential \(b_1\) or \(d^0\) seems to exist, giving a sense to such groups.

### 7. Relations with Barr–Beck homology

In this section, we denote by \((\mathcal{P}\text{-alg}, A)\) the category of \(\mathcal{P}\)-algebras over \(A\): an object of \((\mathcal{P}\text{-alg}, A)\) is a \(\mathcal{P}\)-algebra \(B\) such that there exists a morphism of \(\mathcal{P}\)-algebras \(\phi : B \to A\). A morphism \(\psi : (B, \phi) \to (B', \phi')\) is a morphism of \(\mathcal{P}\)-algebras \(\psi : B \to B'\) such that \(\phi' \circ \psi = \phi\).

**Lemma 7.1.** Let \((B, \phi) \in (\mathcal{P}\text{-alg}, A)\), \((M, \psi)\) be a representation of \(A\) and \((N, \theta)\) be a corepresentation of \(A\). Then \(M\) is a representation of \(B\) and \(N\) is a corepresentation of \(B\).

**Proof.** It suffices to take \(\psi' : \mathcal{P}(2) \otimes B \otimes M \to M\) and \(\theta' : \mathcal{P}(2) \otimes B \otimes N \to N\) defined, with obvious notations, by \(\psi'(\mu, b, m) = \psi(\mu, \phi(b), m)\) and \(\theta'(\mu, b, n) = \theta(\mu, \phi(b), n)\).

Let \(V\) be a vector space; the free \(\mathcal{P}\)-algebra generated by \(V\) is the vector space \(\mathcal{F}_\mathcal{P}(V) = \bigoplus_{n \geq 1} \mathcal{P}(n) \otimes V^\otimes n\) equipped with the following structure of \(\mathcal{P}\)-algebra: for
Let $\mu$ be the category of vector spaces and $T_p : \mathcal{C} \to \mathcal{C}$ be the functor defined, for $V \in \mathcal{C}$, by $T_p V = \mathcal{P}(V)$. Let $\eta : 1_\mathcal{C} \to T_p$ be the natural transformation given by the inclusion and $m : T_p^2 \to T_p$ be the natural transformation given by the composition maps: more precisely, for all $\mu_1 \otimes Z_1, \ldots, \mu_m \otimes Z_m \in \mathcal{P}(n_1) \otimes \Sigma_n V \otimes n_1 \times \cdots \times \mathcal{P}(n_m) \otimes \Sigma_n V \otimes n_m$, $m_V : T_p^2 V \to T_p V$ is defined by

$$m_V(\mu_1 \otimes Z_1, \ldots, \mu_m \otimes Z_m) = \mu_1(Z_1) \otimes \cdots \otimes \mu_m(Z_m),$$

where $\mu_i(Z_i)$ stands for the composition in the $\mathcal{P}$-algebra $T_p V$ (see (7.1)). We now recall some definitions about triples; for more details, the reader should report to [20].

**Definition 7.2.** A triple or monad in the category $\mathcal{C}$ is given by the following data:
- a functor $T : \mathcal{C} \to \mathcal{C}$;
- two natural transformations, $\eta : 1_\mathcal{C} \to T$ and $m : T^2 \to T$, such that the following diagrams commute:

$$
\begin{array}{ccc}
T & \xrightarrow{T_T} & T^2 \\
\downarrow{id} & & \downarrow{id}
\end{array}
\quad
\begin{array}{ccc}
T & \xrightarrow{id} & T \\
\downarrow{m} & & \downarrow{m}
\end{array}
\quad
\begin{array}{ccc}
T & \xrightarrow{T_m} & T^2 \\
\downarrow{id} & & \downarrow{id}
\end{array}
$$

**Definition 7.3.** A cotriple $(G, \varepsilon, \delta)$ in the category $\mathcal{C}$ is given by the following data:
- a functor $G : \mathcal{C} \to \mathcal{C}$;
- two natural transformations, $\varepsilon : 1_\mathcal{C} \to G$ and $\delta : G \to G^2$, such that the diagrams corresponding to (7.2) (where the arrows are reversed) commute.

**Definition 7.4.** Let $(T, \eta, m)$ be a triple in the category $\mathcal{C}$. A $T$-algebra $(V, \alpha)$ is an object $V \in \mathcal{C}$ equipped with a morphism $\alpha : TV \to V$ such that the following diagrams commute:

$$
\begin{array}{ccc}
V & \xrightarrow{id} & V \\
\downarrow{\eta} & & \downarrow{id}
\end{array}
\quad
\begin{array}{ccc}
T^2 V & \xrightarrow{T_T} & TV \\
\downarrow{m} & & \downarrow{\alpha}
\end{array}
\quad
\begin{array}{ccc}
TV & \xrightarrow{\alpha} & V \\
\downarrow{\eta} & & \downarrow{id}
\end{array}
\quad
\begin{array}{ccc}
TV & \xrightarrow{\alpha} & T \\
\downarrow{\eta} & & \downarrow{id}
\end{array}
$$
A morphism of $T$-algebras $f : (V, \alpha) \to (W, \beta)$ is a morphism of $C$, $f : V \to W$ such that:

$$
\begin{array}{ccc}
TV & \xrightarrow{Tf} & TW \\
\downarrow{\alpha} & & \downarrow{\beta} \\
V & \xrightarrow{f} & W
\end{array}
$$

In the sequel, we will denote by $\mathcal{A}^T$ the category of $T$-algebras.

We have the following proposition (see for instance [10]):

**Proposition 7.5.** Let $\mathcal{P}$ be an operad with $\mathcal{P}(1) = k$ then:
(a) $(T_p, \eta, m)$ is a triple in the category $C$;
(b) the category of $T_p$-algebras is equal to the category of $C$-algebras.

### 7.6. Barr–Beck homology

In this paragraph, we briefly recall some notions about Barr–Beck’s theory of homology [3].

Let $(T, \eta, m)$ be a triple and let

$$
F : C \to \mathcal{A}^T, \quad U : \mathcal{A}^T \to C
$$

be the functors defined, with obvious notations, by $F(V) = (TV, m_V)$ (free $T$-algebra functor) and $U(V, \alpha) = V$ (forgetful functor). It is straightforward to verify that these two functors are adjoint. Therefore (see [20]), we have a cotriple $(G = FU, \epsilon, \delta)$ in the category $\mathcal{A}^T$, where $\epsilon : G(V, \alpha) = (TV, m_V) \to (V, \alpha)$ is the multiplication $\alpha$ and $\delta : G(V, \alpha) \to G^2(V, \alpha)$ is $T_{\eta_V}$. For any object $V$, this gives rise to a simplicial object $(X_n)_{n \in \mathbb{N}}$ where $X_n = G^{n+1} \cdot V$: the $i$th face is $\epsilon_i = G^{n-i} \cdot \epsilon G^i$ and the $i$th degeneracy is $\delta_i = G^{n-i} \cdot \delta G^i$. Let $k$-vect denote the category of vector spaces over $k$.

**Definition 7.6.1.** Let $E : \mathcal{A}^T \to k$-vect be a contravariant functor. The cohomology of $V$ with coefficients in $E$ for the cotriple $G$ is the homology of the following complex ($EG^n V$ is classically put in degree $n - 1$):

$$
0 \to EGV \to EG^2V \to \cdots \to EG^n V \xrightarrow{\epsilon_{n+1}} EG^{n+1} V \to \cdots
$$

where the differential is $\delta_n = \sum_{i=0}^{n+1} (-1)^i E \delta_i$. In the sequel, we will denote by $H^\star(V, E)_G$ the groups of cohomology.
Definition 7.6.2. Let $D : \mathcal{T} \to k$-vect be a covariant functor. The homology of $V$ with coefficients in $D$ for the cotriple $G$ is the homology of the following complex $(DG^n V)$ is put in degree $n - 1$:

$$\cdots \to DG^{n+1} V \xrightarrow{b_n} DG^n V \to \cdots \to DG V \to 0$$

where the differential is $b_n = \sum_{i=0}^{n} (-1)^i D_{b_i}$. In the sequel, we will denote by $H_{\ast}(V, D)_G$ the groups of homology.

Now we restrict ourself to the category of $P$-algebras over $A$, where $A$ is still an algebra over the quadratic operad $P$. We can consider the category of $T_P$-algebras over $A$ denoted by $(\mathcal{T}_P, A)$ (the definition is left to the reader). From Lemma 7.5 these two categories are equal. Let us consider the cotriple $G_P = T_P U$: according to Definitions 7.6.1 and 7.6.2, for any contravariant functor $E : (\mathcal{T}_P, A) \to k$-vect and any covariant functor $D : (\mathcal{A}-\text{alg}, A) \to k$-vect we can consider the cohomology $H_{\ast}(-, E)_G$, and the homology $H_{\ast}(-, D)_G$. Let $M$ be a representation of $A$ and $N$ be a corepresentation of $A$. Let $E_M$ be the contravariant functor $E_M : (\mathcal{T}_P, A) \to k$-vect defined by

$$E_M(B) = \text{Hom}_{\mathcal{T}_P(B)}(\Omega^1_P(B/k), M) = \text{Der}_P(B, M).$$

In the same way, let $D_N$ be the covariant functor $D_N : (\mathcal{T}_P, A) \to k$-vect defined by

$$D_N(B) = N \otimes_{\mathcal{T}_P(B)} \Omega^1_P(B/k)$$

(notice that these functors are well defined since we have Lemma 7.1). The main theorem of this section is the following:

Theorem 7.7. Let $P$ be a quadratic operad with $P(1) = k$, $A$ be a $P$-algebra, $M$ be a representation of $A$ and $N$ be a corepresentation of $A$. Then, for all $B \in (\mathcal{T}_P, A)$ and $n = 0, 1, 2$ there are isomorphisms:

$$H^n(B, E_M)_G \simeq H^{n+1}_P(B, M),$$

$$H_n(B, D_N)_G \simeq H_{n+1}^P(B, N).$$

Furthermore, for $n \geq 3$ there are epimorphisms $H^n(B, E_M)_G \to H^{n+1}_P(B, M)$ and monomorphisms $H_{n+1}^P(B, N) \to H_n(B, D_N)_G$.

Proof. We give a proof for cohomology leaving homology to the reader. For $n \in \mathbb{N}$ let $\Gamma^n : (\mathcal{T}_P, A) \to k$-vect be the contravariant functor defined by

$$\Gamma^n(B) = C^{n+1}_P(B, M).$$
Let $d^n_B : \Gamma^n(B) \to \Gamma^{n+1}(B)$ be the differential constructed in Section 3.3.

**Lemma 1.** $d^n : \Gamma^n \to \Gamma^{n+1}$ is a natural transformation.

**Proof of Lemma 1.** For any morphism $f : B \to C$ we have to prove that the following diagram is commutative:

$$
\begin{array}{ccc}
\Gamma^n C & \xrightarrow{d^n_C} & \Gamma^{n+1} C \\
\downarrow \Gamma^n f & & \downarrow \Gamma^{n+1} f \\
\Gamma^n B & \xrightarrow{d^n_B} & \Gamma^{n+1} B
\end{array}
$$

This statement is straightforward: it suffices to use the expression of the differential found in Section 3.3 (see the proof of Proposition 3.3.1). \hfill \Box

Therefore we have a complex of functors

$$0 \to \Gamma^0 \to \Gamma^1 \to \ldots \to \Gamma^n \to \Gamma^{n+1} \to \ldots$$

If we apply it to a $\mathcal{P}$-algebra (over $A$) $B$, we get the complex $(G^*_{\mathcal{P}}(B,M), d^*)$; hence

$H^n(\Gamma^* B) = H_{\mathcal{P}}^{n+1}(B,M)$.

**Lemma 2.** The functor $\Gamma^n$ is $G_{\mathcal{P}}$-representable, i.e. there exists a natural transformation $\theta^n : \Gamma^n G_{\mathcal{P}} \to \Gamma^n$ such that $\theta^n \circ \Gamma^n \varepsilon = 1_{\Gamma^n}$.

**Proof of Lemma 2.** We recall that $\varepsilon : G_{\mathcal{P}}(B, \beta) = (T_{\mathcal{P}} B, m_B) \to (B, \beta)$ is the multiplication $\beta$. Therefore, if we take $\eta : 1_{\mathcal{P}} \to T_{\mathcal{P}}$ and $\theta^n = \Gamma^n \eta$ we get

$$\theta^n \circ \Gamma^n \varepsilon = \Gamma^n (\varepsilon \circ \eta) = \Gamma^n (\beta \circ \eta) = 1_{\Gamma^n}.$$ \hfill \Box

The natural transformation $\theta^n$ gives rise to an homotopy map (see [3] for more details); therefore we have the following corollary:

**Corollary.** $H^p(B, \Gamma^q)_{G_{\mathcal{P}}} = \begin{cases} 
\Gamma^q B & \text{if } p = 0, \\
0 & \text{if } p > 0.
\end{cases}$

**Proof of Theorem 7.7 (Conclusion).** Let $(C^{*,*}, d^*, \partial^*)$ be the double complex defined, for $p, q \geq 0$, by

$$C^{p,q} = \Gamma^p G_{\mathcal{P}}^{q+1} B$$
(notice that, thanks to Lemma 1, it is a double complex and that \( G^{q+1}B \) is a \( \mathcal{P} \)-algebra over \( A \), since we have a morphism from \( GA \) to \( A \)). Let \( E^{p,q}_1 \) and \( E^{p,q}_2 \) be the two spectral sequences associated to this double complex (the first corresponding to the \( p \)-filtration, the second to the \( q \)-filtration). According to the above corollary, each line is exact (except the leftmost term). Therefore, we have:

\[
E^{p,q}_2 = \begin{cases} 
H^{p+1}_\mathcal{P}(B,M) & \text{if } q = 0, \\
0 & \text{if } q > 0.
\end{cases}
\]

The second spectral sequence is given by the following: \( E^{p,q}_2 = H^p(K^*, q) \), where \( K^{p,q} = H^{q+1}_\mathcal{P}(G^{p+1}B, M) \), the differential being induced by \( \partial \). In particular, since \( H^p(-, M) = \text{Der}(-, M) \) (see 3.3.3), we have \( E^{p,0}_2 = H^p(B, EM) \).

Now let us examine the spectral sequence \( E^{p,q}_2 \) more precisely. Since the first spectral sequence collapses, its abutment \( E^{p,q}_\infty \) is \( E^{p,q}_2 \). Furthermore, if \( r > \max(p, q + 1) \) then \( E^{p,q}_r \simeq E^{p,q}_\infty \). Hence, for \( i = 0, 1 \) we have

\[
E^{i,0}_2 \simeq E^{i,0}_\infty \simeq E^{i,0}_2
\]

i.e. for \( i = 0, 1 \) \( H^i(B, EM)_\mathcal{P} \simeq H^{i+1}_\mathcal{P}(B, M) \). We also have \( E^{2,0}_3 = H^3_\mathcal{P}(B, M) \); but \( E^{2,0}_2 \) is the vector space \( \text{Im } d_2 \) where \( d_2 : E^{2,0}_1 \to E^{2,0}_3 \to 0 \) is the canonical differential of the spectral sequence. Let us examine \( E^{2,0}_2 \); we have seen that it is \( H^0(K^{*,1}) \) where \( K^{*,1} \) is the complex such that

\[
K^{p,1} = H^2_\mathcal{P}(G^{p+1}B, M).
\]

**Lemma 3.** For any quadratic operad \( \mathcal{P} \), any vector space \( V \) and any representation \( M \) of \( \mathcal{P}(V) \), we have \( H^2_\mathcal{P}(\mathcal{P}(V), M) = 0 \).

**Proof of Lemma 3.** We have seen that \( H^2_\mathcal{P}(\mathcal{P}(V), M) \simeq H^1(G_\mathcal{P}V, EM)_\mathcal{P} \). In [3], it is proved that if \( p > 0 \) then \( H^p(G_\mathcal{P}V, EM)_\mathcal{P} = 0 \); whence the result. Another proof would consist in using the interpretation of \( H^2_\mathcal{P} \) in terms of extension, given in 3.4.

**Consequence.** For any \( p \geq 0 \), we have \( K^{p,1} = 0 \) and hence \( E^{0,1}_2 = 0 \). Thus \( \text{Im } d_2 = 0 \) and so \( H^2(B, EM)_\mathcal{P} \simeq H^0_\mathcal{P}(B, M) \).

The end of the theorem follows from general properties of spectral sequences: there is always an epimorphism from \( H^p(B, EM)_\mathcal{P} \) to \( H^p_\mathcal{P}(B, M) \). □

**Remark 7.8.** In [1], the author has shown that for \( n = 2,3 \) the groups \( H^n_\mathcal{P}(A, A) \) are adapted to the study of deformations of the \( \mathcal{P} \)-algebra \( A \) (in the sense of Gerstenhaber [8]). In [5], Fox also showed that for \( n = 1, 2 \) the groups \( H^n(V, \text{Der}(-, V))_G \) are adapted to the study of deformations of the \( T \)-algebra \( V \). If we take \( T = T_\mathcal{P} \) the two points of view agree since we have Theorem 7.7.
Remark 7.9. If $\mathcal{P}$ is a Koszul operad (see [11] for more details) then, for any $n \in \mathbb{N}$, we have

$$H^n(B,E_k)_{G^A} \simeq H^{n+1}_\mathcal{P}(B,k)$$

$$H_n(B,D_k)_{G^A} \simeq H^{n+1}_\mathcal{P}(B,k),$$

because in this case, for any $p$ we have $K^{p,q} = 0$. We do not know if this result remains true in the general case, i.e. when $M$ and $N$ are not equal to $k$ (although we believe it is true, we do not have any solid proof).

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References