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Monomial Hopf algebras ☆

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Abstract

Let K be a field of characteristic 0 containing all roots of unity. We classified all the Hopf structures on monomial K-coalgebras, or, in dual version, on monomial K-algebras.

Keywords: Hopf structures; Monomial coalgebras

Introduction

In the representation theory of algebras, one uses quivers and relations to construct algebras, and the resulted algebras are elementary, see Auslander, Reiten, and Smalø [1] and Ringel [15]. The construction of a path algebra has been dualized by Chin and Montgomery [4] to get a path coalgebra. It is then natural to consider subcoalgebras of a path coalgebra, which are all pointed.

There are also several works to construct neither commutative nor cocommutative Hopf algebras via quivers (see, e.g., [5–7,9]). An advantage for this construction is that a natural basis consisting of paths is available, and one can relate the properties of a quiver to the ones of the corresponding Hopf structures.

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In [5] Cibils determined all the graded Hopf structures (with length grading) on the path algebra KZ_n^a of basic cycle Z_n of length n; in [6], Cibils and Rosso studied graded Hopf structures on path algebras; in [9] E. Green and Solberg studied Hopf structures on some special quadratic quotients of path algebras. More recently, Cibils and Rosso [7] introduced the notion of the Hopf quiver of a group with ramification, and then classified all the graded Hopf algebras with length grading on path coalgebras. It turns out that a path coalgebra KQ^c admits a graded Hopf structure (with length grading) if and only if Q is a Hopf quiver (here a Hopf quiver is not necessarily finite).

The cited works above stimulate us to look for finite-dimensional Hopf algebra structures, on more quotients of path algebras, or in dual version, on more subcoalgebras of path coalgebras.

The aim of this paper is to classify all the Hopf algebra structures on a monomial algebra, or equivalently, on a monomial coalgebra.

Since a finite-dimensional Hopf algebra is both Frobenius and coFrobenius, we first look at the structure of monomial Frobenius algebras, or dually, the one of monomial coFrobenius coalgebras. It turns out that each indecomposable coalgebra component of a non-semisimple monomial coFrobenius coalgebra is $C_d(n)$ with $d \geqslant 2$, where $C_d(n)$ is the subcoalgebra of path coalgebra KZ_n^c with basis the set of paths of length strictly smaller than d. See Section 2.

Then by a theorem of Montgomery (Theorem 3.2 in [13]), a non-semisimple monomial Hopf algebra C is a crossed product of a Hopf structure on $C_d(n)$ with a group algebra. Thus, we turn to study the Hopf structures on $C_d(n)$ with $d \geqslant 2$. It turns out that the coalgebra $C_d(n)$, $d \geqslant 2$, admits a Hopf structure if and only if $d \mid n$ (Theorem 3.1). Moreover, when q runs over primitive dth roots of unity, the generalized Taft algebras $A_{n,d}(q)$ gives all the isoclasses of graded Hopf structures on $C_d(n)$ with length grading; while the Hopf structures (not necessarily graded with length grading) on $C_d(n)$ are exactly the algebras denoted by $A(n,d,\mu,q)$, with q a primitive dth root of unity and $\mu \in K$. These algebras $A(n,d,\mu,q)$ have been studied by Radford [14], Andruskiewitsch and Schneider [2]. See Theorem 3.6.

Note that algebra $A(n,d,\mu,q)$ is given by generators and relations. In Section 4, we prove that $A(n,d,\mu,q)$ is the product of KZ_d^a/J^d and $n/d-1$ copies of matrix algebra $M_d(K)$ when $\mu \neq 0$, and the product of n/d copies of KZ_d^a/J^d when $\mu = 0$, see Theorem 4.3. Hence the Gabriel quiver and the Auslander–Reiten quiver of $A(n,d,\mu,q)$ are known.

Finally, we introduce the notion of a group datum. By using the quiver construction of $C_d(n)$, the Hopf structure on it, and Montgomery's theorem (Theorem 3.2 in [13]), we get a one to one correspondence of Galois type between the set of the isoclasses of non-semisimple monomial Hopf K-algebras and the isoclasses of group data over K. This gives a classification of monomial Hopf algebras.

1. Preliminaries

Throughout this paper, K denotes a field of characteristic 0 containing all roots of unity. By an algebra we mean a finite-dimensional associative K-algebra with identity element.

Quivers considered here are always finite. Given a quiver $Q = (Q_0, Q_1)$ with Q_0 the set of vertices and Q_1 the set of arrows, denote by KQ, KQ^a, and KQ^c, the K-space with basis the set of all paths in Q, the path algebra of Q, and the path coalgebra of Q, respectively. Note that they are all graded with respect to length grading. For $\alpha \in Q_1$, let $s(\alpha)$ and $t(\alpha)$ denote respectively the starting and ending vertex of α.

Recall that the comultiplication of the path coalgebra KQ^c is defined by (see [4])

$$\Delta(p) = \sum_{\beta\alpha = p} \beta \otimes \alpha = \alpha_l \cdots \alpha_1 \otimes s(\alpha_1) + \sum_{i=1}^{l-1} \alpha_l \cdots \alpha_{i+1} \otimes \alpha_i \cdots \alpha_1 + t(\alpha_l) \otimes \alpha_l \cdots \alpha_1$$

for each path $p = \alpha_l \cdots \alpha_1$ with each $\alpha_i \in Q_1$; and $\varepsilon(p) = 0$ if $l \geqslant 1$, and 1 if $l = 0$. This is a pointed coalgebra.

Let C be a coalgebra. The set of group-like elements is defined to be

$$G(C) := \{c \in C \mid \Delta(c) = c \otimes c,\ c \neq 0\}.$$

It is clear $\varepsilon(c) = 1$ for $c \in G(C)$. For $x, y \in G(C)$, denote by

$$P_{x,y}(C) := \{c \in C \mid \Delta(c) = c \otimes x + y \otimes c\},$$

the set of x, y-primitive elements in C. It is clear that $\varepsilon(c) = 0$ for $c \in P_{x,y}(C)$. Note that $K(x - y) \subseteq P_{x,y}(C)$. An element $c \in P_{x,y}(C)$ is non-trivial if $c \notin K(x - y)$. Note that $G(KQ^c) = Q_0$; and

Lemma 1.1. *For $x, y \in Q_0$, we have*

$$P_{x,y}(KQ^c) = y(KQ_1)x \oplus K(x - y)$$

where $y(KQ_1)x$ denotes the K-space spanned by all arrows from x to y. In particular, there is a non-trivial x, y-primitive element in KQ^c if and only if there is an arrow from x to y in Q.

An ideal I of KQ^a is admissible if $J^N \subseteq I \subseteq J^2$ for some positive integer $N \geqslant 2$, where J is the ideal generated by all arrows.

An algebra A is elementary if $A/R \cong K^n$ as algebras for some n, where R is the Jacobson radical of A. For an elementary algebra A, there is a (unique) quiver Q, and an admissible ideal I of KQ^a, such that $A \cong KQ^a/I$. See [1,15].

An algebra A is monomial if there exists an admissible ideal I generated by some paths in Q such that $A \cong KQ^a/I$. Dually, we have

Definition 1.2. A subcoalgebra C of KQ^c is called monomial provided that the following conditions are satisfied:

(i) C contains all vertices and arrows in Q;

(ii) C is contained in subcoalgebra $C_d(Q) := \bigoplus_{i=0}^{d-1} KQ(i)$ for some $d \geqslant 2$, where $Q(i)$ is the set of all paths of length i in Q;
(iii) C has a basis consisting of paths.

It is clear by definition that both monomial algebras and monomial coalgebras are finite-dimensional; and A is a monomial algebra if and only if the linear dual A^* is a monomial coalgebra.

In the following, for convenience, we will frequently pass from a monomial algebra to a monomial coalgebra by duality. For this we will use the following:

Lemma 1.3. *The path algebra KQ^a is exactly the graded dual of the path coalgebra KQ^c, i.e.,*

$$KQ^a \cong \left(KQ^c\right)^{\mathrm{gr}};$$

and for each $d \geqslant 2$ there is a graded algebra isomorphism:

$$KQ^a/J^d \cong \left(C_d(Q)\right)^*.$$

1.4. Let $q \in K$ be an nth root of unity. For non-negative integers l and m, the Gaussian binomial coefficient is defined to be

$$\binom{m+l}{l}_q := \frac{(l+m)!_q}{l!_q m!_q}$$

where

$$l!_q := 1_q \cdots l_q, \qquad 0!_q := 1, \qquad l_q := 1 + q + \cdots + q^{l-1}.$$

Observe that $\binom{d}{l}_q = 0$ for $1 \leqslant l \leqslant d-1$ if the order of q is d.

1.5. Denote by Z_n the basic cycle of length n, i.e., an oriented graph with n vertices $e_0, \ldots, e_{n-1}$, and a unique arrow α_i from e_i to e_{i+1} for each $0 \leqslant i \leqslant n-1$. Take the indices modulo n. Denote by p_i^l the path in Z_n of length l starting at e_i. Thus we have $p_i^0 = e_i$ and $p_i^1 = \alpha_i$.

For each nth root $q \in K$ of unity, Cibils and Rosso [7] have defined a graded Hopf algebra structure $KZ_n(q)$ (with length grading) on the path coalgebra KZ_n^c by

$$p_i^l \cdot p_j^m = q^{jl} \binom{m+l}{l}_q p_{i+j}^{l+m},$$

with antipode S mapping p_i^l to $(-1)^l q^{-\frac{l(l+1)}{2} - il} p_{n-l-i}^l$.

1.6. In the following, denote $C_d(Z_n)$ by $C_d(n)$. That is, $C_d(n)$ is the subcoalgebra of KZ_n^c with basis the set of all paths of length strictly less than d.

Since $\binom{m+l}{l}_q = 0$ for $m \leqslant d-1$, $l \leqslant d-1$, $l+m \geqslant d$, it follows that if the order of q is d then $C_d(n)$ is a subHopfalgebra of $KZ_n(q)$. Denote this graded Hopf structure on $C_d(n)$ by $C_d(n,q)$.

Let d be the order of q. Recall that by definition $A_{n,d}(q)$ is an associative algebra generated by elements g and x, with relations

$$g^n = 1, \qquad x^d = 0, \qquad xg = qgx.$$

Then $A_{n,d}(q)$ is a Hopf algebra with comultiplication Δ, counit ε, and antipode S given by

$$\Delta(g) = g \otimes g, \qquad \varepsilon(g) = 1,$$
$$\Delta(x) = x \otimes 1 + g \otimes x, \qquad \varepsilon(x) = 0,$$
$$S(g) = g^{-1} = g^{n-1}, \qquad S(x) = -xg^{-1} = -q^{-1}g^{n-1}x.$$

In particular, if q is an nth primitive root of unity (i.e., $d = n$), then $A_{n,d}(q)$ is the n^2-dimensional Hopf algebra introduced by Taft [17]. For this reason $A_{n,d}(q)$ is called a generalized Taft algebra in [10].

Observe that $C_d(n,q)$ is generated by e_1 and α_0 as an algebra. Mapping g to e_1 and x to α_0, we get a Hopf algebra isomorphism

$$A_{n,d}(q) \cong C_d(n,q).$$

1.7. Let $q \in K$ be an nth root of unity of order d. For each $\mu \in K$, define a Hopf structure $C_d(n,\mu,q)$ on coalgebra $C_d(n)$ by

$$p_i^l \cdot p_j^m = q^{jl} \binom{m+l}{l}_q p_{i+j}^{l+m}, \quad \text{if } l+m < d,$$

and

$$p_i^l \cdot p_j^m = \mu q^{jl} \frac{(l+m-d)!_q}{l!_q m!_q} \left(p_{i+j}^{l+m-d} - p_{i+j+d}^{l+m-d}\right), \quad \text{if } l+m \geqslant d,$$

with antipode

$$S\left(p_i^l\right) = (-1)^l q^{-\frac{l(l+1)}{2} - il} p_{n-l-i}^l,$$

where $0 \leqslant l, m \leqslant d-1$, and $0 \leqslant i, j \leqslant n-1$. This is indeed a Hopf algebra with identity element $p_0^0 = e_0$ and of dimension nd. Note that this is in general not graded with respect to the length grading; and that

$$C_d(n,0,q) = C_d(n,q).$$

In [14] and [2] Radford and Andruskiewitsch–Schneider have considered the following Hopf algebra $A(n,d,\mu,q)$, which as an associative algebra is generated by two elements g and x with relations

$$g^n = 1, \qquad x^d = \mu\big(1 - g^d\big), \qquad xg = qgx,$$

with comultiplication Δ, counit ε, and the antipode S given as in 1.6.

It is clear that

$$A(n,d,0,q) = A_{n,d}(q);$$

and if $d = n$ then $A(n,d,\mu,q)$ is the n^2-dimensional Taft algebra.

Observe that $C_d(n,q,\mu)$ is generated by e_1 and α_0. By sending g to e_1 and x to α_0 we obtain a Hopf algebra isomorphism

$$A(n,d,\mu,q) \cong C_d(n,\mu,q).$$

2. Monomial Frobenius algebras and coFrobenius coalgebras

The aim of this section is to determine the form of monomial Frobenius, or dually, monomial coFrobenius coalgebras, for later application. This is well-known, but it seems that there are no exact references.

Let A be a monomial algebra. Thus, $A \cong KQ^a/I$ for a finite quiver Q, where I is an admissible ideal generated by some paths of lengths $\geqslant 2$. For $p \in KQ^a$, let $\bar{p}$ be the image of p in A. Then the finite set

$$\{\bar{p} \in A \mid p \text{ does not belong to } I\}$$

forms a basis of A. It is easy to see the following

Lemma 2.1. *Let A be a monomial algebra. Then*

(i) *The K-dimension of* $\mathrm{soc}(Ae_i)$ *is the number of the maximal paths starting at vertex i, which do not belong to I.*
(ii) *The K-dimension of* $\mathrm{soc}(e_iA)$ *is the number of the maximal paths ending at vertex i, which do not belong to I.*

Lemma 2.2. *Let A be an indecomposable, monomial algebra. Then A is Frobenius if and only if $A = k$, or $A \cong KZ_n^a/J^d$ for some positive integers n and d, with $d \geqslant 2$.*

Proof. The sufficiency is straightforward.

If A is Frobenius (i.e., there is an isomorphism $A \cong A^*$ as left A-modules, or equivalently, as right A-modules), then the socle of an indecomposable projective left A-module is simple (see, e.g., [8]). It follows from Lemma 2.1 that there is at most one arrow starting at each vertex i. Replacing "left" by "right" we observe that there is at most one arrow ending at each vertex i.

On the other hand, the quiver of an indecomposable Frobenius algebra is a single vertex, or has no sources and sinks (a source is a vertex at which there are no arrows ending; similarly for a sink), see, e.g., [8]. It follows that if $A \neq k$ then the quiver of A is a basic cycle Z_n for some n. However it is well-known that an algebra KZ_n^a/I with I admissible is Frobenius if and only if $I = J^d$ for some $d \geqslant 2$. □

The dual version of Lemma 2.2 gives the following:

Lemma 2.3. *Let A be an indecomposable, monomial coalgebra. Then A is coFrobenius (i.e., A^* is Frobenius) if and only if $A = k$, or $A \cong C_d(n)$ for some positive integers n and d, with $d \geqslant 2$.*

An algebra A is called Nakayama, if each indecomposable projective left and right module has a unique composition series. It is well known that an indecomposable elementary algebra is Nakayama if and only if its quiver is a basic cycle or a linear quiver A_n (see [8]). Note that a finite-dimensional Hopf algebra is Frobenius and coFrobenius (see, e.g., [12, p. 18]).

Corollary 2.4. *An algebra is a monomial Frobenius algebra if and only if it is elementary Nakayama Frobenius. Hence, a Hopf algebra is monomial if and only if it is elementary and Nakayama.*

3. Hopf structures on coalgebra $C_d(n)$

The aim of this section is to give a numerical description such that coalgebra $C_d(n)$ admits Hopf structures (Theorem 3.1), and then classify all the (graded, or not necessarily graded) Hopf structures on $C_d(n)$ (Theorem 3.6).

Theorem 3.1. *Let K be a field of characteristic 0, containing an nth primitive root of unity. Let $d \geqslant 2$ be a positive integer. Then coalgebra $C_d(n)$ admits a Hopf algebra structure if and only if $d \mid n$.*

The sufficiency follows from 1.6, or 1.7. In order to prove the necessity we need some preparations.

Lemma 3.2. *Suppose that the coalgebra $C_d(n)$ admits a Hopf algebra structure. Then*

(i) *The set $\{e_0, \dots, e_{n-1}\}$ of the vertices in $C_d(n)$ forms a cyclic group, say, with identity element $1 = e_0$. Then e_1 is a generator of the group.*
(ii) *Set $g := e_1$. Then up to a Hopf algebra isomorphism we have for any i such that $0 \leqslant i \leqslant n-1$*

$$\alpha_i \cdot g = q\alpha_{i+1} + \kappa_{i+1}\left(g^{i+1} - g^{i+2}\right)$$

and

$$g \cdot \alpha_i = \alpha_{i+1} + \lambda_{i+1}\big(g^{i+1} - g^{i+2}\big),$$

where $q, \lambda_i, \kappa_i \in K$*, with* $q^n = 1$.

Proof. (i) Since $C_d(n)$ is a Hopf algebra, it follows that $G(C_d(n)) = \{e_0, \ldots, e_{n-1}\}$ is a group, say with identity element e_0. Since α_0 is a non-trivial e_0, e_1-primitive element, it follows that $\alpha_0 e_1$ is a non-trivial e_1, e_1^2-primitive element, i.e., there is an arrow in $C_d(n)$ from e_1 to e_1^2. Thus $e_1^2 = e_2$. A similar argument shows that $e_i = e_1^i$ for any i.

(ii) Since both $\alpha_i g$ and $g\alpha_i$ are non-trivial g^{i+1}, g^{i+2}-primitive elements, it follows that

$$\alpha_i \cdot g = w_{i+1}\alpha_{i+1} + \kappa_{i+1}\big(g^{i+1} - g^{i+2}\big)$$

and

$$g \cdot \alpha_i = y_{i+1}\alpha_{i+1} + \lambda'_{i+1}\big(g^{i+1} - g^{i+2}\big)$$

with $w_i, \kappa_i, y_i, \lambda'_i \in K$.

Since $g^n \cdot \alpha_0 = \alpha_0$, it follows that $y_1 \cdots y_n = 1$. Set $\theta_j := y_{j+1} \cdots y_n$, $1 \leqslant j \leqslant n-1$, and $\theta_n := 1$. Define a linear isomorphism $\Theta : C_d(n) \to C_d(n)$ by

$$p_i^l \mapsto (\theta_i \cdots \theta_{i+l-1}) p_i^l.$$

In particular $\Theta(e_i) = e_i$ and $\Theta(\alpha_i) = \theta_i\alpha_i$. Then $\Theta : C_d(n) \to C_d(n)$ is a coalgebra map. Endow $C_d(n) = \Theta(C_d(n))$ with the Hopf algebra structure via the given Hopf algebra structure of $C_d(n)$ and Θ. Then in $\Theta(C_d(n))$ we have

$$\begin{aligned} g \cdot (\theta_i\alpha_i) &= \Theta(g) \cdot \Theta(\alpha_i) = \Theta(g \cdot \alpha_i) \\ &= y_{i+1}\Theta(\alpha_{i+1}) + \lambda'_{i+1}\big(g^{i+1} - g^{i+2}\big) \\ &= y_{i+1}\theta_{i+1}\alpha_{i+1} + \lambda'_{i+1}\big(g^{i+1} - g^{i+2}\big). \end{aligned}$$

Since $\theta_i = y_{i+1}\theta_{i+1}$, it follows that in $\Theta(C_d)$ we have

$$g \cdot \alpha_i = \alpha_{i+1} + \lambda_{i+1}\big(g^{i+1} - g^{i+2}\big)$$

(with $\lambda_{i+1} = \lambda'_{i+1}/\theta_i$). Assume that now in $\Theta(C_d(n))$ we have

$$\alpha_i \cdot g = q_{i+1}\alpha_{i+1} + \kappa_{i+1}\big(g^{i+1} - g^{i+2}\big).$$

Since $\alpha_0 g^n = \alpha_0$, it follows that $q_1 \cdots q_n = 1$. However, $(g \cdot \alpha_i) \cdot g = g \cdot (\alpha_i \cdot g)$ implies $q_i = q_{i+1}$ for each i. Write $q_i = q$. Then $q^n = 1$. This completes the proof. $\square$

Lemma 3.3. *Suppose that there is a Hopf algebra structure on $C_d(n)$. Then up to a Hopf algebra isomorphism we have*

$$p_i^l \cdot p_j^m \equiv q^{jl} \binom{m+l}{l}_q p_{i+j}^{l+m} \quad \bigl(\operatorname{mod} C_{l+m}(n)\bigr)$$

for $0 \leqslant i, j \leqslant n-1$, and for $l, m \leqslant d-1$, where $q \in K$ is an nth root of unity.

Proof. Use induction on $N := l+m$. For $N = 0$ or 1, the formula follows from Lemma 3.2. Assume that the formula holds for $N \leqslant N_0 - 1$. Then for $N = N_0 \geqslant 1$ we have

$$\begin{aligned}
\Delta\bigl(p_i^l \cdot p_j^m\bigr) &= \Delta\bigl(p_i^l\bigr) \cdot \Delta\bigl(p_j^m\bigr) \\
&= \left(\sum_{r=0}^{l} p_{i+r}^{l-r} \otimes p_i^r\right) \cdot \left(\sum_{s=0}^{m} p_{j+s}^{m-s} \otimes p_j^s\right) \\
&= \sum_{k=0}^{N_0} \sum_{r+s=k, 0\leqslant r\leqslant l, 0\leqslant s\leqslant m} p_{i+r}^{l-r} \cdot p_{j+s}^{m-s} \otimes p_i^r \cdot p_j^s \\
&= p_i^l \cdot p_j^m \otimes g^{i+j} + g^{i+j+N_0} \otimes p_i^l \cdot p_j^m \\
&\quad + \sum_{k=1}^{N_0-1} \sum_{r+s=k, 0\leqslant r\leqslant l, 0\leqslant s\leqslant m} p_{i+r}^{l-r} \cdot p_{j+s}^{m-s} \otimes p_i^r \cdot p_j^s.
\end{aligned}$$

By the induction hypothesis for each r and s with $1 \leqslant k := r+s \leqslant N_0 - 1$ we have

$$p_i^r \cdot p_j^s \equiv q^{jr} \binom{k}{r}_q p_{i+j}^k \quad \bigl(\operatorname{mod} C_k(n)\bigr)$$

and

$$p_{i+r}^{l-r} \cdot p_{j+s}^{m-s} \equiv q^{(j+s)(l-r)} \binom{N_0-k}{l-r}_q p_{i+j+k}^{N_0-k} \quad \bigl(\operatorname{mod} C_{N_0-k}(n)\bigr).$$

It follows that

$$\begin{aligned}
\Delta\bigl(p_i^l \cdot p_j^m\bigr) &\equiv p_i^l \cdot p_j^m \otimes g^{i+j} + g^{i+j+N_0} \otimes p_i^l \cdot p_j^m + \Sigma \\
&\quad \left(\operatorname{mod} \bigoplus_{1\leqslant k\leqslant N_0-1} C_{N_0-k}(n) \otimes C_k(n)\right)
\end{aligned}$$

where

$$\begin{aligned}\Sigma &= q^{jl} \sum_{k=1}^{N_0-1} \sum_{r+s=k,0\leqslant r\leqslant l,0\leqslant s\leqslant m} q^{sl-sr} \binom{k}{r}_q \binom{N_0-k}{l-r}_q p_{i+j+k}^{N_0-k} \otimes p_{i+j}^k \\ &= q^{jl} \sum_{k=1}^{N_0-1} \binom{N_0}{l}_q p_{i+j+k}^{N_0-k} \otimes p_{i+j}^k.\end{aligned}$$

Note that in the last equality the following identity has been used (see, e.g., Proposition IV.2.3 in [11]):

$$\sum_{r+s=k} q^{sl-sr} \binom{k}{r}_q \binom{N_0-k}{l-r}_q = \binom{N_0}{l}_q, \quad 0 < k < N_0.$$

Now, put $X := p_i^l p_j^m - q^{jl} \binom{N_0}{l}_q p_{i+j}^{N_0}$. Then by the computation above we have

$$\Delta(X) \equiv X \otimes g^{i+j} + g^{i+j+N_0} \otimes X \quad \Bigl(\operatorname{mod} \bigoplus_{1\leqslant k\leqslant N_0-1} C_{N_0-k}(n) \otimes C_k(n)\Bigr).$$

Let $X = \sum_{v\geqslant 0} c_v$, where c_v is the vth homogeneous component with respect to the length grading. Then we have

$$\sum_v \Delta(c_v) \equiv \sum_v \bigl(c_v \otimes g^{i+j} + g^{i+j+N_0} \otimes c_v\bigr) \quad \Bigl(\operatorname{mod} \bigoplus_{1\leqslant k\leqslant N_0-1} C_{N_0-k}(n) \otimes C_k(n)\Bigr).$$

Since the elements in $C_{N_0-k}(n) \otimes C_k(n)$ are of degrees strictly smaller than N_0, it follows that for $v \geqslant N_0$ we have

$$\Delta(c_v) = c_v \otimes g^{i+j} + g^{i+j+N_0} \otimes c_v.$$

Now for each $v \geqslant N_0 \geqslant 1$, note that in the right hand side of the above equality the terms are of degree $(v, 0)$ or $(0, v)$; but in the left hand side if $c_v \neq 0$ then it really contains a term of degree which is neither $(v, 0)$ nor $(0, v)$. This forces $c_v = 0$ for $v \geqslant N_0$. It follows that

$$p_i^l p_j^m = q^{jl} \binom{N_0}{l}_q p_{i+j}^{N_0} + X \equiv q^{jl} \binom{N_0}{l}_q p_{i+j}^{N_0} \quad \bigl(\operatorname{mod} C_{N_0}(n)\bigr).$$

This completes the proof. □

By a direct analysis from the definition of the Gaussian binomial coefficients we have

Lemma 3.4. *Let $1 \neq q \in K$ be an nth root of unity of order d. Then*

$$\binom{m+l}{l}_q = 0 \quad \text{if and only if} \quad \left[\frac{m+l}{d}\right] - \left[\frac{m}{d}\right] - \left[\frac{l}{d}\right] > 0,$$

where $[x]$ means the integer part of x.

3.5. Proof of Theorem 3.1

Assume that $C_d(n)$ admits a Hopf algebra structure. Let q be the nth root of unity as appeared in Lemma 3.3 with order d_0. It suffices to prove $d = d_0$. Since $C_d(n)$ has a basis p_i^l with $l \leqslant d-1$ and $0 \leqslant i \leqslant n-1$, it follows from Lemma 3.3 that

$$\binom{m+l}{l}_q = 0 \quad \text{for } l, m \leqslant d-1,\ l+m \geqslant d.$$

While by Lemma 3.4

$$\binom{m+l}{l}_q = 0 \quad \text{if and only if} \quad \left[\frac{m+l}{d_0}\right] - \left[\frac{m}{d_0}\right] - \left[\frac{l}{d_0}\right] > 0.$$

(Note that here we have used the assumption that K is of characteristic 0: since K is of characteristic zero, it follows that $\binom{m+l}{l}_1$ can never be zero. Thus $q \neq 1$, and then Lemma 3.4 can be applied.)

Take $l = 1$ and $m = d-1$. Then we have $[d/d_0] - [(d-1)/d_0] > 0$. This means $d_0 \mid d$. Let $d = kd_0$ with k a positive integer. If $k > 1$, then by taking $l = d_0$ and $m = (k-1)d_0$ we get a desired contradiction $\binom{l+m}{l}_q \neq 0$.

Theorem 3.6. *Assume that K is a field of characteristic* 0*, containing an nth primitive root of unity. Let $d \mid n$ with $d \geqslant 2$. Then*

- (i) *Any graded Hopf structure* (*with length grading*) *on $C_d(n)$ is isomorphic to* (*as a Hopf algebra*) *some $C_d(n,q) \cong A_{n,d}(q)$, where $C_d(n,q)$ and $A_{n,d}(q)$ are given as in* 1.6.
- (ii) *Any Hopf structure* (*not necessarily graded*) *on $C_d(n)$ is isomorphic to* (*as a Hopf algebra*) *some $C_d(n,\mu,q) \cong A(n,d,\mu,q)$, where $C_d(n,\mu,q)$ and $A(n,d,\mu,q)$ are given as in* 1.7.
- (iii) *If $A(n_1,d_1,\mu_1,q_1) \simeq A(n_2,d_2,\mu_2,q_2)$ as Hopf algebras, then $n_1 = n_2$, $d_1 = d_2$, $q_1 = q_2$.*

If $d \neq n$, then $A(n,d,\mu_1,q) \simeq A(n,d,\mu_2,q)$ as Hopf algebras if and only if $\mu_1 = \delta^d \mu_2$ for some $0 \neq \delta \in K$, and $A(n,n,\mu_1,q) \simeq A(n,n,\mu_2,q)$ for any $\mu_1,\mu_2 \in K$. In particular, for each n, $C_d(n,q_1)$ is isomorphic to $C_d(n,q_2)$ if and only if $q_1 = q_2$.

Proof. (i) By Lemma 3.3 and by the proof of Theorem 3.1 we see that any graded Hopf algebra on $C_d(n)$ is isomorphic to $C_d(n,q)$ for some root q of unity of order d.

(ii) Assume that $C_d(n)$ is a Hopf algebra. By Lemma 3.2 we have

$$\alpha_0 \cdot e_1 = q e_1 \cdot \alpha_0 + \kappa\left(e_1 - e_1^2\right)$$

for some primitive dth root q. Set $X := \alpha_0 + \frac{\kappa}{q-1}(1-e_1)$. Then $Xe_1 = qe_1X$. Since $\Delta(X) = e_1 \otimes X + X \otimes 1$, it follows that

$$\Delta\big(X^d\big) = \big(\Delta(X)\big)^d = \sum_{i=0}^{d} \binom{d}{i}_q e_{d-i}X^i \otimes X^{d-i} = e_d \otimes X^d + X^d \otimes 1,$$

where in the last equality we have used the fact that

$$\binom{d}{i}_q = 0 \quad \text{for } 1 \leqslant i \leqslant d-1.$$

Since there is no non-trivial $1, e_d$-primitive element in $C_d(n)$, it follows that $X^d = \mu(1-e_1^d)$ for some $\mu \in K$. Hence we obtain an algebra map

$$F : A(n,d,\mu,q) \to C_d(n)$$

such that $F(g) = e_1$ and $F(x) = X$. Since $C_d(n)$ is generated by e_1 and α_0 by Lemma 3.3, it follows that F is surjective, and hence an algebra isomorphism by comparing the K-dimensions. It is clear that F is also a coalgebra map, hence a bialgebra isomorphism, which is certainly a Hopf isomorphism [16].

(iii) If $C_{d_1}(n_1,\mu_1,q_1) \cong C_{d_2}(n_2,\mu_2,q_2)$, then their groups of the group-like elements are isomorphic. Thus $n_1 = n_2$, and hence $d_1 = d_2$ by comparing the K-dimensions. The remaining assertions can be easily deduced. We omit the details. □

Remark 3.7. The following example shows that, the assumption "K is of characteristic 0" is really needed in Theorem 3.1.

Let K be a field of characteristic 2, and let $n \geqslant 2$ be an arbitrary integer. Then each graded Hopf algebra structure on $C_2(n)$ is given by (up to a Hopf algebra isomorphism):

$$g^j\alpha_i = \alpha_i g^j = \alpha_{i+j}, \qquad \alpha_i\alpha_j = 0,$$
$$S(\alpha_i) = \alpha_{n-i-1}, \qquad S\big(g^j\big) = g^{n-j}$$

for all $0 \leqslant i, j \leqslant n-1$.

(In fact, consider the Hopf algebra structure $KZ_n(1)$ on Z_n. Its subcoalgebra $C_2(n)$ is also a subalgebra, which is exactly the given Hopf algebra. On the other hand, for each graded Hopf algebra over $C_2(n)$, the corresponding q in Lemma 3.3 must satisfy $\binom{2}{1}_q = 1 + q = 0$, and hence $q = 1$. Then the assertion follows from Lemma 3.3.)

Remark 3.8. It is easy to determine the automorphism group of the Hopf algebra $A(n,d,\mu,q)$: it is $K - \{0\}$ if $\mu = 0$ or $d = n$, and Z_d otherwise.

4. The Gabriel quiver and the Auslander–Reiten quiver of $A(n, d, \mu, q)$

The aim of this section is to determine the Gabriel quiver and the Auslander–Reiten quiver of algebra $A(n, d, \mu, q) \cong C_d(n, \mu, q)$, where q is an nth root of unity of order d.

We start from the central idempotent decomposition of $A := A(n, d, \mu, q)$.

Lemma 4.1. *The center of A has a linear basis $\{1, g^d, g^{2d}, \dots, g^{n-d}\}$.*

Let $\omega \in K$ be a root of unity of order n/d. Then we have the central idempotent decomposition $1 = c_0 + c_1 + \cdots + c_t$ with $c_i = (d/n)\sum_{j=0}^{t}(\omega^i g^d)^j$ for all $0 \leqslant i \leqslant t$, where $t = n/d - 1$.

Proof. By 1.7 the dimension of A is nd, thus $\{g^i x^j \mid 0 \leqslant i \leqslant n-1, 0 \leqslant j \leqslant d-1\}$ is a basis of A. An element $c = \sum a_{ij} g^i x^j$ is in the center of A if and only if $xc = cx$ and $gc = cg$. By comparing the coefficients, we get $a_{ij} = 0$ unless $j = 0$ and $d \mid i$. Obviously, g^d is in the center. It follows that the center of A has a basis $\{1, g^d, g^{2d}, \dots, g^{n-d}\}$.

Since $\sum_{i=0}^{t}(\omega^j)^i = 0$ for each $1 \leqslant j \leqslant t$, it follows that

$$c_0 + c_1 + \cdots + c_t = \frac{d}{n}\sum_{j=0}^{t} g^{dj} \sum_{i=0}^{t}\bigl(\omega^j\bigr)^i = \frac{d}{n}\left(\sum_{i=0}^{t} 1 + \sum_{j=1}^{t} g^{dj}\right) = \frac{d}{n}(t+1) = 1;$$

and

$$\begin{aligned}
c_i c_i' &= \frac{d^2}{n^2}\sum_{0\leqslant j, j'\leqslant t} g^{d(j+j')}\omega^{ij+i'j'} \\
&= \frac{d^2}{n^2}\sum_{k=0}^{2t} g^{dk}\omega^{i'k} \sum_{0\leqslant j\leqslant \min\{k,t\}, 0\leqslant k-j\leqslant t} \omega^{(i-i')j} \\
&= \frac{d^2}{n^2}\left(\sum_{k=0}^{t} g^{dk}\omega^{i'k}\sum_{0\leqslant j\leqslant k}\omega^{(i-i')j} + \sum_{k=t+1}^{2t} g^{dk}\omega^{i'k}\sum_{k-t\leqslant j\leqslant t}\omega^{(i-i')j}\right) \\
&= \frac{d^2}{n^2}\left(\sum_{k=0}^{t} g^{dk}\omega^{i'k}\sum_{0\leqslant j\leqslant k}\omega^{(i-i')j} + \sum_{k'=0}^{t-1} g^{dk'}\omega^{i'k'}\sum_{1+k'\leqslant j\leqslant t}\omega^{(i-i')j}\right) \\
&= \frac{d^2}{n^2}\left(g^{dt}\omega^{i't}\sum_{0\leqslant j\leqslant t}\omega^{(i-i')j} + \sum_{k=0}^{t-1} g^{dk}\omega^{i'k}\sum_{0\leqslant j\leqslant t}\omega^{(i-i')j}\right) \\
&= \frac{d^2}{n^2}\left(g^{dt}\omega^{i't}\delta_{i,i'}(t+1) + \sum_{k=0}^{t-1} g^{dk}\omega^{i'k}\delta_{i,i'}(t+1)\right) \\
&= (t+1)\frac{d^2}{n^2}\delta_{i,i'}\sum_{k=0}^{t} g^{dk}\omega^{i'k} = \delta_{i,i'}c_i
\end{aligned}$$

where $\delta_{i,i'}$ is the Kronecker symbol. This completes the proof. □

Lemma 4.2. *Let $B = B(d, \lambda, q)$ be an algebra generated by g and x with relations $\{g^d = 1, x^d = \lambda, xg = qgx\}$, where $\lambda, q \in K$, and q is a root of unity of order d.*

(i) *If $\lambda = 0$, then $B \simeq KZ_d^a/J^d$.*
(ii) *If $\lambda \neq 0$, then $B \simeq M_d(K)$.*

Proof. (i) Note that if $\lambda = 0$, then $B \simeq A(d, d, 0, q) \cong C_d(d, 0, q)$, which is a d^2-dimensional Taft algebra. By the self-duality of the Taft algebras (see [5, Proposition 3.8]) we have algebra isomorphisms

$$B \cong A(d, d, 0, q) \simeq A(d, d, 0, q)^* \simeq C_d(d, 0, q)^* \simeq KZ_d^a/J^d.$$

(ii) If $\lambda \neq 0$, then define an algebra homomorphism $\phi : B \to M_d(K)$:

$$\phi(g) = \begin{pmatrix} 1 & & & & \\ & q & & & \\ & & q^2 & & \\ & & & \ddots & \\ & & & & q^{d-1} \end{pmatrix}$$

and

$$\phi(x) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ \lambda & & & & 0 \end{pmatrix}.$$

Note that ϕ is well-defined. It is easy to check that $\phi(g)$ and $\phi(x)$ generate the algebra $M_d(K)$. Thus ϕ is a surjective map. However, the dimension of B is at most d^2, thus ϕ is an algebra isomorphism. $\square$

Now we are ready to prove the main result of this section.

Theorem 4.3. *Write $A = A(n, d, \mu, q)$ and $t = n/d - 1$.*

(i) *If $\mu \neq 0$, then $A \simeq KZ_d^a/J^d \times M_d(K) \times \cdots \times M_d(K)$ (with t copies of $M_d(K)$).*
(ii) *If $\mu = 0$, then $A \simeq KZ_d^a/J^d \times KZ_d^a/J^d \times \cdots \times KZ_d^a/J^d$ (with n/d copies of KZ_d^a/J^d).*

Proof. By Lemma 4.1 we have $A \cong c_0A \times c_1A \times \cdots \times c_tA$ as algebras. Write $A_i = c_iA$. Note that $c_ig^d = \omega^{-i}c_i$ for all $0 \leqslant i \leqslant t$. It follows that $\{c_ig^kx^j \mid 0 \leqslant k \leqslant d-1, 0 \leqslant j \leqslant$

$d-1\}$ is a linear basis of A_i. Let $\omega_0 \in K$ be an nth primitive root of unity such that $\omega_0^d = \omega$. Obviously, as an algebra each A_i is generated by $\omega_0^i c_i g$ and $c_i x$, satisfying

$$\left(\omega_0^i c_i g\right)^d = c_i, \qquad (c_i x)^d = c_i \mu\left(1 - g^d\right) = c_i \mu\left(1 - \omega^{-i}\right)$$

and

$$(c_i x)\left(\omega_0^i c_i g\right) = q\left(\omega_0^i c_i g\right)(c_i x).$$

Note that c_i is the identity of A_i. Thus we have an algebra homomorphism

$$\theta_i : B\left(d, \mu\left(1 - \omega^{-i}\right), q\right) \to A_i$$

such that $\theta_i(g) = \omega_0^i c_i g$ and $\theta_i(x) = c_i x$. A simple dimension argument shows that θ_i is an algebra isomorphism. Note that $\mu(1 - \omega^{-i}) = 0$ if and only if $\mu = 0$ or $i = 0$. Then the assertion follows from Lemma 4.2. $\square$

Corollary 4.4. *The Gabriel quiver of algebra $A(n, d, \mu, q)$ is the disjoint union of a basic d-cycle and t isolated vertices if $\mu \neq 0$, and the disjoint union of n/d basic d-cycles if $\mu = 0$.*

Since the Auslander–Reiten quiver $\Gamma(KZ_d^a/J^d)$ is well-known (see, e.g., [1, p. 111]), it follows that the Auslander–Reiten quiver of $A(n, d, \mu, q)$ is clear.

5. Hopf structures on monomial algebras and coalgebras

The aim of is section is to classify non-semisimple monomial Hopf K-algebras, by establishing a one-to-one correspondence between the set of the isoclasses of non-semisimple monomial Hopf K-algebras and the isoclasses of group data over K.

Theorem 5.1.

(i) *Let A be a monomial algebra. Then A admits a Hopf algebra structure if and only if $A \cong k \times \cdots \times k$ as an algebra, or*

$$A \cong KZ_n^a/J^d \times \cdots \times KZ_n^a/J^d$$

as an algebra, for some $d \geqslant 2$ dividing n.

(ii) *Let C be a monomial coalgebra. Then C admits a Hopf algebra structure if and only if $C \cong k \oplus \cdots \oplus k$ as a coalgebra, or*

$$C \cong C_d(n) \oplus \cdots \oplus C_d(n)$$

as a coalgebra, for some $d \geqslant 2$ dividing n.

Proof. By duality it suffices to prove one of them. We prove (ii).

If $C = C_1 \oplus \cdots \oplus C_l$ as a coalgebra, where each $C_i \cong C_1$ as coalgebras, and C_1 admits Hopf structure H_1, then $H_1 \otimes KG$ is a Hopf structure on C, where G is any group of order l. This gives the sufficiency.

Let C be a monomial coalgebra admitting a Hopf structure. Since a finite-dimensional Hopf algebra is coFrobenius, it follows from Lemma 2.3 that as a coalgebra C has the form $C = C_1 \oplus \cdots \oplus C_l$ with each C_i indecomposable as coalgebra, and $C_i = k$ or $C_i = C_{d_i}(n_i)$ for some n_i and $d_i \geqslant 2$.

We claim that if there exists a $C_i = k$, then $C_j = k$ for all j. Thus, if $C \neq k \oplus \cdots \oplus k$, then C is of the form

$$C = C_{d_1}(n_1) \oplus \cdots \oplus C_{d_l}(n_l)$$

as a coalgebra, with each $d_i \geqslant 2$.

(Otherwise, let $C_j = C_d(n)$ for some j. Let α be an arrow in C_j from x to y. Let h be the unique group-like element in $C_i = k$. Since the set $G(C)$ of the group-like elements of C forms a group, it follows that there exists an element $k \in G(C)$ such that $h = kx$. Then $k\alpha$ is a h, ky-primitive element in C. But according to the coalgebra decomposition $C = C_1 \oplus \cdots \oplus C_l$ with $C_i = Kh$, C has no h, ky-primitive elements. A contradiction.)

Assume that the identity element 1 of $G(C)$ is contained in $C_1 = C_{d_1}(n)$. It follows from a theorem of Montgomery [13, Theorem 3.2] that C_1 is a subHopfalgebra of C, and that

$$g_i^{-1} C_{d_i}(n_i) = C_{d_i}(n_i) g_i^{-1} = C_{d_1}(n_1)$$

for any $g_i \in G(C_{d_i}(n_i))$ and for each i. By comparing the numbers of group-like elements in $g_i^{-1} C_{d_i}(n_i)$ and in $C_{d_1}(n_1)$ we have $n_i = n_1 = n$ for each i. While by comparing the K-dimensions we see that $d_i = d_1 = d$ for each i. Now, since $C_1 = C_d(n)$ is a Hopf algebra, it follows from Theorem 3.1 that d divides n. $\square$

5.2. For convenience, we call a Hopf structure on a monomial coalgebra C a monomial Hopf algebra. Note that a monomial Hopf algebra is not necessarily graded with length grading, by Lemma (iii) below.

Lemma. *Let C be a non-semisimple, monomial Hopf algebra.*

(i) *Let C_1 be the indecomposable coalgebra component containing the identity element* 1*. Then $G(C_1)$ is a cyclic group contained in the center of $G(C)$.*
(ii) *There exists a unique element $g \in C$ such that there is a non-trivial $1, g$-primitive element in C. The element g is a generator of $G(C_1)$.*
(iii) *As an algebra, C is generated by $G(C)$ and a non-trivial $1, g$-primitive element x, satisfying*

$$x^d = \mu\big(g^d - 1\big)$$

for some $\mu \in K$, where $d = \dim_K C_1 / o(g)$, $o(g)$ is the order of g.

(iv) *There exists a one-dimensional K-representation χ of G such that*

$$x \cdot h = \chi(h)h \cdot x, \quad \forall h \in G,$$

and $\mu = 0$ if $o(g) = d$ (note that $d = o(\chi(g))$); and $\chi^d = 1$ if $\mu \neq 0$ and $g^d \neq 1$.

Proof. (i) Note that C_1 is a subHopfalgebra of C by Theorem 3.2 in [13]. By Theorem 5.1(ii) we have $C_1 \cong C_d(n)$ as a coalgebra. It follows from Lemma 3.3 that $G(C_1)$ is a cyclic group. By Theorem 5.1(ii) we can identify each indecomposable coalgebra component C_i of C with $C_d(n)$. For any $h \in G(C)$ with $h \in C_i$, note that $h\alpha_0$ is a non-trivial h, he_1-primitive element in C_i, and $\alpha_0 h$ is a non-trivial h, e_1h-primitive element in C_i. This implies that there is an arrow in $C_i = C_d(n)$ from h to he_1, and that there is an arrow in C_i from h to e_1h. Thus by the structure of a basic cycle we have $he_1 = e_1h$. While e_1 is a generator of $G(C_1)$. Thus, $G(C_1)$ is contained in the center of $G(C)$.

(ii) One can see this assertion from Theorem 5.1(ii) by identifying C_1 with $C_d(n)$, and the claimed g is exactly e_1 in $C_d(n)$.

(iii) By Theorem 3.2 in [13], as an algebra, C is generated by C_1 and $G(C)$. By the proof of Theorem 3.1(ii) C_1 is generated by $g = e_1$ and a non-trivial $1, e_1$-primitive element x, satisfying the given relation, together with

$$xe_1 = qe_1x$$

with q a primitive dth root of unity.

(iv) For any $h \in G$, since both $x \cdot h$ and $h \cdot x$ are non-trivial h, gh-primitive elements in C (note $gh = hg$), it follows that there exists K-functions χ and χ' on G such that

$$x \cdot h = \chi(h)h \cdot x + \chi'(h)(1 - g)h.$$

We claim that χ is a one-dimensional representation of G and $\chi' = 0$.

By $x \cdot (h_1 \cdot h_2) = (x \cdot h_1) \cdot h_2$, one infers that

$$\chi(h_1 \cdot h_2) = \chi(h_1)\chi(h_2)$$

and

$$\chi'(h_1 \cdot h_2) = \chi(h_1)\chi'(h_2) + \chi'(h_1).$$

Since $\chi(g) = q$ and $\chi'(g) = 0$, it follows that $\chi'(h \cdot g) = \chi'(h)$ for all $h \in G$. Thus, we have

$$\chi'(h) = \chi'(h \cdot g) = \chi'(g \cdot h) = \chi(g)\chi'(h),$$

which implies $\chi' = 0$.

Since $x^d = \mu(1 - g^d)$, it follows that one can make a choice such that $\mu = 0$ if $d = n$. By $x^d \cdot h = \chi^d(h)h \cdot x^d$ and $x^d = \mu(g^d - 1)$ we see $\chi^d = 1$ if $\mu \neq 0$ and $g^d \neq 1$. □

In order to classify non-semisimple monomial Hopf K-algebras, we introduce the notion of group data.

Definition 5.3. A group datum $\alpha = (G, g, \chi, \mu)$ over K consists of

(i) a finite group G, with an element g in its center;
(ii) a one-dimensional K-representation χ of G; and
(iii) an element $\mu \in K$, such that $\mu = 0$ if $o(g) = o(\chi(g))$, and that if $\mu \neq 0$ then $\chi^{o(\chi(g))} = 1$.

Definition 5.4. Two group data $\alpha = (G, g, \chi, \mu)$ and $\alpha' = (G', g', \chi', \mu')$ are said to be isomorphic, if there exist a group isomorphism $f : G \to G^{'}$ and some $0 \neq \delta \in K$ such that $f(g) = g'$, $\chi = \chi' f$ and $\mu = \delta^d \mu'$.

Lemma 5.2 permits us to introduce the following notion.

Definition 5.5. Let C be a non-semisimple monomial Hopf algebra. A group datum $\alpha = (G, g, \chi, \mu)$ is called an induced group datum of C provided that

(i) $G = G(C)$;
(ii) there exists a non-trivial $1, g$-primitive element x in C such that

$$x^d = \mu\bigl(1 - g^d\bigr), \qquad xh = \chi(h)hx, \quad \forall h \in G,$$

where d is the multiplicative order of $\chi(g)$.

For example, $(Z_n, \bar{1}, \chi, \mu)$ with $\chi(\bar{1}) = q$ is an induced group datum of $A(n, d, \mu, q)$ (as defined in 1.7).

Lemma 5.6.

(i) *Let C, C' be non-semisimple monomial Hopf algebras, $f : C \to C'$ a Hopf algebra isomorphism, and $\alpha = (G, g, \chi, \mu)$ an induced group datum of C. Then $f(\alpha) = (f(G), f(g), \chi f^{-1}, \mu)$ is an induced group datum of C'.*
(ii) *If $\alpha = (G, g, \chi, \mu)$ and $\beta = (G', g', \chi', \mu')$ both are induced group data of a non-semisimple monomial Hopf algebra C, then α is isomorphic to β.*

Thus, we have a map α from the set of the isoclasses of non-semisimple monomial Hopf K-algebras to the set of the isoclasses of group data over K, by assigning each non-semisimple monomial Hopf algebra C to its induced group datum $\alpha(C)$.

Proof. The assertion (i) is clear by definition.

(ii) By definition we have $G = G(C) = G'$. By definition there exists a non-trivial $1, g$-element x, and also a non-trivial $1, g'$-element x'. But according to Theorem 5.1(ii) such g and g' turn out to be unique, i.e., $g = g' = e_1$ if we identify C_1 with $C_d(n)$. And according to the coalgebra structure of C, and of $C_1 \cong C_d(n)$, we have

$$x = \delta x' + \kappa(1-g)$$

for some $\delta \neq 0, \kappa \in K$. It follows that

$$x \cdot h = \chi(h)h \cdot x = \chi(h)\delta h \cdot x' + \chi(h)\kappa h \cdot (1-g)$$

and

$$x \cdot h = \bigl(\delta x' + \kappa(1-g)\bigr) \cdot h = \delta\chi'(h)h \cdot x' + \kappa h \cdot (1-g)$$

and hence $\chi = \chi'$ and $\kappa = 0$. Thus

$$\mu\bigl(1-g^d\bigr) = x^d = (\delta x')^d = \delta^d \mu'\bigl(1-g^d\bigr),$$

i.e., $\mu = \delta^d \mu'$, which implies that α and β are isomorphic. □

5.7. For a group datum $\alpha = (G, g, \chi, \mu)$ over K, define $A(\alpha)$ to be an associative algebra with generators x and all $h \in G$, with relations

$$x^d = \mu\bigl(1-g^d\bigr), \qquad xh = \chi(h)hx, \quad \forall h \in G,$$

where $d = o(\chi(g))$. One can check that $\dim_K A(\alpha) = |G|d$ by Bergman's diamond lemma in [3] (here the condition "$\chi^d = 1$ if $\mu \neq 0$" is needed). Endow $A(\alpha)$ with comultiplication Δ, counit ε, and antipode S by

$$\Delta(x) = g \otimes x + x \otimes 1, \qquad \varepsilon(x) = 0,$$
$$\Delta(h) = h \otimes h, \qquad \varepsilon(h) = 1, \quad \forall h \in G,$$
$$S(x) = g^{-1}x, \qquad S(h) = h^{-1}, \quad \forall h \in G.$$

It is straightforward to verify that $A(\alpha)$ is indeed a Hopf algebra.

Lemma 5.8.

(i) *For each group datum $\alpha = (G, g, \chi, \mu)$ over K, $A(\alpha)$ is a non-semisimple monomial Hopf K-algebra, with the induced group datum α.*
(ii) *If α and β are isomorphic group data, then $A(\alpha)$ and $A(\beta)$ are isomorphic as Hopf algebras.*

Thus, we have a map A from the set of the isoclasses of group data over K to the set of the isoclasses of non-semisimple monomial Hopf K-algebras, by assigning each group datum α to $A(\alpha)$.

Proof. (i) Since $\dim_k A(\alpha) = |G|d$, it follows that $\{hx^i \mid h \in G, i \leqslant d\}$ is a basis for $A(\alpha)$. Let $\{a_1 = 1, \dots, a_l\}$ be a set of representatives of cosets of G respect to G_1. For each $1 \leqslant i \leqslant l$, let A_i be the K-span of the set $\{a_i g^j x^k \mid 0 \leqslant j \leqslant n-1, 0 \leqslant k \leqslant d-1\}$, where $n = |G_1|$. It is straightforward to verify that

$$A(\alpha) = A_1 \oplus \cdots \oplus A_l$$

as a coalgebra, and $A_i \cong A_j$ as coalgebras for all $1 \leqslant i, j \leqslant l$. Note that there is a coalgebra isomorphism $A_1 \cong C_d(n)$, by sending $g^i x^j$ to $(j!_q) p_i^j$, where p_i^j is the path starting at e_i and of length j. This proves that

$$A(\alpha) \cong C_d(n) \oplus \cdots \oplus C_d(n)$$

as coalgebras.

(ii) Let $\alpha = (G, g, \chi f, \delta^d \mu) \cong \beta = (f(G), f(g), \chi, \mu)$ with a group isomorphism $f : G \to G'$. Then $F : A(\alpha) \to A(\beta)$ given by $F(x) = \delta x'$, $F(h) = f(h)$, $h \in G$, is a surjective algebra map, and hence an isomorphism by comparing the K-dimensions. This is also a coalgebra map, and hence a Hopf algebra isomorphism. □

The following theorem gives a classification of non-semisimple, monomial Hopf K-algebras via group data over K.

Theorem 5.9. *The maps α and A above gives a one to one correspondence between sets*

$$\{\textit{the isoclasses of non-semisimple monomial Hopf } K\textit{-algebras}\}$$

and

$$\{\textit{the isoclasses of group data over } K\}.$$

Proof. By Lemmas 5.6 and 5.8, it remains to prove that $C \cong A(\alpha(C))$ as Hopf algebras, which are straightforward by constructions. □

5.10. A group datum $\alpha = (G, g, \chi, \mu)$ is said to be trivial, if $G = \langle g \rangle \times N$, and the restriction of χ to N is trivial.

Corollary. *Let $\alpha = (G, g, \chi, \mu)$ be a group datum over K. Then $A(\alpha)$ is isomorphic to $A(o(g), o(\chi(g)), \mu, \chi(g)) \otimes KN$ as Hopf algebras, if and only if α is trivial with $G = \langle g \rangle \times N$, where $A(o(g), o(\chi(g)), \mu, \chi(g))$ is as defined in* 1.7.

Proof. If α is trivial with $G = \langle g \rangle \times N$, then

$$\alpha\big(A\big(o(g), o\big(\chi(g)\big), \mu, \chi(g)\big) \otimes KN\big) = \alpha,$$

it follows from Theorem 5.9 that

$$A(\alpha) \cong A\big(o(g), o\big(\chi(g)\big), \mu, \chi(g)\big) \otimes KN.$$

Conversely, we then have

$$\alpha = \alpha\big(A(\alpha)\big) = \alpha\big(A\big(o(g), o\big(\chi(g)\big), \mu, \chi(g)\big) \otimes KN\big)$$

is trivial. □

Remark 5.11. It is easy to determine the automorphism group of $A(\alpha)$ with $\alpha = (G, g, \chi, \mu)$: it is $K^* \times \Gamma$ if $\mu = 0$, and $Z_d \times \Gamma$ if $\mu \neq 0$, where $\Gamma := \{f \in \mathrm{Aut}(G) \mid f(g) = g, \chi f = \chi\}$.

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References

[1] M. Auslander, I. Reiten, S.O. Smalø, Representation Theory of Artin Algebras, in: Cambridge Stud. Adv. Math, vol. 36, Cambridge Univ. Press, Cambridge, 1995.
[2] N. Andruskiewitsch, H.J. Schneider, Lifting of quantum linear spaces and pointed Hopf algebras of order p^3, J. Algebra 209 (1998) 658–691.
[3] G.M. Bergman, The diamond lemma for ring theory, Adv. Math. 29 (1978) 178–218.
[4] W. Chin, S. Montgomery, Basic coalgebras, in: Modular Interfaces (Riverside, CA, 1995), in: AMS/IP Stud. Adv. Math., vol. 4, Amer. Math. Soc., Providence, RI, 1997, pp. 41–47.
[5] C. Cibils, A quiver quantum group, Comm. Math. Phys. 157 (1993) 459–477.
[6] C. Cibils, M. Rosso, Algebres des chemins quantique, Adv. Math. 125 (1997) 171–199.
[7] C. Cibils, M. Rosso, Hopf quivers, J. Algebra 254 (2) (2002) 241–251.
[8] Yu.A. Drozd, V.V. Kirichenko, Finite Dimensional Algebras, Springer-Verlag, Berlin–Heidelberg–New York–Tokyo, 1993.
[9] E.L. Green, Ø. Solberg, Basic Hopf algebras and quantum groups, Math. Z. 229 (1998) 45–76.
[10] H.L. Huang, H.X. Chen, P. Zhang, Generalized Taft algebras, Algebra Colloq., in press.
[11] C. Kassel, Quantum Groups, in: Grad. Texts in Math., vol. 155, Springer-Verlag, New York, 1995.
[12] S. Montgomery, Hopf Algebras and Their Actions on Rings, in: CBMS Reg. Conf. Ser. Math., vol. 82, Amer. Math. Soc., Providence, RI, 1993.
[13] S. Montgomery, Indecomposable coalgebras, simple comodules and pointed Hopf algebras, Proc. Amer. Math. Soc. 123 (1995) 2343–2351.
[14] D.E. Radford, On the coradical of a finite-dimensional Hopf algebra, Proc. Amer. Math. Soc. 53 (1) (1975) 9–15.
[15] C.M. Ringel, Tame Algebras and Integral Quadratic Forms, in: Lecture Notes in Math., vol. 1099, Springer-Verlag, 1984.
[16] M.E. Sweedler, Hopf Algebras, Benjamin, New York, 1969.
[17] E.J. Taft, The order of the antipode of finite dimensional Hopf algebras, Proc. Natl. Acad. Sci. USA 68 (1971) 2631–2633.