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Banach frames in coorbit spaces consisting of elements which are invariant under symmetry groups

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Abstract

This paper is concerned with the construction of atomic decompositions and Banach frames for subspaces of certain Banach spaces consisting of elements which are invariant under some symmetry group. These Banach spaces—called coorbit spaces—are related to an integrable group representation. The construction is established via a generalization of the well-established Feichtinger–Gröchenig theory. Examples include radial wavelet-like atomic decompositions and frames for radial Besov–Triebel–Lizorkin spaces, as well as radial Gabor frames and atomic decompositions for radial modulation spaces.

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1. Introduction

The study of time–frequency analysis and wavelet analysis of functions on $\mathbb{R}^d$ that are invariant under a symmetry group was started in [17]. There the author raised the question whether it is possible to exploit

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the symmetry in order to reduce complexity, improve approximation quality, etc., in Gabor or wavelet analysis.

Imagine that a function f on $\mathbb{R}^d$, which has some symmetries, is represented by a Gabor or wavelet expansion. Then the functions (translates and dilates or modulations of a single function) in the expansion will not all (actually nearly none of them) obey the same symmetry properties as f. So one might ask whether it is possible to find a Gabor-like frame or wavelet-like frame (for the subspace of $L^2(\mathbb{R}^d)$ consisting of invariant functions) such that each frame element itself is invariant under the symmetry group.

In case of radial symmetry in $\mathbb{R}^d$, Epperson and Frazier successfully constructed radial wavelet frames which even serve as atomic decompositions for subspaces of Besov spaces and Triebel–Lizorkin spaces consisting of radial functions [5]. Kühn et al. used this radial atomic decomposition to establish results concerning compact embeddings of radial Besov spaces in [15]. In dimension 3 radial orthonormal wavelets were constructed in [19] using the concept of a multiresolution analysis. However, concerning radial Gabor frames there seems nothing to be known up to now.

Both wavelet theory and time–frequency analysis can be treated simultaneously using representation theory of locally compact groups. In this abstract setting the theory for the continuous transform in the presence of invariance under a general symmetry group was developed in [17]. The symmetry group is realized as compact automorphism group of the locally compact group whose representation coefficients generate the continuous transform. As examples, the continuous wavelet transform and the short time Fourier transform (STFT) of radial functions on $\mathbb{R}^d$ were discussed in detail. A radial function can be described by some function on the positive halfline $\mathbb{R}_+$ and it turned out in [17] that the continuous wavelet transform and the STFT of a radial function can be computed by an integral transform on $\mathbb{R}_+$, which involves a generalized translation in case of the wavelet transform and some kind of a generalized combined translation and modulation (formula (4.4) in [17]) in case of the STFT. Both of these "generalized operations" are given as integrals and in particular the generalized combined translation/modulation turns out to be quite complicated.

The (stable) discretization of the "radial wavelet transform" and the "radial STFT" actually means the construction of frames, where each frame element is given as some generalized translation or as some generalized translation/modulation of a single function. In order to attack the discretization problem, the first idea would probably be to proceed analogously to the classical wavelet and Gabor theory. And in fact, in case of radial wavelets in $\mathbb{R}^3$ this approach was successful [19]. However, in arbitrary dimension and for radial Gabor frames the direct approach seems hopeless because of the complicated form of the combined generalized translation/modulation. So one has to look for different approaches.

In the classical setting (i.e., without symmetry group) Feichtinger–Gröchenig theory has proven to provide a general and very flexible way to construct coherent atomic decompositions and Banach frames for certain Banach spaces, called coorbit spaces, which are related to the continuous transform [8–10,12]. This approach makes heavy use of group theory and, thus, is quite abstract. However, the final outcome is a very elegant solution to the discretization problem. In particular, regular and irregular Gabor and wavelet frames are included as examples. Moreover, not only Hilbert space theory is covered but also atomic decompositions and Banach frames of Besov–Triebel–Lizorkin spaces and of modulation spaces are provided. So it also provides a new aspect of the theory of function spaces.

Motivated by its success, it seemed very promising to attack the problem of constructing frames, where each frame element is invariant under some symmetry group, by generalizing the Feichtinger–Gröchenig theory. And in fact, this paper presents the results of this approach. As in [8–10,12] we make use of coorbit spaces $\mathsf{Co}Y$. These are Banach spaces related to the corresponding wavelet transform, which is given by matrix coefficients of some integrable unitary group representation of a locally compact group $\mathcal{G}$. Typically the coorbit spaces are smoothness spaces of distributions, for example Sobolev spaces. Since here we are only interested in elements (distributions), which are invariant under a symmetry group $\mathcal{A}$, we consider the subspaces $\mathsf{Co}Y_{\mathcal{A}}$ consisting only of those. We will then proceed analogously to the classical papers of Feichtinger and Gröchenig [9,10,12] and shall finally establish coherent atomic decompositions and Banach frames for $\mathsf{Co}Y_{\mathcal{A}}$ (Theorems 7.1–7.3). We emphasize that every element of this atomic decomposition or Banach frame by itself will be invariant under $\mathcal{A}$. In particular, radial wavelet frames and radial Gabor frames will be covered by the corresponding theorems as examples. Since in case of the Heisenberg group (with the STFT as corresponding transform) the coorbit spaces are the modulation spaces, we obtain atomic decompositions for radial modulation spaces, which were not known before.

We remark that Dahlke et al. developed a generalization of Feichtinger–Gröchenig theory into another direction [2,3]. In their approach the parameter space of the transform is not a group anymore but a homogeneous space. A further generalization was recently provided by Fornasier and Rauhut [11]. Their starting point is an abstract continuous frame.

The paper is organized as follows. In Section 2 we introduce notation and certain preliminaries. Here, we try to keep as close as possible to the classical papers [9,10,12] and to [17] in order to make comparison easy. In Section 3 we define the coorbit spaces and their subspaces of invariant elements and state some elementary properties. In order to establish the atomic decompositions we shall need a so-called invariant bounded uniform partition of unity (IBUPU) as one of the main tools. We show in Section 4 that such IBUPUs exist for every locally compact (σ-compact) group and every compact automorphism group. As another important tool we will need Wiener amalgam spaces on $\mathcal{G}$ and their subspaces of invariant elements. These will be discussed in Section 5. The atomic decompositions and Banach frames will be established using certain operators on functions on $\mathcal{G}$ that approximate the convolution. As in [12] we will use three different approximation operators which will lead to an atomic decomposition, to a Banach frame and to the existence of a 'dual' frame. These operators will be discussed in detail in Section 6. Finally, in Section 7, after all preparation, we shall establish atomic decompositions and Banach frames. For reasons of length, the detailed discussion of examples will be postponed.

2. Notation and preliminaries

Let $\mathcal{G}$ be a locally compact group and $\mathcal{A}$ be a compact automorphism group of $\mathcal{G}$, such that $\mathcal{A}$ acts continuously on $\mathcal{G}$, i.e., the mapping $\mathcal{G} \times \mathcal{A} \to \mathcal{G}$, $(x, A) \mapsto Ax$ is continuous. We denote the left Haar measures on $\mathcal{G}$ and $\mathcal{A}$ by μ and ν, where ν is assumed to be normalized. However, we usually write $\mathrm{d}x$ and $\mathrm{d}A$ in integrals. The modular function on $\mathcal{G}$ is denoted by Δ and the left and right translation operators on $\mathcal{G}$ by $L_y F(x) = F(y^{-1}x)$ and $R_y F(x) = F(xy)$. Furthermore, we define two involutions by $F^{\vee}(x) = F(x^{-1})$ and $F^{\nabla}(x) = \overline{F(x^{-1})}$. The action of $\mathcal{A}$ on functions on $\mathcal{G}$ is denoted by $F_A(x) = F(A^{-1}x)$, $A \in \mathcal{A}$, and the action on measures $\tau \in M(\mathcal{G})$, the space of complex bounded Radon measures on $\mathcal{G}$ (the dual space of $C_0(\mathcal{G})$), by $\tau_A(F) = \tau(F_{A^{-1}})$, $A \in \mathcal{A}$, $\tau \in M(\mathcal{G})$, $F \in C_0(\mathcal{G})$.

The functions (measures) which satisfy $F_A = F$ for all $A \in \mathcal{A}$ are called invariant (under $\mathcal{A}$). A standard argument shows that the Haar-measure μ and the modular function Δ are invariant under any compact automorphism group. For a function (measure) space Y on $\mathcal{G}$ we denote the subspace of invariant elements by $Y_{\mathcal{A}} := \{F \in Y, F_A = F \text{ for all } A \in \mathcal{A}\}$. An invariant function on $\mathcal{G}$ can be interpreted as a function on $\mathcal{K} := \mathcal{A}(\mathcal{G})$ the space of all orbits of the form $\mathcal{A}x := \{Ax, A \in \mathcal{A}\}$, $x \in \mathcal{G}$. The orbit space $\mathcal{K}$ becomes a topological space by inheriting the topology of $\mathcal{G}$ in a natural way [1,14,17].

For some positive measurable weight function m on $\mathcal{G}$ we define the weighted space $L^p_m := \{F \text{ measurable}, Fm \in L^p\}$ with norm $\|F \mid L^p_m\| := \|Fm \mid L^p\|$ where the L^p-spaces on $\mathcal{G}$ are defined as usual.

We recall some facts about the convolution of invariant functions from [17].

- The convolution of two invariant functions (measures) is again invariant, in particular $M_{\mathcal{A}}(\mathcal{G}) \cong M(\mathcal{K})$ is a closed subalgebra of $M(\mathcal{G})$ and $L^1_{\mathcal{A}}(\mathcal{G}) \cong L^1(\mathcal{K}, \tilde{\mu})$ is a closed subalgebra of $L^1(\mathcal{G})$, where $\tilde{\mu}$ is the projection of the Haar measure onto $\mathcal{K}$, i.e., $\int_{\mathcal{K}} F(\mathcal{A}x)\,\mathrm{d}\tilde{\mu}(\mathcal{A}x) = \int_{\mathcal{G}} F(x)\,\mathrm{d}\mu(x)$.
- Define the generalized left translation by
$$\mathcal{L}_y F(x) := \int_{\mathcal{A}} F\big(A\big(y^{-1}\big)x\big)\,\mathrm{d}A = \varepsilon_{\mathcal{A}y} * F(x)$$
whenever this expression is well defined a.e., for instance for $F \in C(\mathcal{G})$. Here, $\varepsilon_{\mathcal{A}y}(F) := \int_{\mathcal{A}} F(Ay)\,\mathrm{d}A$ denotes the 'invariant Dirac' measure. Then $\mathcal{L}_y$ maps invariant functions onto invariant ones, and the convolution of two invariant functions F, G may be expressed by the formula
$$F * G(x) = \int_{\mathcal{G}} F(y)\mathcal{L}_y G(x)\,\mathrm{d}\mu(y) = \int_{\mathcal{K}} F(\mathcal{A}y)\mathcal{L}_{\mathcal{A}y} G(x)\,\mathrm{d}\tilde{\mu}(\mathcal{A}y) \tag{2.1}$$
whenever the convolution is defined.
- Define an involution on $\mathcal{K}$ by $(\mathcal{A}x)^{\tilde{}} := \mathcal{A}(x^{-1})$. Then $(\mathcal{K}, *, \tilde{})$ is a hypergroup, more precisely an orbit hypergroup (see also [1,14]).

In this paper we will work with Banach spaces of functions on $\mathcal{G}$ which will usually be denoted by Y. Similarly as in [12] we will make the following assumptions on Y.

(1) Y is continuously embedded into $L^1_{\mathrm{loc}}(\mathcal{G})$, the locally integrable functions on $\mathcal{G}$.
(2) Y is solid, i.e., if $F \in L^1_{\mathrm{loc}}(\mathcal{G})$, $G \in Y$ and $|F(x)| \leqslant |G(x)|$ a.e., then $F \in Y$ and $\|F \mid Y\| \leqslant \|G \mid Y\|$.
(3) Y is invariant under left and right translations. Hence, we may define the two functions $u(x) := \|L_x \mid Y \to Y\|$ and $v(x) := \|R_{x^{-1}} \mid Y \to Y\| \Delta(x^{-1})$. Clearly, $u(xy) \leqslant u(x)u(y)$ and $v(xy) \leqslant v(x)v(y)$, i.e., u and v are submultiplicative. Additionally, we require that u and v are continuous. Under these assumptions, as pointed out in [9,18], we have
$$L^1_u * Y \subset Y, \qquad \|F * G \mid Y\| \leqslant \big\|F \mid L^1_u\big\| \|G \mid Y\| \quad \text{for all } F \in L^1_u,\ G \in Y \tag{2.2}$$
and
$$Y * L^1_v \subset Y, \qquad \|F * G \mid Y\| \leqslant \|F \mid Y\| \big\|G \mid L^1_v\big\| \quad \text{for all } F \in Y,\ G \in L^1_v. \tag{2.3}$$
(4) $\mathcal{A}$ acts continuously on Y. Without loss of generality we may assume that $u(Ax) = u(x)$ and $v(Ax) = v(x)$ for all $A \in \mathcal{A}$. (In case this is not true define an invariant norm on Y by $\|F \mid Y\|' := \int_{\mathcal{A}} \|F_A \mid Y\|\,\mathrm{d}A$. Since $\mathcal{A}$ acts continuously on Y, this is an equivalent norm on Y.) Then $Y_{\mathcal{A}}$ is a closed nontrivial subspace of Y. (To see that there is a nontrivial element contained in Y start with a positive nonzero function F in Y and let $F'(x) := \int_{\mathcal{A}} F(Ax)\,\mathrm{d}A$, which clearly is invariant.)

Examples of such spaces include L_m^p-spaces with invariant moderate weight function m, certain mixed norm spaces on $\mathcal{G}$, etc., see also [12].

With the Banach space Y, we will always associate the weight function

$$w(x) := \max\{u(x), u(x^{-1}), v(x), v(x^{-1})\Delta(x^{-1})\}.$$

Then, as a consequence of our assumptions on Y, w is continuous, $w(xy) \leqslant w(x)w(y)$, $w(x) \geqslant 1$, and $w(Ax) = w(x)$ for all $A \in \mathcal{A}$ and $x \in \mathcal{G}$. Furthermore, by (2.3) we have

$$Y * L_w^1 \subset Y, \qquad \|F * G \mid Y\| \leqslant \|F \mid Y\| \|G \mid L_w^1\|. \tag{2.4}$$

We further assume that we are given a unitary, irreducible (strongly continuous) representation π of $\mathcal{G}$ on some Hilbert space $\mathcal{H}$ and some unitary (strongly continuous) representation σ of $\mathcal{A}$ (not necessarily irreducible) on the same Hilbert space $\mathcal{H}$, such that the following basic relation is satisfied (see also [17, 18]),

$$\pi(A(x))\sigma(A) = \sigma(A)\pi(x). \tag{2.5}$$

In other words, we require that all the representations $\pi_A := \pi \circ A$ are unitarily equivalent to π and that the intertwining operators $\sigma(A)$ form a representation of $\mathcal{A}$.

For $f \in \mathcal{H}$ we let $f_A = \sigma(A) f$ and $\mathcal{H}_{\mathcal{A}} := \{f \in \mathcal{H},\ f_A = f$ for all $A \in \mathcal{A}\}$, the closed(!) subspace of invariant elements. We always assume that $\mathcal{H}_{\mathcal{A}}$ is not trivial. The wavelet transform or voice transform is defined by

$$V_g f(x) := \langle f, \pi(x) g\rangle.$$

It maps $\mathcal{H}$ into $C^b(\mathcal{G})$, the space of bounded continuous functions on $\mathcal{G}$. With an element $g \in \mathcal{H}_{\mathcal{A}}$ we denote by $\tilde{V}_g$ the restriction of V_g to $\mathcal{H}_{\mathcal{A}}$. We recall some facts from [17].

- For $f, g \in \mathcal{H}_{\mathcal{A}}$ the function $\tilde{V}_g f$ is invariant under $\mathcal{A}$, i.e., $\tilde{V}_g$ maps $\mathcal{H}_{\mathcal{A}}$ into $C_{\mathcal{A}}^b(\mathcal{G})$.
- For $x \in \mathcal{G}$ we define

  $$\tilde{\pi}(x) := \int_{\mathcal{A}} \pi(Ax)\,\mathrm{d}A$$

  in a weak sense. This operator maps $\mathcal{H}_{\mathcal{A}}$ onto $\mathcal{H}_{\mathcal{A}}$ and depends only on the orbit of x under $\mathcal{A}$, i.e., $\tilde{\pi}(Bx) = \tilde{\pi}(x)$ for all $B \in \mathcal{A}$. Furthermore, we have

  $$\tilde{V}_g f(x) = \langle f, \tilde{\pi}(x) g\rangle_{\mathcal{H}_{\mathcal{A}}}. \tag{2.6}$$
- The operators $\tilde{\pi}(x)$ form an irreducible representation of the orbit hypergroup $\mathcal{K}$.
- We have the following covariance principle

  $$\tilde{V}_g(\tilde{\pi}(x) f) = \mathcal{L}_x \tilde{V}_g f.$$

We further require that π is integrable which means that there exists a nonzero element $g \in \mathcal{H}$ such that $\int_{\mathcal{G}} |V_g g(x)|\,\mathrm{d}x < \infty$. This implies that π is square-integrable, i.e., there exists $g \in \mathcal{H}$ such that $\int_{\mathcal{G}} |V_g f(x)|^2\,\mathrm{d}x < \infty$ for all $f \in \mathcal{H}$. Such a g (corresponding to the square-integrability condition) is called admissible. We list some further properties from [4] and [17] that hold under the square-integrability condition.

- There exists a positive, densely defined operator S such that the domain $\mathcal{D}(S)$ of S consists of all admissible vectors and the orthogonality relation

$$\int_{\mathcal{G}} V_{g_1} f_1(x) \overline{V_{g_2} f_2(x)}\, \mathrm{d}x = \langle Sg_2, Sg_1\rangle \langle f_1, f_2\rangle$$

  holds for all $f_1, f_2 \in \mathcal{H}$, $g_1, g_2 \in \mathcal{D}(S)$.
- As a consequence, if $\|Sg\| = 1$, we have the reproducing formula

$$V_g f = V_g f * V_g g \tag{2.7}$$

  and, of course, the same formula holds also for $\tilde{V}_g$.
- The space $\operatorname{span}\{\pi(x)f,\ x \in \mathcal{G}\}$ is dense in $\mathcal{H}$ for any nonzero $f \in \mathcal{H}$ and $\operatorname{span}\{\tilde{\pi}(x)f,\ x \in \mathcal{K}\}$ is dense in $\mathcal{H}_\mathcal{A}$ for any nonzero $f \in \mathcal{H}_\mathcal{A}$.
- The operator S commutes with the action of $\mathcal{A}$, i.e., $\sigma(A)S = S\sigma(A)$ for all $A \in \mathcal{A}$. Furthermore, $\mathcal{D}_\mathcal{A}(S) := \mathcal{D}(S) \cap \mathcal{H}_\mathcal{A}$ is dense in $\mathcal{H}_\mathcal{A}$ and S maps $\mathcal{D}_\mathcal{A}(S)$ into $\mathcal{H}_\mathcal{A}$.
- For $g \in \mathcal{D}_\mathcal{A}(S)$ with $\|Sg\| = 1$ we have the following inversion formula on $\mathcal{H}_\mathcal{A}$:

$$f = \int_{\mathcal{K}} \tilde{V}_g f(y) \tilde{\pi}(y) g\, \mathrm{d}\tilde{\mu}(y), \quad f \in \mathcal{H}_\mathcal{A}, \tag{2.8}$$

  where the integral is understood in a weak sense.

Example 2.1. Consider the similitude group $\mathcal{G} = \mathbb{R}^d \rtimes (\mathbb{R}_+^* \times SO(d))$ with $d \geqslant 2$ where $\mathbb{R}_+^*$ denotes the multiplicative group of positive real numbers. We introduce the following operators on $L^2(\mathbb{R}^d)$:

$$T_x f(t) = f(t - x), \qquad D_a f(t) = a^{-d/2} f(t/a), \qquad U_R f(t) = f\big(R^{-1}t\big)$$

for $t, x \in \mathbb{R}^d$, $a \in \mathbb{R}_+^*$, $R \in SO(d)$, $f \in L^2(\mathbb{R}^d)$. Then the operators

$$\pi(x, a, R) = T_x D_a U_R, \quad (x, a, R) \in \mathbb{R}^d \rtimes \big(\mathbb{R}_+^* \times SO(d)\big) = \mathcal{G}$$

form an irreducible unitary square-integrable representation of the similitude group on $\mathcal{H} = L^2(\mathbb{R}^d)$. The corresponding voice transform is the continuous wavelet transform

$$V_g f(x, a, R) = \big\langle f, \pi(x, a, R) g\big\rangle = a^{-d/2} \int_{\mathbb{R}^d} f(t) \overline{g\big(a^{-1} R^{-1}(t - x)\big)}\, \mathrm{d}t.$$

The compact subgroup $\mathcal{A} = SO(d)$ of $\mathcal{G}$ acts on $\mathcal{G}$ by inner automorphisms. It is trivial to check that the restriction $\sigma = \pi|_{SO(d)}$ is a representation of $SO(d)$ on $L^2(\mathbb{R}^d)$ satisfying (2.5). The space $H_\mathcal{A}$ of invariant vectors is then given by the space of radial L^2-functions, $L^2_{\mathrm{rad}}(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d),\ f(R^{-1}t) = f(t) \text{ for all } R \in SO(d)\}$. The operators $\tilde{\pi}(x, a, R)$ depend only on $|x|$ and a and they are given by

$$\tilde{\pi}(x, a, R) = \tau_{|x|} D_a,$$

where τ_s, $s \in [0, \infty)$, denotes a generalized translation which is defined by

$$\tau_s f(t) = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} f(t - s\xi)\, \mathrm{d}S(\xi), \quad t \in \mathbb{R}^d.$$

Here, S^{d-1} denotes the unit sphere in $\mathbb{R}^d$, $|S^{d-1}| = (2\pi^{d/2})/(\Gamma(d/2))$ its surface area and $\mathrm{d}S$ the surface measure. This operator maps radial functions onto radial ones. As a consequence of (2.6), the continuous wavelet transform of a radial function with respect to a radial wavelet can be computed by an integral over $[0,\infty)$ involving the operators τ_s. By writing the radial function $f \in L^2_{\mathrm{rad}}(\mathbb{R}^d)$ as $f(t) = f_0(|t|)$, for some function f_0 on $[0,\infty)$, we obtain

$$\tau_s f(t) = \frac{|S^{d-2}|}{|S^{d-1}|} \int_{-1}^{1} f_0\left(\sqrt{s^2 - 2s|t|r - |t|^2}\right)\left(1-r^2\right)^{(d-3)/2}\,\mathrm{d}r.$$

For further details and for an example connected to time–frequency analysis of radial functions we refer to [17,18].

For technical reasons we further assume without loss of generality that $\mathcal{G}$ is σ-compact.

3. Coorbit spaces

Given a function space Y on $\mathcal{G}$ with associated weight function w the set of analyzing vectors is defined by

$$\mathbb{A}_w := \left\{g \in \mathcal{H},\ V_g g \in L^1_w(\mathcal{G})\right\}$$

and its subspace of invariant elements by

$$\mathbb{A}_w^{\mathcal{A}} := \mathbb{A}_w \cap \mathcal{H}_{\mathcal{A}} = \left\{g \in \mathcal{H}_{\mathcal{A}},\ \tilde{V}_g g \in L^1_w(\mathcal{G})\right\}.$$

We shall always assume that $\mathbb{A}_w^{\mathcal{A}}$ is not trivial and consider only those weights w (respectively, function spaces Y) for which this is the case. Since π is irreducible, the elements $\pi(x)g$, $x \in \mathcal{G}$, span a dense subspace of $\mathcal{H}$ and

$$V_{\pi(x)g}\left(\pi(x)g\right)(y) = \left\langle \pi(x)g, \pi(y)\pi(x)g\right\rangle = V_g g\left(x^{-1}yx\right) = L_x R_x V_g g(y).$$

Since L^1_w is left and right invariant, we conclude that $\pi(x)g \in \mathbb{A}_w$ whenever $g \in \mathbb{A}_w$. Hence, $\mathbb{A}_w$ is a dense subspace of $\mathcal{H}$ and $\mathbb{A}_w^{\mathcal{A}}$ is a dense subspace of $\mathcal{H}_{\mathcal{A}}$.

Fixing an arbitrary nonzero vector $g \in \mathbb{A}_w^{\mathcal{A}}$ the space $\mathcal{H}^1_w$ is defined by

$$\mathcal{H}^1_w := \left\{f \in \mathcal{H},\ V_g f \in L^1_w\right\}$$

with norm

$$\left\|f \mid \mathcal{H}^1_w\right\| := \left\|V_g f \mid L^1_w\right\|.$$

Its subspace of invariant elements is given by

$$\left(\mathcal{H}^1_w\right)_{\mathcal{A}} := \mathcal{H}_{\mathcal{A}} \cap \mathcal{H}^1_w = \left\{f \in \mathcal{H}_{\mathcal{A}},\ \tilde{V}_g f \in L^1_w\right\}.$$

In [9] it is proven that the definition of $\mathcal{H}^1_w$ is independent of the choice of $g \in \mathbb{A}_w$ with equivalent norms for different g. Clearly, $\mathbb{A}_w \subset \mathcal{H}^1_w$ and $\mathbb{A}_w^{\mathcal{A}} \subset (\mathcal{H}^1_w)_{\mathcal{A}}$ and, hence, $\mathcal{H}^1_w$ is dense in $\mathcal{H}$ and $(\mathcal{H}^1_w)_{\mathcal{A}}$ is dense in $\mathcal{H}_{\mathcal{A}}$.

As an appropriate reservoir of elements for the coorbit spaces we take the space $(\mathcal{H}^1_w)^\urcorner$ of all continuous conjugate linear functionals on $\mathcal{H}^1_w$ (the anti-dual space). We extend the bracket $\langle\cdot,\cdot\rangle$ to

$(\mathcal{H}_w^1)^\urcorner \times \mathcal{H}_w^1$ by means of $\langle f, g\rangle = f(g)$. We remark that by taking the anti-dual instead of the usual dual we can formally use the bracket in the same way as in the Hilbert space $\mathcal{H}$ and all formulas carry over without change. Note that the anti-dual can always be identified with the dual via the mapping $J:(\mathcal{H}_w^1)' \to (\mathcal{H}_w^1)^\urcorner$, $J(f)(h) = \overline{f(h)}$, $h \in \mathcal{H}_w^1$. Further, we also extend the bracket on $L^2(\mathcal{G})$ by $\langle F, G\rangle = \int_\mathcal{G} F(x)\overline{G(x)}\,\mathrm{d}x$ for $F \in L^\infty_{1/w}(\mathcal{G})$, $G \in L^1_w(\mathcal{G})$.

With the usual identification of elements in $\mathcal{H}_w^1$ with elements in the anti-dual we have the continuous embeddings

$$\mathcal{H}_w^1 \subset \mathcal{H} \subset \left(\mathcal{H}_w^1\right)^\urcorner.$$

We also need the anti-dual $((\mathcal{H}_w^1)_\mathcal{A})^\urcorner$. Define a map $\tilde{}:((\mathcal{H}_w^1)_\mathcal{A})^\urcorner \to (\mathcal{H}_w^1)^\urcorner$ by $\tilde{f}(g) := f(\int_\mathcal{A} g_A\,\mathrm{d}A)$, $g \in \mathcal{H}_w^1$, where $\int_\mathcal{A} g_A\,\mathrm{d}A$ defines an element of $(\mathcal{H}_w^1)_\mathcal{A}$ in a weak sense. The map $\tilde{}$ establishes an isometric isomorphism between $((\mathcal{H}_w^1)_\mathcal{A})^\urcorner$ and $((\mathcal{H}_w^1)^\urcorner)_\mathcal{A}$, the space of all functionals f in $(\mathcal{H}_w^1)^\urcorner$ that satisfy $f(g_A) = f(g)$ for all $A \in \mathcal{A}$ and $g \in \mathcal{H}_w^1$. We may therefore unambiguously write $(\mathcal{H}_w^1)^\urcorner_\mathcal{A}$.

Since $V_g(\pi(x)g) = L_x V_g g$ and since L_w^1 is translation invariant, all elements $\pi(x)g$, $x \in \mathcal{G}$, are contained in $\mathcal{H}_w^1$ whenever $g \in \mathcal{H}_w^1$. Hence, the action of π on $\mathcal{H}_w^1$ can be extended to $(\mathcal{H}_w^1)^\urcorner$ by the usual rule $(\pi(x)f)(g) = f(\pi(x^{-1})g)$ for $f \in (\mathcal{H}_w^1)^\urcorner$, $g \in \mathcal{H}_w^1$ and it is reasonable to extend the voice transform to $(\mathcal{H}_w^1)^\urcorner$ by

$$V_g f(x) := \langle f, \pi(x)g\rangle = f\left(\pi(x)g\right), \quad f \in \left(\mathcal{H}_w^1\right)^\urcorner,\ g \in \mathcal{H}_w^1.$$

Clearly, in the same way $\tilde{V}_g$ extends to $(\mathcal{H}_w^1)^\urcorner_\mathcal{A}$.

For more details on $\mathcal{H}_w^1$ and $(\mathcal{H}_w^1)^\urcorner$ we refer to [9]. The results there carry over to the subspaces $(\mathcal{H}_w^1)_\mathcal{A}$ and $(\mathcal{H}_w^1)^\urcorner_\mathcal{A}$.

Definition 3.1. For a fixed nonzero $g \in \mathbb{A}_w^\mathcal{A}$ we define the coorbit of Y under the representation π by

$$\mathsf{Co}Y := \left\{f \in \left(\mathcal{H}_w^1\right)^\urcorner,\ V_g f \in Y\right\}$$

with natural norm

$$\|f \mid \mathsf{Co}Y\| := \|V_g f \mid Y\|.$$

Further, the closed subspace of invariant elements is defined by

$$\mathsf{Co}Y_\mathcal{A} := \left(\mathcal{H}_w^1\right)^\urcorner_\mathcal{A} \cap \mathsf{Co}Y = \left\{f \in \left(\mathcal{H}_w^1\right)^\urcorner_\mathcal{A},\ \tilde{V}_g f \in Y_\mathcal{A}\right\},$$

with induced norm.

It is proven in [9] that $\mathsf{Co}Y$ is a Banach space which is independent of $g \in \mathbb{A}_w$ (with equivalent norms for different gs) and in some sense there is also independence of the weight function w. Namely, if w_2 is another weight function with $w(x) \leqslant Cw_2(x)$, then replacing $(\mathcal{H}_w^1)^\urcorner$ in the definition of $\mathsf{Co}Y$ with $(\mathcal{H}_{w_2}^1)^\urcorner$ results in the same space. Clearly, the analogous statements hold for $\mathsf{Co}Y_\mathcal{A}$.

A central role is played by the following proposition, which is an easy adaption of Proposition 4.3 in [9], by using the fact that the convolution preserves the $\mathcal{A}$-invariance.

Proposition 3.1 (Correspondence principle). (a) *Given $g \in \mathbb{A}_w^\mathcal{A}$ with $\|Sg\| = 1$, a function $F \in Y_\mathcal{A}$ is of the form $\tilde{V}_g f$ for some $f \in \mathsf{Co}Y_\mathcal{A}$ if and only if F satisfies the reproducing formula $F = F * \tilde{V}_g g$.*

(b) $\tilde{V}_g : \mathrm{Co}Y_{\mathcal{A}} \to Y_{\mathcal{A}}$ *establishes an isometric isomorphism between* $\mathrm{Co}Y_{\mathcal{A}}$ *and the closed subspace* $Y_{\mathcal{A}} * \tilde{V}_g g$ *of* $Y_{\mathcal{A}}$, *whereas* $F \mapsto F * \tilde{V}_g g$ *defines a bounded projection from* $Y_{\mathcal{A}}$ *onto this subspace.*

(c) *Every invariant function* $F = F * \tilde{V}_g g$ *is continuous, belongs to* $L^\infty_{1/w}(\mathcal{G})$ *and the evaluation mapping may also be written as* $F(x) = \langle F, L_x \tilde{V}_g g\rangle = \langle F, \mathcal{L}_x \tilde{V}_g g\rangle$.

We remark that in all places where the convolution appears one should have formula (2.1) in mind.

Examples of coorbit spaces include the homogeneous Besov spaces $\dot{B}^{p,q}_s(\mathbb{R}^d)$, the homogeneous Triebel–Lizorkin spaces $\dot{F}^{p,q}_s(\mathbb{R}^d)$ and the modulation spaces $M^{p,q}_s(\mathbb{R}^d)$. The first two examples are connected to the similitude group $\mathcal{G} = \mathbb{R}^d \rtimes (\mathbb{R}^*_+ \times SO(d))$ and the third example is connected to the Heisenberg group, for details see [8,12], and [20] for the corresponding characterizations of Besov–Triebel–Lizorkin spaces. When the automorphism group is $SO(d)$ the corresponding coorbit spaces $\mathrm{Co}Y_{SO(d)}$ include subspaces of $\dot{B}^{p,q}_s(\mathbb{R}^d)$, $\dot{F}^{p,q}_s(\mathbb{R}^d)$, or $M^{p,q}_s(\mathbb{R}^d)$ consisting of radially symmetric distributions on $\mathbb{R}^d$. For details on how $SO(d)$ acts on the Heisenberg group or the similitude group we refer to [17].

4. Invariant bounded uniform partitions of unity

Our main task is to find atomic decompositions of the invariant coorbit spaces $\mathrm{Co}Y_{\mathcal{A}}$, i.e., we look for discretizations of the inversion formula (2.8) for $\tilde{V}_g$. In [9] the concept of a bounded uniform partition of unity has been proven useful. In order to adapt this tool to our case we require that all functions belonging to the partition of unity are invariant under $\mathcal{A}$. This leads to the following definition.

Definition 4.1. A collection of functions $\Psi = (\psi_i)_{i\in I}$, $\psi_i \in C_0(\mathcal{G})$, is called $\mathcal{A}$-invariant bounded uniform partition of unity of size U (for short U-$\mathcal{A}$-IBUPU), if the following conditions are satisfied:

(1) $0 \leqslant \psi_i(x) \leqslant 1$ for all $i \in I$ and $x \in \mathcal{G}$,
(2) $\sum_{i\in I} \psi_i(x) = 1$ for all $x \in \mathcal{G}$,
(3) $\psi_i(Ax) = \psi_i(x)$ for all $x \in \mathcal{G},\ A \in \mathcal{A},\ i \in I$,
(4) there is a relatively compact neighborhood $U = \mathcal{A}(U)$ of the unit e and there are elements $(x_i)_{i\in I} \subset \mathcal{G}$ such that

$$\operatorname{supp}\psi_i \subset \mathcal{A}(x_i U) = \bigcup_{A\in\mathcal{A}} A(x_i U),$$

(5) $\sup_{z\in\mathcal{G}} \#\{i \in I \mid z \in \mathcal{A}(x_i Q)\} \leqslant C_Q < \infty$ for all compact sets $Q \subset \mathcal{G}$.

We remark that condition (5) is equivalent to condition (5′):

$$\sup_{j\in I} \#\{i \in I \mid \operatorname{supp}\psi_i \cap \operatorname{supp}\psi_j \neq \emptyset\} \leqslant C < \infty.$$

If the automorphism group is trivial, i.e., $\mathcal{A} = \{e\}$, then the definition above reduces to the one of a BUPU in the sense of [9].

In the sequel we will prove the existence of arbitrarily fine IBUPUs on every locally compact group. A first step is the following lemma whose proof is an adaption of the one in [16].

Lemma 4.1. *Let $\mathcal{A}$ be a compact automorphism group of a locally compact, σ-compact group $\mathcal{G}$ and let $V = V^{-1} = \mathcal{A}(V)$ be a relatively compact neighborhood of $e \in \mathcal{G}$ with nonvoid interior. Then there exists a countable subset $X = (x_i)_{i\in I} \subset \mathcal{G}$ with the following properties*:

(1) $\mathcal{G} = \bigcup_{i\in I} \mathcal{A}(x_i V)$.
(2) *For all compact sets $K_1, K_2 \subset \mathcal{G}$ there exists a constant $C > 0$ such that*

$$\sup_{y\in\mathcal{G}} \#\bigl\{i \in I,\ \mathcal{A}(yK_1) \cap \mathcal{A}(x_i K_2) \neq \emptyset\bigr\} \leqslant C < \infty.$$

Moreover, X can be chosen such that, for any set $W = W^{-1} = \mathcal{A}(W)$ with $W^2 \subset V$, we have

$$\mathcal{A}(x_i W) \cap \mathcal{A}(x_j W) = \emptyset \quad \textit{for all } i, j \in I,\ i \neq j. \tag{4.1}$$

Proof. For property (1) we first consider the case that $\mathcal{G} = \bigcup_{n=1}^{\infty} V^n$. We choose $x_1 := e$. Now form $K^{(2)} := \overline{V^2} \setminus V$. If $K^{(2)} = \emptyset$ (only possible if $\mathcal{G}$ is compact), then we are done, because we have $\mathcal{G} = V$. Otherwise choose $x_2 \in K^{(2)}$ and form $K^{(3)} := \overline{V^2} \setminus (V \cup \mathcal{A}(x_2 V))$. If $K^{(3)} \neq \emptyset$ choose $x_3 \in K^{(3)}$. Continuing in this way one obtains

$$\overline{V^2} \subset \bigcup_{i=1}^{N_2} \mathcal{A}(x_i V)$$

with $x_j \notin \bigcup_{i=1}^{j-1} \mathcal{A}(x_i V)$. Let us estimate the size of N_2. If $W = W^{-1} = \mathcal{A}(W)$ is a relatively compact neighborhood of e, with $W^2 \subset V$, then at most $|\overline{V^2}W|/|W|$ of such $x_i W$ fit into $(\overline{V^2})W$. Then $\bigcup_{i=1}^{N_2} x_i W^2$ and $\bigcup_{i=1}^{N_2} \mathcal{A}(x_i W^2)$ are coverings of $\overline{V^2}$. Hence, $N_2 \leqslant |\overline{V^2}W|/|W|$.

Now consider $K^{(N_2+1)} = \overline{V^3} \setminus \bigcup_{i=1}^{N_2} \mathcal{A}(x_i V)$ and choose $x_{N_2+1} \in K^{(N_2+1)}$ (if $K^{(N_2+1)} \neq \emptyset$). Inductively we obtain a covering

$$\mathcal{G} = \bigcup_{i=1}^{\infty} \mathcal{A}(x_i V).$$

If $\mathcal{G}$ is compact, then the covering is finite. It is easy to see that property (4.1) holds for the set $X = (x_i)_{i\in I}$.

In the general case we may write $\mathcal{G} = \bigcup_{s\in S'} s\mathcal{G}_0$ (disjoint union) where $\mathcal{G}_0 = \bigcup_{n=1}^{\infty} V^n$ is an open and closed subgroup of $\mathcal{G}$ (consisting of (possibly several) connected components of $\mathcal{G}$ including the connected component of the identity). Since $\mathcal{G}$ is σ-compact, the set $S' \subset \mathcal{G}$ is countable. However, it is not clear whether $\mathcal{A}$ keeps invariant each connected component $s\mathcal{G}_0$. To take care of this fact we form $\mathcal{G}_s := \mathcal{A}(s)\mathcal{G}_0$. Now, we may write $\mathcal{G} = \bigcup_{s\in S} \mathcal{G}_s$ (disjoint union) for some subset $S \subset S'$ and treat every $\mathcal{G}_s$ similarly as above. Namely, start with $x_1^s := s$ and put $K_s^{(2)} := \mathcal{A}(s\overline{V^2}) \setminus \mathcal{A}(sV)$ (this really is a subset of $\mathcal{G}_s$ by our construction!) and take $x_2^s \in K_s^{(2)}$ and so on. The rest is analogous to the above construction.

Let us now prove that property (2) holds for the set X constructed above. Suppose that $z \in \mathcal{A}(yK_1) \cap \mathcal{A}(x_i K_2) \neq \emptyset$ with $y \in \mathcal{G}$ for some $i \in I$. Then $z = A_1(y)k_1 = A_2(x_i)k_2$ with $A_1, A_2 \in \mathcal{A}$ and $k_j \in \mathcal{A}(K_j)$, $j = 1, 2$. Denoting $A_{i,y} = A_1^{-1}A_2$ we immediately deduce $A_{i,y}(x_i) \in y\mathcal{A}(K_1K_2^{-1})$ and, hence, $\mathcal{A}_{i,y}(x_i)W \subset y\mathcal{A}(K_1K_2^{-1})W$. The property (4.1) implies in particular $x_i W \cap x_j W = \emptyset$. Furthermore, the number of nonoverlapping sets of the form xW that fit into $y\mathcal{A}(K_1K_2^{-1})W$ is obviously bounded by $|\mathcal{A}(K_1K_2^{-1})W|/|W|$. Altogether we obtain

$$\#\{i \in I,\ \mathcal{A}(yK_1) \cap \mathcal{A}(x_i K_2) \neq \emptyset\} \leqslant \#\{i \in I,\ A_{i,y}(x_i)W \subset y\mathcal{A}(K_1 K_2^{-1})W\} \leqslant \frac{|\mathcal{A}(K_1 K_2^{-1})W|}{|W|}.$$

This completes the proof. □

A set X with the property (1) in Lemma 4.1 is called *V-dense* and a set X with property (2) *relatively separated*. If both properties hold, then X is called *well-spread* (*with respect to* $\mathcal{A}$).

Now we are ready to settle the problem of existence of IBUPUs.

Theorem 4.2. *Let $\mathcal{G}$ be a locally compact, σ-compact group, $\mathcal{A}$ be a compact automorphism group of $\mathcal{G}$ and $U = \mathcal{A}(U)$ be an open relatively compact neighborhood of $e \in \mathcal{G}$. Then there exists a U-$\mathcal{A}$-IBUPU in the sense of Definition* 4.1.

Proof. Choose $V = V^{-1} = \mathcal{A}(V)$ such that $V^2 \subset U$ and $X = (x_i)_{i\in I}$ according to Lemma 4.1 with the additional property (4.1) (where we construct X with respect to V and not with respect to U!). For every $i \in I$ let $\phi_i \in C_c(\mathcal{G})$ be such that $\phi_i(x) = 1$ for $x \in \mathcal{A}(x_i V)$, $\operatorname{supp}\phi_i \subset \mathcal{A}(x_i U)$, $0 \leqslant \phi_i(x) \leqslant 1$, for all $x \in \mathcal{G}$ and $\phi_i(Ax) = \phi_i(x)$ for all $A \in \mathcal{A}$, $x \in \mathcal{G}$. (Such a function exists: Take any function p_i that satisfies all properties except the invariance and put $\phi_i(x) = \int_{\mathcal{A}} p_i(Ax)\,\mathrm{d}A$. Then ϕ_i is invariant and still satisfies all other properties.) By property (2) in Lemma 4.1 (applied for $K_1 = K_2 = U$) and since the sets $\operatorname{supp}\phi_i$ cover $\mathcal{G}$, we have

$$1 \leqslant \Phi(x) := \sum_{i\in I} \phi_i(x) \leqslant C < \infty.$$

Now set $\psi_i(x) := \phi_i(x)/\Phi(x) \in C_c(\mathcal{G})$ yielding $\sum_{i\in I} \psi_i(x) = 1$ for all $x \in \mathcal{G}$ and $\operatorname{supp}\psi_i = \operatorname{supp}\phi_i \subset \mathcal{A}(x_i U)$. The invariance under $\mathcal{A}$ of the functions ψ_i is clear and the finite overlap property (5) follows from property (2) in Lemma 4.1. □

5. Wiener amalgam spaces

As another tool we shall need Wiener amalgam spaces. The idea of these spaces is to measure local and global properties of a function at the same time. For their definition, let B be a Banach space of functions (measures) on $\mathcal{G}$ and Y be a solid, left and right invariant *BF*-space. Using a nonzero *window function* $k \in C_c(\mathcal{G})$ (most commonly a function that satisfies $0 \leqslant k(x) \leqslant 1$ and $k(x) = 1$ for x in some compact neighborhood of the identity) we define the *control function* by

$$K(F, k, B)(x) := \|(L_x k)F\|_B, \quad x \in \mathcal{G}, \tag{5.1}$$

where F is locally contained in B, in symbols $F \in B_{\mathrm{loc}}$. The Wiener amalgam $W(B, Y)$ is now defined by

$$W(B, Y) := \{F \in B_{\mathrm{loc}},\ K(F, k, B) \in Y\}$$

with norm

$$\|F \mid W(B, Y)\| := \|K(F, k, B) \mid Y\|.$$

It has been shown in [7] that these spaces are two-sided invariant Banach spaces which do not depend on the particular choice of the window function k. Moreover, different window functions define equivalent norms. For the various properties of Wiener amalgam spaces see [6,7,9,10,13].

Replacing the left translation L_x with the right translation R_x in the definition (5.1) of the control function leads to right Wiener amalgam spaces $W^R(B,Y)$.

We state two convolution properties that will be essential for our purpose.

Proposition 5.1. (a) (*Proposition* 3.10 *in* [9].) *Under our general assumptions relating Y and w we have*

$$W(M,Y) * W^R\big(C_0, L^1_w\big) \subset Y, \qquad \|\mu * G \mid Y\| \leqslant C\big\|\mu \mid W(M,Y)\big\|\,\big\|G \mid W^R\big(C_0,L^1_w\big)\big\|.$$

(b) (*Theorem* 7.1(b) *in* [10].) *There exists a constant $D > 0$ such that*

$$Y * W\big(C_0, L^1_w\big) \subset W(C_0,Y), \qquad \big\|F * G \mid W(C_0,Y)\big\| \leqslant D\|F \mid Y\|\,\big\|G \mid W\big(C_0,L^1_w\big)\big\|.$$

Note that a function F is contained in $W(C_0,L^1_w)$, if and only if $F^\vee$ is contained in $W^R(C_0,L^1_w)$ and $\|F \mid W^R(C_0,L^1_w)\| = \|F \mid W(C_0,L^1_w)\|$.

As always throughout this paper we further assume that $\mathcal{A}$ acts isometrically on Y and B. Then $\mathcal{A}$ clearly acts also isometrically on $W(B,Y)$ and we may define the closed subspace

$$W_{\mathcal{A}}(B,Y) := \big\{F \in W(B,Y),\ F_A = F \text{ for all } A \in \mathcal{A}\big\},$$

and analogously for the right Wiener amalgams. Since the convolution of two $\mathcal{A}$-invariant functions (measures) is again $\mathcal{A}$-invariant, we may replace each function (measure) space in Proposition 5.1 by its subspace of invariant functions.

We will need two sequence spaces related to Wiener amalgams. Later on these will serve for the characterization of coorbit spaces via atomic decompositions and Banach frames. For a well-spread family $X = (x_i)_{i\in I}$ with respect to $\mathcal{A}$, a relatively compact set $U = \mathcal{A}(U)$ with nonvoid interior and a solid BF space Y we define

$$Y^b_{\mathcal{A}} := Y^b_{\mathcal{A}}(X) := \left\{(\lambda_i)_{i\in I},\ \sum_{i\in I} |\lambda_i| \chi_{\mathcal{A}(x_iU)} \in Y\right\}$$

with natural norm

$$\big\|(\lambda_i)_{i\in I} \mid Y^b_{\mathcal{A}}\big\| := \left\|\sum_{i\in I} |\lambda_i \mid \chi_{\mathcal{A}(x_iU)}| Y\right\|,$$

where $\chi_{\mathcal{A}(x_iU)}$ denotes the characteristic function of the set $\mathcal{A}(x_iU)$. Further let

$$a_i := \big|\mathcal{A}(x_iU)\big|$$

and define the space

$$Y^d_{\mathcal{A}} := Y^d_{\mathcal{A}}(X) := \big\{(\lambda_i)_{i\in I},\ \big(a_i^{-1}\lambda_i\big)_{i\in I} \in Y^b_{\mathcal{A}}\big\}$$

with norm

$$\big\|(\lambda_i)_{i\in I} \mid Y^d_{\mathcal{A}}\big\| := \big\|\big(a_i^{-1}\lambda_i\big)_{i\in I} \mid Y^b_{\mathcal{A}}\big\|.$$

(According to the later use of these spaces, 'd' stands for (atomic) decomposition and 'b' stands for Banach frame.) Note that the numbers a_i are always finite, since U is relatively compact and $\mathcal{A}$ is compact, hence, $\mathcal{A}(x_iU)$ is relatively compact. By solidity of Y it is immediate that also $Y^d_{\mathcal{A}}$ and $Y^b_{\mathcal{A}}$ are solid. Note that $a_i,\ i \in I$, is constant in case of the trivial automorphism group, and then both spaces $Y^b_{\mathcal{A}}$ and

$Y_{\mathcal{A}}^d$ coincide, of course. Similarly to the classical case one shows that $Y_{\mathcal{A}}^b$ and $Y_{\mathcal{A}}^d$ do not depend on the particular choice of the set U and different sets define equivalent norms. The following lemma is useful for this task.

Lemma 5.2. *Let $U = \mathcal{A}(U)$ and $V = \mathcal{A}(V)$ be invariant relatively compact neighborhoods of the identity. Then there exist constants $C_1, C_2 > 0$ such that $C_1|\mathcal{A}(xV)| \leqslant |\mathcal{A}(xU)| \leqslant C_2|\mathcal{A}(xV)|$ for all $x \in \mathcal{G}$.*

Proof. By compactness there exists a finite number of points $y_j \in \mathcal{G}$, $j = 1, \ldots, n$, such that $V \subset \bigcup_{j=1}^n U y_j$. Since $V = \mathcal{A}(V)$ and $U = \mathcal{A}(U)$, we have

$$\mathcal{A}(xV) = (\mathcal{A}x)V \subset \bigcup_{j=1}^n (\mathcal{A}x)Uy_j = \bigcup_{j=1}^n \mathcal{A}(xU)y_j$$

yielding

$$\left|\mathcal{A}(xV)\right| \leqslant \sum_{j=1}^n \left|\mathcal{A}(xU)y_j\right| \leqslant \sum_{j=1}^n \Delta(y_j)\left|\mathcal{A}(xU)\right| = C_1^{-1}\left|\mathcal{A}(xU)\right|.$$

Exchanging the roles of U and V yields a reversed inequality. □

If $Y = L_m^p(\mathcal{G})$, $1 \leqslant p \leqslant \infty$, with invariant moderate weight function m, then $Y_{\mathcal{A}}^b(X) = l_{\nu_p}^p(I)$ and $Y_{\mathcal{A}}^d(X) = l_{m_p}^p(I)$ where

$$\nu_p(i) := m(x_i)a_i^{1/p}, \qquad m_p(i) := m(x_i)a_i^{1/p-1}$$

and $\|(\lambda_i)_{i\in I} \mid l_m^p(I)\| = (\sum_{i\in I} |\lambda_i|^p m(i)^p)^{1/p}$ with the usual modification for $p = \infty$. We have in particular $\nu_\infty(i) = m_1(i) = m(x_i)$.

Let us now derive a different characterization of $Y_{\mathcal{A}}^d$. To this end, for a positive window function k which is invariant under $\mathcal{A}$, we define the function

$$m_k(x,z) := K(\varepsilon_{\mathcal{A}x}, k, M)(z) = \left\|(L_z k)\varepsilon_{\mathcal{A}x}\right\|_M = \int_{\mathcal{A}} k\big(z^{-1}A(x)\big)\,\mathrm{d}A = \mathcal{L}_z k(x) = \mathcal{L}_x k^\vee(z).$$

Since k is assumed to be invariant, m_k is invariant in both variables. Further, if we have $\operatorname{supp} k \subset U$, then $\operatorname{supp} m_k(\cdot, z) \subset \mathcal{A}(zU)$ and $\operatorname{supp} m_k(x, \cdot) \subset \mathcal{A}(xU^{-1})$. Moreover, if $k = k^\vee$, then $m_k(x,z) = m_k(z,x)$.

If $k = \chi_U$ is the characteristic function of some set $U = \mathcal{A}(U)$, then $m_{\chi_U} =: m_U$ has a geometric interpretation, i.e., $m_U(x,z)$ is the size of the set

$$K_U(x,z) := \left\{A \in \mathcal{A} \mid z^{-1}Ax \in U\right\}$$

(measured with the Haar-measure of $\mathcal{A}$), which can be interpreted as the normalized 'surface measure' of $\mathcal{A}x \cap zU$ in the orbit ('surface') $\mathcal{A}x$. We provide a technical lemma concerning the function m_U.

Lemma 5.3. *Let $U = U^{-1} = \mathcal{A}U$ and $Q = Q^{-1} = \mathcal{A}Q$ be open relatively compact subsets of $\mathcal{G}$. Then*

$$m_U(x,z) \leqslant m_{U^3Q}(y,z) \quad \text{for all } y \in \mathcal{A}(zUQ),\ x \in \mathcal{G}. \tag{5.2}$$

Proof. If $x \notin \operatorname{supp} m_U(\cdot, z) \subset \mathcal{A}(zU)$, there is nothing to prove. Because of the $\mathcal{A}$-invariance of m_U and m_{U^3Q} it suffices to prove that $m_U(x,z) \leqslant m_{U^3Q}(y,z)$ holds, if $x \in zU$, $y \in zUQ$. The latter means $x = zu_x$ and $y = zu_y q$ for some elements $u_x, u_y \in U$, $q \in Q$. Hence, $x = yq^{-1}u_y^{-1}u_x =: yq^{-1}v \in yQU^2$. Now suppose $A \in K_U(x,z)$, i.e., $z^{-1}Ax \in U$ implying $z^{-1}A(yq^{-1}v) \in U$. This gives $z^{-1}A(y) \in UA(v^{-1})A(q) \subset U^3Q$, because $\mathcal{A}U = U$ and $\mathcal{A}Q = Q$ by assumption. Hence, $K_U(x,z) \subset K_{U^3Q}(y,z)$ and $m_U(x,z) \leqslant m_{U^3Q}(y,z)$. □

Now we are ready to prove the announced characterization.

Lemma 5.4. *There are constants $C_1, C_2 > 0$ such that*

$$C_1 \big\| (\lambda_i)_{i\in I} \mid Y_{\mathcal{A}}^d \big\| \leqslant \bigg\| \sum_{i\in I} |\lambda_i \mid m_k(x_i, \cdot)| Y_{\mathcal{A}} \bigg\| \leqslant C_2 \big\| (\lambda_i)_{i\in I} \mid Y_{\mathcal{A}}^d \big\|, \tag{5.3}$$

i.e., the expression in the middle defines an equivalent norm on $Y_{\mathcal{A}}^d$.

Proof. We claim that it suffices to prove (5.3) for characteristic functions $k = \chi_U$ for a relatively compact neighborhood U of $e \in \mathcal{G}$ satisfying $U = \mathcal{A}(U) = U^{-1}$. Indeed, if k is an arbitrary nonzero and positive function in $(C_c)_{\mathcal{A}}(\mathcal{G})$, then there exists a neighborhood $U = U^{-1} = \mathcal{A}(U) \subset \mathcal{G}$ of e and constants $C_1, C_2 > 0$ such that

$$C_1 \chi_U(x) \leqslant (L_y k)(x) \leqslant C_2 \chi_{\operatorname{supp} L_y k} \quad \text{for all } x \in \mathcal{G}$$

for some suitable $y \in \mathcal{G}$. The set $V := \mathcal{A}(\operatorname{supp}(L_y k) \cup (\operatorname{supp}(L_y k))^{-1})$ is a relatively compact neighborhood of e satisfying $V = V^{-1} = \mathcal{A}(V)$ and $\chi_{\operatorname{supp} L_y k} \leqslant \chi_V$. This implies $C_1 m_U(x,z) \leqslant m_{L_y k}(x,z) \leqslant C_2 m_V(x,z)$ for all $x, z \in \mathcal{G}$. Since $m_{L_y k}(x,z) = m_k(x, zy)$ and Y is right translation invariant, this shows the claim.

So we assume $U = U^{-1} = \mathcal{A}(U)$ to be a relatively compact neighborhood of e. By invariance of the Haar measure under left translation and under the action of $\mathcal{A}$ we obtain

$$\begin{aligned}
|U| &= \int_{\mathcal{G}} \chi_{x_i U}(x)\,\mathrm{d}x = \int_{\mathcal{A}} \int_{\mathcal{G}} \chi_{A(x_i)U}(x) \chi_{\mathcal{A}(x_i U)}(x)\,\mathrm{d}x\,\mathrm{d}A \\
&= \int_{\mathcal{G}} \int_{\mathcal{A}} \chi_{U^{-1}}\big(x^{-1}A(x_i)\big)\,\mathrm{d}A\, \chi_{\mathcal{A}(x_i U)}(x)\,\mathrm{d}x = \int_{\mathcal{A}(x_i U)} m_U(x_i, x)\,\mathrm{d}x \\
&= \int_{\mathcal{A}(x_i U)} m_U(x, x_i)\,\mathrm{d}x \leqslant \int_{\mathcal{A}(x_i U)} m_{U^4}(y, x_i)\,\mathrm{d}x = \big|\mathcal{A}(x_i U)\big| m_{U^4}(x_i, y)
\end{aligned}$$

for all $y \in \mathcal{A}(x_i U^2)$ by choosing $Q = U$ in inequality (5.2). Thus we have

$$|U| \chi_{\mathcal{A}(x_i U)}(y) \leqslant |U| \chi_{\mathcal{A}(x_i U^2)}(y) \leqslant \big|\mathcal{A}(x_i U)\big| m_{U^4}(x_i, y) \quad \text{for all } y \in \mathcal{G}.$$

To obtain a reversed inequality we choose again $Q = U$. For all $x \in \mathcal{G}$, Lemma 5.3 yields

$$\big|\mathcal{A}\big(x_i U^2\big)\big| m_U(x_i, x) = \int_{\mathcal{A}(x_i U^2)} m_U(x, x_i)\,\mathrm{d}y \leqslant \int_{\mathcal{A}(x_i U^2)} m_{U^4}(y, x_i)\,\mathrm{d}y$$

$$= \int_{\mathcal{A}(x_i U^2)} \int_{\mathcal{A}} \chi_{x_i U^4}\big(A(y)\big)\,\mathrm{d}A\,\mathrm{d}y \leqslant \int_{\mathcal{A}} \int_{\mathcal{A}(x_i U^4)} \chi_{x_i U^4}(y)\,\mathrm{d}y\,\mathrm{d}A = \big|U^4\big|. \tag{5.4}$$

Here, we used again the invariance of the Haar measure under $\mathcal{A}$. By the relation $\operatorname{supp} m_U(x_i,\cdot) \subset \mathcal{A}(x_i U)$, we obtain

$$\big|\mathcal{A}\big(x_i U^2\big)\big| m_U(x_i, y) \leqslant \big|U^4\big| \chi_{\mathcal{A}(x_i U)}(y) \quad \text{for all } y \in \mathcal{G}.$$

By solidity of Y and since the definition of $Y^d_{\mathcal{A}}$ does not depend on the choice of the set U, with equivalent norms for different choices (see also Lemma 5.2), we finally get inequality (5.3). □

As an easy consequence we obtain the following.

Lemma 5.5. *For some well-spread family $X = (x_i)_{i\in I}$, the measure*

$$\mu_\Lambda := \sum_{i\in I} \lambda_i \varepsilon_{\mathcal{A}x_i}$$

is contained in $W_{\mathcal{A}}(M, Y)$ if and only if $\Lambda = (\lambda_i)_{i\in I}$ is contained in $Y^d_{\mathcal{A}}(X)$ and there are constants $C_1, C_2 \leqslant 0$ such that

$$C_1 \big\| \Lambda \mid Y^d_{\mathcal{A}} \big\| \leqslant \big\| \mu_\Lambda \mid W_{\mathcal{A}}(M, Y) \big\| \leqslant C_2 \big\| \Lambda \mid Y^d_{\mathcal{A}} \big\|.$$

Proof. Clearly, the supports of the $L_z k \varepsilon_{\mathcal{A}x_i}$, $i \in I$, are not overlapping for any $z \in \mathcal{G}$. Hence, for the control function applied to μ_Λ, we obtain

$$K(\mu_\Lambda, k, M)(z) = \Big\| \sum_{i\in I} \lambda_i L_z k \varepsilon_{\mathcal{A}x_i} \Big\|_M = \sum_{i\in I} |\lambda_i| m_k(x_i, z).$$

From this the assertion follows easily with Lemma 5.4. □

We summarize some further statements concerning Wiener amalgam spaces and our newly defined sequence spaces in the following lemma.

Lemma 5.6. (a) *If the bounded functions with compact support are dense in Y, then the finite sequences are dense in $Y^d_{\mathcal{A}}$ and in $Y^b_{\mathcal{A}}$.*

(b) *Let U be some relatively compact neighborhood of $e \in \mathcal{G}$ and let $r(i) := |\mathcal{A}(x_i U)| w(x_i)$. Then $Y^d_{\mathcal{A}}$ is continuously embedded into $l^\infty_{1/r}$.*

(c) *If $G \in W^R_{\mathcal{A}}(C_0, L^1_w)$ and $(x_i)_{i\in I}$ is well-spread* (*with respect to $\mathcal{A}$*)*, then $(\mathcal{L}_{x_i} G(x))_{i\in I} \in l^1_r$ for all $x \in \mathcal{G}$ with r as in* (b).

Proof. The assertion (a) is immediate. For (b) observe that by solidity and left translation invariance of Y we obtain

$$\|\chi_U \mid Y\| = \|L_{x_i^{-1}} \chi_{x_i U} \mid Y\| \leqslant w(x_i) \|\chi_{x_i U} \mid Y\| \leqslant w(x_i) \|\chi_{\mathcal{A}(x_i U)} \mid Y\|.$$

This gives

$$|\lambda_i|\big|\mathcal{A}(x_iU)\big|^{-1}\|\chi_U \mid Y\| \leqslant w(x_i)\big\||\lambda_i|\big|\mathcal{A}(x_iU)\big|^{-1}\chi_{\mathcal{A}(x_iU)} \mid Y\big\|$$
$$\leqslant w(x_i)\bigg\|\sum_{j\in I}|\lambda_j|\big|\mathcal{A}(x_jU)\big|^{-1}\chi_{\mathcal{A}(x_jU)} \mid Y\bigg\| = w(x_i)\big\|(\lambda_j)_{j\in I} \mid Y^d_{\mathcal{A}}\big\|$$

and the claim is shown.

For (c) recall (e.g., from the proof of Proposition 3.10 in [9], see also Proposition 3.7 in [9]) that $G \in W^R(C_0, L^1_w)$ has a decomposition $G = \sum_{n\in N} R_{z_n} G_n$ with $\operatorname{supp} G_n \subset Q = Q^{-1} = \mathcal{A}(Q)$ (compact) and

$$\sum_{n\in N}\|G_n\|_\infty w(z_n) \leqslant C\big\|G \mid W^R\big(C_0, L^1_w\big)\big\|.$$

By the definition of m_Q we have $|\mathcal{L}_{x_i} G_n(x)| = |\varepsilon_{\mathcal{A}(x_i)} * (\chi_Q G_n)(x)| \leqslant \|G_n\|_\infty m_Q(x_i, x)$. Hence, we obtain the estimation

$$\sum_{i\in I}\big|\mathcal{L}_{x_i}G(x)\big|w(x_i)\big|\mathcal{A}(x_iU)\big| \leqslant \sum_{i\in I}\sum_{n\in N}\big|\varepsilon_{\mathcal{A}(x_i)} * R_{z_n}G_n(x)\big|w(x_i)\big|\mathcal{A}(x_iU)\big|$$
$$\leqslant \sum_{n\in N}\sum_{i\in I_{x,n}}\|G_n\|_\infty m_Q(x_i, xz_n)w(x_i)\big|\mathcal{A}(x_iU)\big|.$$

The inner sum runs over the finite index set

$$I_{x,n} = \big\{i \in I,\ x_i \in \mathcal{A}(xz_nQ)\big\}.$$

Since $(x_i)_{i\in I}$ is well spread, we have $|I_{x,n}| \leqslant C_Q < \infty$ uniformly for all x, n. For each $i \in I_{x,n}$, we may write $x_i = xz_nq_i$ for some $q_i \in Q$, which implies $w(x_i) \leqslant w(x)w(z_n)w(q_i)$. Further, it follows from (5.4) that $m_Q(x_i, xz_n) \leqslant C'|\mathcal{A}(x_iU)|^{-1}$ for some suitable constant $C' > 0$. Thus, we finally obtain

$$\sum_{i\in I}\big|\mathcal{L}_{x_i}G(x)\big|w(x_i)\big|\mathcal{A}(x_iU)\big| \leqslant w(x)C'C_Q\sup_{q\in Q}w(q)\sum_{n\in N}\|G_n\|_\infty w(z_n) < \infty \tag{5.5}$$

which completes the proof. □

Note that (5.5) implies that the function $x \mapsto \sum_{i\in I}\mathcal{L}_{x_i}G(x)w(x_i)|\mathcal{A}(x_iU)|$ is contained in $L^\infty_{1/w}(\mathcal{G})$. Essential in later estimations will be the following inequalities.

Lemma 5.7. *Suppose $F \in W_{\mathcal{A}}(C_0, Y)$ and $\Psi = (\psi_i)_{i\in I}$ to be some U-$\mathcal{A}$-IBUPU with corresponding well-spread set $X = (x_i)_{i\in I}$. Then*

$$\bigg\|\sum_{i\in I}F(x_i)\psi_i \mid W_{\mathcal{A}}(C_0, Y)\bigg\| \leqslant \gamma(U)\big\|F \mid W_{\mathcal{A}}(C_0, Y)\big\|$$

and

$$\big\|\big(F(x_i)\big)_{i\in I} \mid Y^b_{\mathcal{A}}\big\| \leqslant \gamma(U)C\big\|F \mid W_{\mathcal{A}}(C_0, Y)\big\| \tag{5.6}$$

for constants $\gamma(U), C < \infty$. If U varies through a family of subsets of some compact $U_0 \subset \mathcal{G}$, then $\gamma(U)$ is uniformly bounded by some constant γ_0.

Proof. We proceed similarly as in [12, Lemma 4.4]. Without loss of generality we assume that a characteristic function χ_Q for some relatively compact neighborhood $Q = Q^{-1} = \mathcal{A}(Q)$ of $e \in \mathcal{G}$ is taken for the definition of the norm of $W(C_0, Y)$. For the control function, we obtain

$$K\left(\sum_{i\in I}\left|F(x_i)\right|\psi_i, \chi_Q, C_0\right)(x) = \left\|(L_x\chi_Q)\sum_{i\in I}\left|F(x_i)\right|\psi_i\right\|_\infty =: H(x).$$

The sum in the last expression runs only over the finite index set

$$I_x := \left\{i \in I,\ xQ \cap \mathcal{A}(x_iU) \neq \emptyset\right\} = \left\{i \in I,\ \mathcal{A}(x_i) \cap xQU^{-1} \neq \emptyset\right\}.$$

Since F is $\mathcal{A}$ invariant and since $(\psi_i)_{i\in I}$ is a partition of unity, we therefore have

$$H(x) \leqslant \left\|(L_x\chi_{QU^{-1}})F\right\|_\infty = K(F, \chi_{QU^{-1}}, C_0)(x).$$

Since different window functions define equivalent norms on $W(C_0, Y)$ (see also [7]), there exists a constant $\gamma(U)$ such that

$$\left\|K(F, \chi_{QU^{-1}}, C_0) \mid Y\right\| \leqslant \gamma(U)\left\|K(F, \chi_Q, C_0) \mid Y\right\|. \tag{5.7}$$

We finally obtain

$$\begin{aligned}\left\|\sum_{i\in I}\left|F(x_i)\right|\psi_i \mid W_\mathcal{A}(C_0, Y)\right\| &= \left\|K\left(\sum_{i\in I}\left|F(x_i)\right|\psi_i, \chi_Q, C_0\right) \,\Big|\, Y_\mathcal{A}\right\| \\ &\leqslant \left\|K(F, \chi_{QU^{-1}}, C_0) \mid Y_\mathcal{A}\right\| \leqslant \gamma(U)\left\|K(F, \chi_Q, C_0) \mid Y_\mathcal{A}\right\| = \gamma(U)\left\|F \mid W_\mathcal{A}(C_0, Y)\right\|.\end{aligned}$$

To prove inequality (5.6) one proceeds analogously using

$$\left\|\left(F(x_i)\right)_{i\in I} \mid Y^b_\mathcal{A}\right\| \leqslant \left\|\sum_{i\in I} F(x_i)\chi_{\mathcal{A}(x_iU)} \mid W_\mathcal{A}(C_0, Y)\right\|,$$

which is easily seen with the finite overlap property of the well-spread family $(x_i)_{x\in I}$.

In order to show the assertion on $\gamma(U)$ we need to give a proof of (5.7) that provides an estimation of the constant $\gamma(U)$ (which is actually hard to extract from the proof in [7]). Since QU^{-1} is relatively compact, there exists a covering $QU^{-1} \subset \bigcup_{k=1}^n z_kQ$ for some points $z_k \in \mathcal{G}$. If $V = V^{-1}$ is such that $V^2 \subset Q$, then the points z_k, $k = 1, \ldots, n$, can be chosen such that

$$n \leqslant \frac{|QU^{-1}V|}{|V|}. \tag{5.8}$$

Indeed, choose a maximal set of points $z_k \in QU^{-1}$, $k = 1, \ldots, n$, such that the sets $z_kV \subset QU^{-1}V$ are mutually disjoint. Then the maximal number n is given by (5.8) and the sets z_kV^2 (and also the sets z_kQ) cover QU^{-1}. Therefore, we can derive the estimates

$$\begin{aligned}K(F, \chi_{QU^{-1}}, C_0)(x) = \left\|(L_x\chi_{QU^{-1}})F\right\|_\infty &\leqslant \left\|\sum_{k=1}^n (L_x\chi_{z_kQ})F\right\|_\infty \\ &\leqslant \sum_{k=1}^n \left\|(L_{xz_k}\chi_Q)F\right\|_\infty = \sum_{k=1}^n R_{z_k}K(F, \chi_Q, C_0)(x),\end{aligned}$$

and

$$\|K(F,\chi_{QU^{-1}},C_0)\mid Y\| \leqslant \sum_{k=1}^{n}\|R_{z_k}K(F,\chi_Q,C_0)\mid Y\| \leqslant \sum_{k=1}^{n} w(z_k)\|K(F,\chi_Q,C_0)\mid Y\|.$$

Thus, we obtain

$$\gamma(U) \leqslant \sum_{k=1}^{n} w(z_k) \leqslant n \sup_{z\in QU^{-1}} w(z) \leqslant \frac{|QU^{-1}V|}{|V|} \sup_{z\in QU^{-1}} w(z).$$

If U runs through a family of subsets of some U_0, then $\gamma(U)$ is clearly bounded. □

To conclude this section we apply the previous lemma in order to make a statement on sample values of $V_g f$ when f is contained in some coorbit space. For this purpose, we introduce the 'better' space of analyzing vectors

$$\mathbb{B}_w^{\mathcal{A}} := \{g \in \mathbb{A}_w^{\mathcal{A}},\ \tilde{V}_g g \in W_{\mathcal{A}}^R(C_0, L_w^1)\}. \tag{5.9}$$

Theorem 5.8. *Suppose $g \in \mathbb{B}_w^{\mathcal{A}}$. Then $\tilde{V}_g f \in W_{\mathcal{A}}(C_0, Y)$ for all $f \in \mathsf{Co}Y_{\mathcal{A}}$. If $X = (x_i)_{i\in I}$ is a U-dense well-spread family, then*

$$\|(\tilde{V}_g f(x_i))_{i\in I} \mid Y_{\mathcal{A}}^b\| \leqslant \gamma(U) C \|f \mid \mathsf{Co}Y_{\mathcal{A}}\|,$$

where the constant C depends only on g.

Proof. Without loss of generality we may assume $\|Sg\| = 1$. By Proposition 3.1, we have $\tilde{V}_g f = \tilde{V}_g f * \tilde{V}_g g$. Combined with the result of Proposition 5.1(b), we obtain

$$\|\tilde{V}_g f \mid W_{\mathcal{A}}(C_0,Y)\| = \|\tilde{V}_g f * \tilde{V}_g g \mid W_{\mathcal{A}}(C_0,Y)\| \leqslant D\|\tilde{V}_g f \mid Y_{\mathcal{A}}\| \|\tilde{V}_g g \mid W_{\mathcal{A}}^R(C_0, L_w^1)\|.$$

Lemma 5.7 finally leads to

$$\|(\tilde{V}_g f(x_i))_{i\in I} \mid Y_{\mathcal{A}}^b\| \leqslant \gamma(U)\|\tilde{V}_g f \mid W_{\mathcal{A}}(C_0,Y)\| \leqslant \gamma(U) D \|\tilde{V}_g g \mid W_{\mathcal{A}}^R(C_0,L_w^1)\| \|f \mid \mathsf{Co}Y_{\mathcal{A}}\|. \quad \square$$

6. Discretization of convolutions

In this section we study several approximations of the convolution operator on $Y_{\mathcal{A}}$, which acts as the identity on $Y_{\mathcal{A}} * G$, i.e., the image of $\mathsf{Co}Y_{\mathcal{A}}$ under $\tilde{V}_g$. For $G \in W_{\mathcal{A}}^R(C_0, L_w^1)$ (later we use $G = \tilde{V}_g g$), we define

$$T : Y_{\mathcal{A}} \to Y_{\mathcal{A}}, \quad TF := F * G = \int_{\mathcal{G}} F(y)\mathcal{L}_y G \, \mathrm{d}y.$$

For some arbitrary $\mathcal{A}$-IBUPU $\Psi = (\psi_i)_{i\in I}$ we approximate T by one of the following operators:

$$T_\Psi F := \sum_{i\in I}\langle F, \psi_i\rangle \mathcal{L}_{x_i} G, \qquad S_\Psi F := \sum_{i\in I} F(x_i)\psi_i * G, \qquad U_\Psi F := \sum_{i\in I} c_i F(x_i)\mathcal{L}_{x_i} G,$$

where $c_i = \int_{\mathcal{G}} \psi_i(x)\,\mathrm{d}x$.

Let us first consider the operator T_Ψ. We show that T_Ψ is a bounded operator from $Y_{\mathcal{A}}$ to $Y_{\mathcal{A}}$ by splitting it into the analysis operator $F \mapsto (\langle F, \psi_i\rangle)_{i\in I}$ and synthesis operator $(\lambda_i)_{i\in I} \mapsto \sum_{i\in I} \lambda_i \mathcal{L}_{x_i} G$ and treating each part separately.

Proposition 6.1. *Let $U = U^{-1} = \mathcal{A}(U)$ be a relatively compact neighborhood of $e \in \mathcal{G}$. For any U-$\mathcal{A}$-IBUPU $(\psi_i)_{i\in I}$ and corresponding well-spread family $X = (x_i)_{i\in I}$ the linear coefficient mapping $F \mapsto \Lambda = (\lambda_i)_{i\in I}$, where $\lambda_i := \langle F, \psi_i\rangle$ is a bounded operator from $Y_{\mathcal{A}}$ into $Y^d_{\mathcal{A}}(X)$, i.e.,*

$$\left\| \Lambda \mid Y^d_{\mathcal{A}} \right\| \leqslant C \|F \mid Y_{\mathcal{A}}\|.$$

The constant can be chosen to be $C = C_1^{-1} \|\chi_{V^3U} | L^1_w(\mathcal{G})\| < \infty$ where $k = \chi_V$ is chosen as window function for the definition of the norm of $Y^d_{\mathcal{A}}$ and C_1 is the constant from Lemma 5.4.

Proof. Let $F \in Y_{\mathcal{A}}$ and χ_V be the window function for the definition of $Y^d_{\mathcal{A}}$ for some open relatively compact set $V = V^{-1} = \mathcal{A}V$. As a consequence of $\operatorname{supp} m_V(\cdot, y) \subset \mathcal{A}(yV)$, the function

$$H(F, y) := \sum_{i\in I} \langle |F|, \psi_i\rangle m_V(x_i, y)$$

is a finite sum over the index set $I_y := \{i,\ x_i \in \mathcal{A}(yV)\}$ for every $y \in \mathcal{G}$. Hence, by using Lemma 5.3 in the first inequality, we obtain

$$\begin{aligned} H(F, y) &= \sum_{i\in I_y} \int_{\mathcal{G}} \left|F(x)\right| \psi_i(x)\, \mathrm{d}x\, m_V(x_i, y) = \int_{\mathcal{A}(yVU)} \left|F(x)\right| \sum_{i\in I_y} \psi_i(x) m_V(x_i, y)\, \mathrm{d}x \\ &\leqslant \int_{\mathcal{A}(yVU)} \left|F(x)\right| m_{V^3U}(x, y)\, \mathrm{d}x \leqslant \int_{\mathcal{G}} \left|F(x)\right| \int_{\mathcal{A}} \chi_{V^3U}\big(y^{-1}A(x)\big)\, \mathrm{d}A\, \mathrm{d}x \\ &= \int_{\mathcal{A}} \int_{\mathcal{G}} L_y \chi_{V^3U}(Ax) \left|F(Ax)\right| \mathrm{d}x\, \mathrm{d}A = \int_{\mathcal{G}} L_y \chi_{V^3U}(x) \left|F(x)\right| \mathrm{d}x = |F| * \chi^{\vee}_{V^3U}(y). \end{aligned}$$

By solidity of Y, Lemma 5.4 and (2.4) we finally conclude

$$\left\| \Lambda \mid Y^d_{\mathcal{A}} \right\| \leqslant C_1^{-1} \left\| H(F, \cdot) \mid Y_{\mathcal{A}} \right\| \leqslant C_1^{-1} \left\| |F| * \chi^{\vee}_{V^3U} \mid Y_{\mathcal{A}} \right\| \leqslant C_1^{-1} \|F \mid Y_{\mathcal{A}}\| \left\| \chi_{V^3U} \mid L^1_w(\mathcal{G}) \right\|. \qquad \square$$

Proposition 6.2. *Let $X = (x_i)_{i\in I}$ be a well-spread set in $\mathcal{G}$ (with respect to $\mathcal{A}$) and let $G \in W^R_{\mathcal{A}}(C_0, L^1_w)$. Then the mapping*

$$\Lambda = (\lambda_i)_{i\in I} \mapsto \sum_{i\in I} \lambda_i \mathcal{L}_{x_i} G$$

is a bounded, linear operator from $Y^d_{\mathcal{A}}(X)$ into $Y_{\mathcal{A}}$ satisfying

$$\left\| \sum_{i\in I} \lambda_i \mathcal{L}_{x_i} G \mid Y_{\mathcal{A}} \right\| \leqslant C \left\| G \mid W^R_{\mathcal{A}}\big(C_0, L^1_w\big) \right\| \left\| \Lambda \mid Y^d_{\mathcal{A}} \right\|$$

with some constant C independent of Λ. The sum always converges pointwise, and in the norm of Y, if the finite sequences are dense in $Y^d_{\mathcal{A}}$.

Proof. Put $\mu_\Lambda = \sum_{i\in I} \lambda_i \varepsilon_{\mathcal{A}x_i}$. By Lemma 5.5 this measure is contained in $W_{\mathcal{A}}(M, Y)$. Furthermore, we have $\sum \lambda_i \mathcal{L}_{x_i} G = \mu_\Lambda * G$. Hence, by Proposition 5.1(a) and again Lemma 5.5, we have

$$\left\| \sum_{i\in I} \lambda_i \mathcal{L}_{x_i} G \mid Y_{\mathcal{A}} \right\| \leqslant C \| \mu_\Lambda \mid W_{\mathcal{A}}(M, Y) \| \| G \mid W_{\mathcal{A}}^R(C_0, L_w^1) \| \leqslant C C_2 \| \Lambda \mid Y_{\mathcal{A}}^d \| \| G \mid W_{\mathcal{A}}^R(C_0, L_w^1) \|.$$

If the finite sequences are dense in $Y_{\mathcal{A}}^d$, the norm convergence in Y is clear. Since $Y_{\mathcal{A}}^d \subset l_{1/r}^\infty$ (Lemma 5.6(b)) and $(\mathcal{L}_{x_i} G(x))_{i\in I} \in l_r^1$ (Lemma 5.6(c)) for all $x \in \mathcal{G}$, where $r(i) = w(x_i)|\mathcal{A}(x_i U)|$, the pointwise convergence follows by l_r^1–$l_{1/r}^\infty$-duality. □

Corollary 6.3. *Suppose that Ψ is a U-$\mathcal{A}$-IBUPU and χ_V is taken as window function for the definition of the norm of $Y_{\mathcal{A}}^d$. Further assume $G \in W_{\mathcal{A}}^R(C_0, L_w^1)$. Then T_Ψ is bounded from Y into Y with operator norm*

$$\|T_\Psi \mid Y \to Y\| \leqslant C \| \chi_{VU^3} \mid L_w^1(\mathcal{G}) \| \| G \mid W^R(C_0, L_w^1) \|$$

where C is some constant independent of G, U and V.

Proof. The assertion follows from Propositions 6.1 and 6.2. □

If $U \subset U_0$, then a U-IBUPU is also an U_0-IBUPU. Hence, we immediately obtain the following corollary.

Corollary 6.4. *The family of operators $(T_\Psi)_\Psi$ where Ψ runs through a system of U_0-IBUPUs is uniformly bounded.*

We shall make use of the following maximal function (see also Definition 4.5 in [12]).

Definition 6.1. If $U \subset \mathcal{G}$ is a relatively compact neighborhood of e, then

$$G_U^\#(x) := \sup_{u\in U} |G(ux) - G(x)|$$

is the *U-oscillation* of G.

We remark that $G_U^\#$ is invariant under $\mathcal{A}$ whenever G is invariant and $U = \mathcal{A}(U)$. In [12] one finds the following lemma.

Lemma 6.5 [12, Lemma 4.6]. (a) *A function G is in $W^R(L^\infty, L_w^1)$ if and only if $G \in L_w^1$ and $G_U^\# \in L_w^1$ for some (and hence for all) open relatively compact neighborhood U of e.*

(b) *If, in addition, G is continuous (i.e., $G \in W^R(C_0, L_w^1)$), then*

$$\lim_{U\to\{e\}} \| G_U^\# \mid L_w^1 \| = 0. \tag{6.1}$$

(c) *If $y \in xU$, then $|L_y G - L_x G| \leqslant L_y G_U^\#$ holds pointwise.*

Corollary 6.6. *If G is $\mathcal{A}$-invariant and $y \in \mathcal{A}(xU)$, then $|\mathcal{L}_y G - \mathcal{L}_x G| \leqslant \mathcal{L}_y G_U^\#$ holds pointwise.*

Proof. Since $y \mapsto \mathcal{L}_y G$ is invariant under $\mathcal{A}$, it is enough to consider $y \in xU$. In this case, Lemma 6.5(c) implies

$$|\mathcal{L}_y G - \mathcal{L}_x G|(z) = \left| \int_{\mathcal{A}} G\big(y^{-1}Az\big) - G\big(x^{-1}Az\big)\,\mathrm{d}A \right| \leqslant \int_{\mathcal{A}} \big|L_y G(Az) - L_x G(Az)\big|\,\mathrm{d}A$$
$$\leqslant \int_{\mathcal{A}} L_y G_U^{\#}(Az)\,\mathrm{d}A = \mathcal{L}_y G_U^{\#}(z). \qquad \square$$

For the following we consider families of operators T_Ψ where Ψ runs through a system of IBUPUs. We write $\Psi \to 0$, if, for the corresponding neighborhoods U of e, we have $U \to \{e\}$.

Theorem 6.7. *Assume that $\Psi = (\psi_i)_{i\in I}$ is a U-$\mathcal{A}$-IBUPU for some set $U = \mathcal{A}(U)$ and $G \in W_{\mathcal{A}}^R(C_0, L_w^1)$. Then we have*

$$\|T - T_\Psi \mid Y_{\mathcal{A}} \to Y_{\mathcal{A}}\| \leqslant \big\|G_U^{\#} \mid L_w^1\big\|$$

and, as s consequence of (6.1),

$$\lim_{\Psi \to 0} \|T - T_\Psi \mid Y_{\mathcal{A}} \to Y_{\mathcal{A}}\| = 0.$$

Proof. We have

$$|TF - T_\Psi F| = \left| \sum_{i\in I} \int_{\mathcal{G}} F(y)\psi_i(y)(\mathcal{L}_y G - \mathcal{L}_{x_i} G)\,\mathrm{d}y \right| \leqslant \sum_{i\in I} \int_{\mathcal{G}} \big|F(y)\big|\psi_i(y)|\mathcal{L}_y G - \mathcal{L}_{x_i} G|\,\mathrm{d}y.$$

Since $\operatorname{supp} \psi_i \in \mathcal{A}(x_i U)$, we obtain from Corollary that 6.6

$$|TF - T_\Psi F| \leqslant \sum_{i\in I} \int_{\mathcal{G}} \big|F(y)\big|\psi_i(y)\mathcal{L}_y G_U^{\#}\,\mathrm{d}y = \int_{\mathcal{G}} \big|F(y)\big|\mathcal{L}_y G_U^{\#}\,\mathrm{d}y = |F| * G_U^{\#}$$

and, finally, by (2.4)

$$\|TF - T_\Psi F \mid Y_{\mathcal{A}}\| \leqslant \|F \mid Y_{\mathcal{A}}\| \big\|G_U^{\#} \mid L_w^1\big\|.$$

This gives the estimate for the operator norm. $\square$

Let us now consider the operators S_Ψ and U_Ψ. Let us first prove their boundedness.

Proposition 6.8. *Suppose that Ψ is a U-$\mathcal{A}$-IBUPU.*

(a) *If $G \in (L_w^1)_{\mathcal{A}}$, then S_Ψ is a bounded operator from $W_{\mathcal{A}}(C_0, Y)$ into $Y_{\mathcal{A}}$ and*

$$\big\|S_\Psi \mid W_{\mathcal{A}}(C_0, Y) \to Y_{\mathcal{A}}\big\| \leqslant \gamma(U)\big\|G \mid L_w^1\big\|,$$

where $\gamma(U)$ is the constant from Lemma 5.7.

(b) *If $G \in W_{\mathcal{A}}^R(C_0, L_w^1)$, then U_Ψ is a bounded operator from $W_{\mathcal{A}}(C_0, Y)$ into $Y_{\mathcal{A}}$ and*

$$\big\|U_\Psi \mid W_{\mathcal{A}}(C_0, Y) \to Y_{\mathcal{A}}\big\| \leqslant \gamma(U)\big(\big\|G \mid L_w^1\big\| + \big\|G_U^{\#} \mid L_w^1\big\|\big), \tag{6.2}$$

where $\gamma(U)$ is again the constant from Lemma 5.7.

Proof. (Analogously to the proof of Proposition 4.8 in [12].) (a) We use the convolution relation (2.4), the norm estimate $\|F \mid Y\| \leqslant \|F \mid W(C_0, Y)\|$ (Lemma 3.9(a) in [9]) and Lemma 5.7 to obtain for $F \in W_{\mathcal{A}}(C_0, Y)$

$$\|S_\Psi F \mid Y_{\mathcal{A}}\| = \left\| \left(\sum_{i\in I} F(x_i)\psi_i \right) * G \mid Y_{\mathcal{A}} \right\| \leqslant \left\| \sum_{i\in I} F(x_i)\psi_i \mid Y_{\mathcal{A}} \right\| \|G \mid L^1_w\|$$
$$\leqslant \left\| \sum_{i\in I} F(x_i)\psi_i \mid W_{\mathcal{A}}(C_0, Y) \right\| \|G \mid L^1_w\| \leqslant \gamma(U) \|F \mid W_{\mathcal{A}}(C_0, Y)\| \|G \mid L^1_w\|. \tag{6.3}$$

(b) Since $\operatorname{supp} \psi_i \subset \mathcal{A}(x_i U)$, we may estimate by Corollary 6.6

$$|c_i \mathcal{L}_{x_i} G - \psi_i * G| = \left| \int_{\mathcal{G}} \psi_i(y)(\mathcal{L}_{x_i} G - \mathcal{L}_y G)\, dy \right| \leqslant \int_{\mathcal{G}} \psi_i(y) \mathcal{L}_y G^\#_U \, dy = \psi_i * G^\#_U.$$

Hence,

$$\|U_\Psi F - S_\Psi F \mid Y_{\mathcal{A}}\| = \left\| \sum_{i\in I} F(x_i)(c_i \mathcal{L}_{x_i} G - \psi_i * G) \mid Y_{\mathcal{A}} \right\| \leqslant \left\| \left(\sum_{i\in I} |F(x_i)| \psi_i \right) * G^\#_U \mid Y_{\mathcal{A}} \right\|.$$

As in (6.3) we obtain

$$\|U_\Psi F - S_\Psi F \mid Y_{\mathcal{A}}\| \leqslant \gamma(U) \|F \mid W_{\mathcal{A}}(C_0, Y)\| \|G^\#_U \mid L^1_w\| \tag{6.4}$$

giving (6.2) by the triangle inequality and (6.3). □

For the analysis of the operator S_Ψ we need to restrict to the subspace $Y_{\mathcal{A}} * G$, where in the original setting $G = \tilde{V}_g g$ with $\|Sg\| = 1$ implying $G = G^\nabla = G * G$.

Theorem 6.9. *Suppose that $G \in W^R_{\mathcal{A}}(C_0, L^1_w)$ with $G = G^\nabla = G * G$ and that Ψ is a U-$\mathcal{A}$-IBUPU. Then*

$$\|T - S_\Psi \mid Y_{\mathcal{A}} * G \to Y_{\mathcal{A}} * G\| \leqslant \|G^\#_U \mid L^1_w\| \|G \mid L^1_w\|.$$

In particular, we have $\lim_{\Psi \to 0} \|T - S_\Psi \mid Y_{\mathcal{A}} * G \to Y_{\mathcal{A}} * G\| = 0$.

Proof. (Similar to the proof of Theorem 4.11 in [12].) Suppose $F \in Y_{\mathcal{A}} * G$. Using the reproducing property $F * G = F$ and the convolution relation (2.4) we obtain

$$\|TF - S_\Psi F \mid Y_{\mathcal{A}}\| \leqslant \left\| F - \sum_{i\in I} F(x_i)\psi_i \mid Y_{\mathcal{A}} \right\| \|G \mid L^1_w\|.$$

Since $F \in Y_{\mathcal{A}} * G \subset W_{\mathcal{A}}(C_0, Y)$ (Proposition 5.1(b)), the expression on the right-hand side is well defined by Lemma 5.7. Moreover, if $y \in \mathcal{A}(x_i U)$, one obtains as in [12] (additionally using the $\mathcal{A}$-invariance of F) $|F(y) - F(x_i)| \leqslant |F| * (G^\#_U)^\vee(y)$ and, hence,

$$\left| F(y) - \sum_{i\in I} F(x_i)\psi_i(y) \right| \leqslant \sum_{i\in I} |F(y) - F(x_i)| \psi_i(y) \leqslant \sum_{i\in I} |F| * \left(G^\#_U\right)^\vee(y) \psi_i(y)$$
$$= |F| * \left(G^\#_U\right)^\vee(y).$$

Finally, this gives

$$\|TF - S_\Psi F \mid Y_\mathcal{A}\| \leqslant \big\||F| * \big(G_U^\#\big)^\vee \mid Y_\mathcal{A}\big\| \big\|G \mid L_w^1\big\| \leqslant \|F \mid Y_\mathcal{A}\| \big\|G_U^\# \mid L_w^1\big\| \big\|G \mid L_w^1\big\|.$$

The last assertion of the theorem follows with Lemma 6.5(b). □

Theorem 6.10. *Suppose that $G \in W_\mathcal{A}^R(C_0, L_w^1)$ with $G = G^\nabla = G * G$ and let Ψ be a U-$\mathcal{A}$-IBUPU. Then*

$$\|T - U_\Psi \mid Y_\mathcal{A} * G \to Y_\mathcal{A} * G\| \leqslant \big\|G_U^\# \mid L_w^1\big\| \big(\big\|G \mid L_w^1\big\| + \gamma(U) D \big\|G \mid W_\mathcal{A}^R\big(C_0, L_w^1\big)\big\|\big),$$

where $\gamma(U)$ is the constant from Lemma 5.7 *and D is the constant in Proposition* 5.1(b). *In particular, we have* $\lim_{\Psi \to 0} \|T - U_\Psi \mid Y_\mathcal{A} * G \to Y_\mathcal{A} * G\| = 0$.

Proof. (Analogous to the proof of Theorem 4.13 in [12].) Suppose $F \in Y_\mathcal{A} * G$. Using the reproducing formula $F * G = F$, (6.4) and Proposition 5.1(b) we obtain

$$\begin{aligned}\|U_\Psi F - S_\Psi F \mid Y_\mathcal{A}\| &\leqslant \gamma(U) \big\|F \mid W_\mathcal{A}(C_0, Y)\big\| \big\|G_U^\# \mid L_w^1\big\| \\ &= \gamma(U) \big\|F * G \mid W_\mathcal{A}(C_0, Y)\big\| \big\|G_U^\# \mid L_w^1\big\| \\ &\leqslant \gamma(U) D \|F \mid Y_\mathcal{A}\| \big\|G \mid W_\mathcal{A}^R\big(C_0, L_w^1\big)\big\| \big\|G_U^\# \mid L_w^1\big\|.\end{aligned}$$

Together with Theorem 6.9 and the triangle inequality we obtain the desired estimation. Since $\gamma(U) \leqslant \gamma_0$ when U runs through a family of subsets of some U_0 (Lemma 5.7), the last assertion follows from Lemma 6.5(b). □

7. Atomic decompositions and Banach frames

After all preparation we establish atomic decompositions and Banach frames for the coorbit spaces $\mathsf{Co}Y_\mathcal{A}$ in this section. As usual Y has an associated weight function w. Also recall definition (5.9) of $\mathbb{B}_w^\mathcal{A}$. We remark that one can easily adapt the proof of Lemma 6.1 in [8] to show that $\mathbb{B}_w^\mathcal{A}$ is dense in $\mathcal{H}_\mathcal{A}$. In particular, there exist nontrivial vectors in $\mathbb{B}_w^\mathcal{A}$. Analogously to Theorem T in [12] we obtain the following.

Theorem 7.1. *Suppose that $g \in \mathbb{B}_w^\mathcal{A}$ with $\|Sg\| = 1$ and let $G := \tilde{V}_g g$. Choose further a relatively compact neighborhood $U = U^{-1} = \mathcal{A}(U)$ of $e \in \mathcal{G}$ such that*

$$\big\|G_U^\# \mid L_w^1\big\| < 1. \tag{7.1}$$

Then for any U-dense well-spread family $X = (x_i)_{i \in I}$ (with respect to $\mathcal{A}$), the coorbit space $\mathsf{Co}Y_\mathcal{A}$ has the following atomic decomposition: if $f \in \mathsf{Co}Y_\mathcal{A}$, then

$$f = \sum_{i \in I} \lambda_i(f) \tilde{\pi}(x_i) g,$$

where the sequence of coefficients $\Lambda(f) = (\lambda_i(f))_{i \in I}$ depends linearly on f and satisfies

$$\big\|\Lambda(f) \mid Y_\mathcal{A}^d\big\| \leqslant C_1 \|f \mid \mathsf{Co}Y_\mathcal{A}\|,$$

with a constant depending only on g.

Conversely, if $\Lambda = (\lambda_i)_{i\in I} \in Y^d_{\mathcal{A}}$, *then* $f = \sum_{i\in I} \lambda_i \tilde{\pi}(x_i)g$ *is contained in* $\mathrm{Co}Y_{\mathcal{A}}$ *and*

$$\|f \mid \mathrm{Co}Y_{\mathcal{A}}\| \leqslant C_2 \big\|\Lambda \mid Y^d_{\mathcal{A}}\big\|.$$

The sum converges in the norm of $\mathrm{Co}Y_{\mathcal{A}}$, *if the finite sequences are dense in* $Y^d_{\mathcal{A}}$ *and in the weak-*$*$*-topology of* $(\mathcal{H}^1_w)^\urcorner_{\mathcal{A}}$ *otherwise.*

Proof. The restriction of the operator $TF := F * G$ to the closed subspace $Y_{\mathcal{A}} * G$ is the identity, since $G = G * G$ by the reproducing formula (2.7). By the assumption on $G^\#_U$ and Theorem 6.7 we have $\|T - T_\Psi \mid Y_{\mathcal{A}} * G \to Y_{\mathcal{A}} * G\| < 1$ and, hence, T_Ψ is invertible on $Y_{\mathcal{A}} * G$ (by means of the von Neumann series). Further, if $f \in \mathrm{Co}Y_{\mathcal{A}}$, then $\tilde{V}_g f \in Y_{\mathcal{A}} * G$ and

$$\tilde{V}_g f = T_\Psi T_\Psi^{-1} \tilde{V}_g f = \sum_{i\in I} \big\langle T_\Psi^{-1} \tilde{V}_g f, \psi_i \big\rangle \mathcal{L}_{x_i} V_g g.$$

Since $\mathcal{L}_{x_i} \tilde{V}_g g = \tilde{V}_g(\tilde{\pi}(x_i)g)$ and since $\tilde{V}_g$ is an isometric isomorphism between $\mathrm{Co}Y_{\mathcal{A}}$ and $Y_{\mathcal{A}} * G$ (Proposition 3.1), we obtain

$$f = \sum_{i\in I} \big\langle T_\Psi^{-1} \tilde{V}_g f, \psi_i \big\rangle \tilde{\pi}(x_i)g.$$

Set $\lambda_i := \langle T_\Psi^{-1} \tilde{V}_g f, \psi_i \rangle$. From the relation $T_\Psi^{-1} \tilde{V}_g f \in Y_{\mathcal{A}} * G \subset Y_{\mathcal{A}}$ and Proposition 6.1, we conclude

$$\big\|(\lambda_i)_{i\in I} \mid Y^d_{\mathcal{A}}\big\| \leqslant C \big\|T_\Psi^{-1} \tilde{V}_g f \mid Y_{\mathcal{A}}\big\| \leqslant C \big\|T_\Psi^{-1} \mid Y_{\mathcal{A}} \to Y_{\mathcal{A}}\big\| \|f \mid \mathrm{Co}Y_{\mathcal{A}}\|.$$

For a converse inequality we apply $\tilde{V}_g$ to the series to obtain

$$F(x) := \tilde{V}_g\bigg(\sum_{i\in I} \lambda_i \tilde{\pi}(x_i)g\bigg)(x) = \sum_{i\in I} \lambda_i \mathcal{L}_{x_i} G(x). \tag{7.2}$$

Since $Y^d_{\mathcal{A}} \subset l^\infty_{1/r}$, with $r(i) = w(x_i)|\mathcal{A}(x_i U)|$ and $G \in W^R_{\mathcal{A}}(C_0, L^1_w)$, the right-hand side of (7.2) converges pointwise and defines a function in $L^\infty_{1/w}(\mathcal{G})$ by (5.5). By Theorem 4.1(v) in [9] the pointwise convergence of the partial sums of F implies the weak-$*$-convergence of $f := \sum_{i\in I} \lambda_i \tilde{\pi}(x_i)g$. Once f is identified with an element of $(\mathcal{H}^1_w)^\urcorner_{\mathcal{A}}$ it belongs to $\mathrm{Co}Y_{\mathcal{A}}$ by Proposition 6.2 (which also implies the stated type of convergence). The constant C_2 equals $C\|G \mid W^R_{\mathcal{A}}(C_0, L^1_w)\|$, where C is the constant from Proposition 5.1. □

The next theorem establishes the existence of Banach frames for $\mathrm{Co}Y_{\mathcal{A}}$ analogously to Theorem S in [12]. In contrast to the preceding theorem the corresponding sequence space will be $Y^b_{\mathcal{A}}$ instead of $Y^d_{\mathcal{A}}$, which is a difference to the classical theory [12], where the corresponding spaces for atomic decompositions and Banach frames coincide.

Theorem 7.2. *Suppose that* $g \in \mathbb{B}^{\mathcal{A}}_w$ *with* $\|Sg\| = 1$ *and set* $G := \tilde{V}_g g$. *Choose further a relatively compact neighborhood* $U = U^{-1} = \mathcal{A}(U)$ *of* $e \in \mathcal{G}$ *such that*

$$\big\|G^\#_U \mid L^1_w\big\| < \frac{1}{\|G \mid L^1_w\|}. \tag{7.3}$$

Then for any U*-dense well-spread family* $X = (x_i)_{i\in I}$ *in* $\mathcal{G}$ *the set* $\{\tilde{\pi}(x_i)g,\ i \in I\}$ *is a Banach frame for* $\mathrm{Co}Y_{\mathcal{A}}$. *This means that*

(a) *$f \in \mathsf{Co}Y_{\mathcal{A}}$ if and only if $(\langle f, \tilde{\pi}(x_i)g\rangle)_{i\in I} \in Y^b_{\mathcal{A}}$;*
(b) *there exist constants $C_1, C_2 > 0$ depending on $g \in \mathbb{B}^{\mathcal{A}}_w$ such that*

$$C_1\|f \mid \mathsf{Co}Y_{\mathcal{A}}\| \leqslant \left\|\left(\langle f, \tilde{\pi}(x_i)g\rangle\right)_{i\in I} \mid Y^b_{\mathcal{A}}\right\| \leqslant C_2\|f \mid \mathsf{Co}Y_{\mathcal{A}}\|;$$

(c) *there exists a bounded linear operator $\Omega : Y^b_{\mathcal{A}} \to \mathsf{Co}Y_{\mathcal{A}}$, such that $\Omega((\langle f, \tilde{\pi}(x_i)\rangle)_{i\in I}) = f$ for all $f \in \mathsf{Co}Y$. If the finite sequences are dense in $Y^b_{\mathcal{A}}$, then this reconstruction is performed by the series*

$$f = \sum_{i\in I}\langle f, \tilde{\pi}(x_i)g\rangle e_i \tag{7.4}$$

with elements $e_i \in (\mathcal{H}^1_w)_{\mathcal{A}},\ i \in I$, and with convergence in $\mathsf{Co}Y_{\mathcal{A}}$.

Proof. By Theorem 6.9 condition (7.3) implies that S_Ψ is invertible on $Y_{\mathcal{A}} * G$. For $F = \tilde{V}_g f$ it therefore holds

$$F = S_\Psi^{-1}S_\Psi F = S_\Psi^{-1}\left(\sum_{i\in I} F(x_i)\psi_i * G\right). \tag{7.5}$$

By the correspondence principle (Proposition 3.1(b)) we obtain

$$f = \tilde{V}_g^{-1}S_\Psi^{-1}\left(\sum_{i\in I}\langle f, \tilde{\pi}(x_i)g\rangle\psi_i * G\right). \tag{7.6}$$

This is a reconstruction of f from the coefficients $(\langle f, \tilde{\pi}(x_i)g\rangle)_{i\in I}$. The reconstruction operator may be written as $\Omega = \tilde{V}_g^{-1}S_\Psi^{-1}TH$, where $H : Y^b_{\mathcal{A}} \to Y$ is defined by $H((\lambda_i)_{i\in I}) := \sum_{i\in I}\lambda_i\psi_i$. Since $\psi_i \leqslant \chi_{\mathcal{A}(x_iU)}$, the operator H is bounded by definition of $Y^b_{\mathcal{A}}$. Hence, also Ω is bounded as the composition of bounded operators.

Letting $Y = L^\infty_{1/w}$, we see that any $f \in \mathsf{Co}(L^\infty_{1/w})_{\mathcal{A}} = (\mathcal{H}^1_w)^\urcorner_{\mathcal{A}}$ (Corollary 4.4(a) in [9]) can be reconstructed as in (7.6). Now, if $(\tilde{V}_g f(x_i))_{i\in I} \in Y^b_{\mathcal{A}}$ holds for $f \in (\mathcal{H}^1_w)^\urcorner_{\mathcal{A}}$, the series in (7.5) converges to a function in $W_{\mathcal{A}}(C_0, Y) * G \subset Y_{\mathcal{A}} * G$ by Lemma 5.7. By the invertibility of S_Ψ on $Y_{\mathcal{A}} * G$ the function $\tilde{V}_g f$ is therefore contained in $Y_{\mathcal{A}} * G$, hence $f \in \mathsf{Co}Y_{\mathcal{A}}$. Together with Theorem 5.8 this shows (a).

From (7.5) we obtain the equivalence of norms,

$$\begin{aligned}
\|f \mid \mathsf{Co}Y_{\mathcal{A}}\| = \|F \mid Y_{\mathcal{A}}\| &\leqslant \left\|S_\Psi^{-1} \mid Y_{\mathcal{A}} * G \to Y_{\mathcal{A}} * G\right\|\left\|\sum_{i\in I} F(x_i)\psi_i * G \mid Y_{\mathcal{A}}\right\| \\
&\leqslant \left\|S_\Psi^{-1}\right\|\left\|\sum_{i\in I} F(x_i)\psi_i \mid Y_{\mathcal{A}}\right\|\left\|G \mid L^1_w\right\| \leqslant \left\|S_\Psi^{-1}\right\|\left\|\sum_{i\in I}\left|F(x_i)\right|\chi_{\mathcal{A}(x_iU)} \mid Y_{\mathcal{A}}\right\|\left\|G \mid L^1_w\right\| \\
&= \left\|S_\Psi^{-1}\right\|\left\|G \mid L^1_w\right\|\left\|\left(F(x_i)\right)_{i\in I} \mid Y^b_{\mathcal{A}}\right\| \leqslant \gamma(U)C\left\|S_\Psi^{-1}\right\|\left\|G \mid L^1_w\right\|\|f \mid \mathsf{Co}Y_{\mathcal{A}}\|.
\end{aligned}$$

Here, we used (2.4), the definition of $Y^b_{\mathcal{A}}(X)$, and Theorem 5.8.

The proof of (7.4) in case that the finite sequences are dense in $Y^b_{\mathcal{A}}$ is completely analogous to the proof of Theorem S in [12] and, therefore, omitted. □

Finally the next theorem establishes the existence of 'dual' frames.

Theorem 7.3. *Suppose that $g \in \mathbb{B}_w^{\mathcal{A}}$ with $\|Sg\| = 1$ and set $G := \tilde{V}_g g$. Choose further a relatively compact neighborhood $U = U^{-1} = \mathcal{A}(U)$ of $e \in \mathcal{G}$ such that*

$$\|G_U^\# \mid L_w^1\| \big(\|G \mid L_w^1\| + \gamma(U) D \|G \mid W_{\mathcal{A}}^R(C_0, L_w^1)\|\big) < 1. \tag{7.7}$$

Then for any U-dense and relatively separated family $X = (x_i)_{i \in I}$ the set $\{\tilde{\pi}(x_i)g,\ i \in I\}$ is both a set of atoms and a Banach frame for $\mathsf{Co}Y_{\mathcal{A}}$. Moreover, there exists a 'dual frame' $\{e_i,\ i \in I\} \subset (\mathcal{H}_w^1)_{\mathcal{A}}$ such that

(a) *the following norms are equivalent*:

$$\|f \mid \mathsf{Co}Y_{\mathcal{A}}\| \cong \big\|\big(\langle f, e_i\rangle\big)_{i \in I} \mid Y_{\mathcal{A}}^d\big\| \cong \big\|\big(\langle f, \tilde{\pi}(x_i)g\rangle\big)_{i \in I} \mid Y_{\mathcal{A}}^b\big\|; \tag{7.8}$$

(b) *for $f \in \mathsf{Co}Y_{\mathcal{A}}$ we have*

$$f = \sum_{i \in I} \langle f, e_i\rangle \tilde{\pi}(x_i) g,$$

with norm convergence in $\mathsf{Co}Y_{\mathcal{A}}$ if the finite sequences are dense in $Y_{\mathcal{A}}^d$, and with weak-$*$-convergence otherwise;

(c) *if the finite sequences are dense in $Y_{\mathcal{A}}^b$, then the decomposition*

$$f = \sum_{i \in I} \big\langle f, \tilde{\pi}(x_i) g\big\rangle e_i$$

is valid for $f \in \mathsf{Co}Y_{\mathcal{A}}$.

Proof. Similarly as in the two previous proofs condition (7.7) implies, by Theorem 6.10, that the operator U_Ψ is invertible on $Y_{\mathcal{A}} * G$. For $f \in \mathsf{Co}Y_{\mathcal{A}}$ and $F = \tilde{V}_g f$ we have

$$F = U_\Psi U_\Psi^{-1} F = \sum_{i \in I} \big(U_\Psi^{-1} F\big)(x_i) c_i \mathcal{L}_{x_i} G \tag{7.9}$$

and

$$F = U_\Psi^{-1} U_\Psi F = U_\Psi^{-1}\bigg(\sum_{i \in I} F(x_i) c_i \mathcal{L}_{x_i} G\bigg). \tag{7.10}$$

Now one proceeds similarly to the proofs of Theorems 7.1 and 7.2, i.e., (7.9) leads to an atomic decomposition of $\mathsf{Co}Y_{\mathcal{A}}$ and (7.10) leads to Banach frames. However, the norm estimates are slightly different, since the numbers c_i are not bounded from above in general as it is the case in the classical theory [12].

So starting from (7.9) we define $\lambda_i(f) := c_i (U_\Psi^{-1} \tilde{V}_g f)(x_i)$ yielding $f = \sum_{i \in I} \lambda_i(f) \tilde{\pi}(x_i) g$. Moreover, since $\operatorname{supp} \psi_i \subset \mathcal{A}(x_i U)$, we have $c_i \leqslant a_i = |\mathcal{A}(x_i V)|$ if $U \subset V$, and we assume, without loss of generality, that such a set V is chosen for the definition of $Y_{\mathcal{A}}^b$. Further, we have $U_\Psi^{-1} F \in W_{\mathcal{A}}(C_0, Y) \cap Y_{\mathcal{A}} * G$ by Proposition 6.8. Altogether we obtain, by using Lemma 5.7 and Proposition 5.1(b), that

$$\begin{aligned}
\big\|\big(\lambda_i(f)\big)_{i \in I} \mid Y_{\mathcal{A}}^d\big\| &\leqslant \big\|\big(\big(U_\Psi^{-1} \tilde{V}_g f\big)(x_i)\big)_{i \in I} \mid Y_{\mathcal{A}}^b\big\| \leqslant \big\|U_\Psi^{-1} \tilde{V}_g f \mid W_{\mathcal{A}}(C_0, Y)\big\| \\
&= \big\|\big(U_\Psi^{-1} \tilde{V}_g f\big) * G \mid W_{\mathcal{A}}(C_0, Y)\big\| \leqslant D \big\|U_\Psi^{-1} \tilde{V}_g f \mid Y_{\mathcal{A}}\big\| \big\|G \mid W_{\mathcal{A}}^R\big(C_0, L_w^1\big)\big\| \\
&\leqslant \big\|U_\Psi^{-1} \mid Y_{\mathcal{A}} * G \to Y_{\mathcal{A}} * G\big\| \big\|G \mid W_{\mathcal{A}}^R\big(C_0, L_w^1\big)\big\| \|f \mid \mathsf{Co}Y_{\mathcal{A}}\|.
\end{aligned} \tag{7.11}$$

The converse norm estimate is the same as in the proof of Theorem 7.1.

Beginning with (7.10), the norm estimate in the proof of the Banach frame property is obtained by

$$\begin{aligned}\|f \mid \mathrm{Co}Y_{\mathcal{A}}\| &= \|\tilde{V}_g f \mid Y_{\mathcal{A}}\| = \left\| U_\Psi^{-1}\left(\sum_{i\in I} c_i \tilde{V}_g f(x_i)\mathcal{L}_{x_i}G\right) \;\Big|\; Y_{\mathcal{A}}\right\| \\ &\leqslant \left\|U_\Psi^{-1} \mid Y_{\mathcal{A}} * G \to Y_{\mathcal{A}} * G\right\| \left\| \sum_{i\in I} c_i \tilde{V}_g f(x_i)\varepsilon_{\mathcal{A}(x_i)} * G \mid W_{\mathcal{A}}(C_0, Y)\right\| \\ &\leqslant \left\|U_\Psi^{-1}\right\| \left\|\sum_{i\in I} c_i \tilde{V}_g f(x_i)\varepsilon_{\mathcal{A}x_i} \mid W_{\mathcal{A}}(M, Y)\right\| \left\|G \mid W_{\mathcal{A}}^R\left(C_0, L_w^1\right)\right\| \\ &\leqslant C\left\|U_\Psi^{-1}\right\| \left\|G \mid W_{\mathcal{A}}^R\left(C_0, L_w^1\right)\right\| \left\|\left(\langle f, \tilde{\pi}(x_i)g\rangle\right)_{i\in I} \mid Y_{\mathcal{A}}^b\right\| \\ &\leqslant C'\gamma(U)\left\|U_\Psi^{-1}\right\| \left\|G \mid W_{\mathcal{A}}^R\left(C_0, L_w^1\right)\right\|^2 \|f \mid \mathrm{Co}Y_{\mathcal{A}}\|.\end{aligned}$$

Here, we used Proposition 5.1(a), Lemma 5.4, $c_i \leqslant a_i$ and again Theorem 5.8.

Now set $E_i := c_i U_\Psi^{-1}(\mathcal{L}_{x_i}G)$, then $E_i \in (L_w^1)_{\mathcal{A}} * G$ and $E_i = \tilde{V}_g(e_i)$ for some unique $e_i \in (\mathcal{H}_w^1)_{\mathcal{A}}$. The identity $f = \sum_{i\in I}\langle f, \tilde{\pi}(x_i)g\rangle e_i$ follows from (7.10), provided that the finite sequences are dense in $Y_{\mathcal{A}}^b$.

As in [12, Theorem U] we claim that

$$\lambda_i(f) = c_i\left(U_\Psi^{-1}V_g f\right)(x_i) = \langle f, e_i\rangle,$$

Combined with the correspondence principle, this yields $f = \sum_{i\in I}\langle f, e_i\rangle\tilde{\pi}(x_i)g$ (with weak-∗-convergence, and, if the finite sequences are dense in $Y_{\mathcal{A}}^d$, with norm convergence). For the sake of completeness we repeat Gröchenig's arguments [12].

Since $U_\Psi^{-1}F \in Y_{\mathcal{A}} * G$, Proposition 3.1(c) gives $U_\Psi^{-1}F(x_i) = \langle U_\Psi^{-1}F, \mathcal{L}_{x_i}G\rangle$. It follows that U_Ψ satisfies $\langle U_\Psi F, H\rangle = \langle F, U_\Psi H\rangle$ for all $F \in Y * G,\ H \in L_w^1 * G$:

$$\begin{aligned}\langle U_\Psi F, H\rangle &= \sum_{i\in I} c_i F(x_i)\langle\mathcal{L}_{x_i}G, H\rangle = \sum_{i\in I} c_i\langle F, \mathcal{L}_{x_i}G\rangle\langle\mathcal{L}_{x_i}G, H\rangle \\ &= \sum_{i\in I} c_i\overline{H(x_i)\langle\mathcal{L}_{x_i}G, F\rangle} = \langle F, U_\Psi H\rangle.\end{aligned}$$

Hence, the same relation applies to $U_\Psi^{-1} = \sum_{n=0}^{\infty}(Id - U_\Psi)^n$ and we conclude $\langle U_\Psi^{-1}F, \mathcal{L}_{x_i}G\rangle = \langle F, U_\Psi^{-1}\mathcal{L}_{x_i}G\rangle$. Finally,

$$c_i\left(U_\Psi^{-1}F\right)(x_i) = \left\langle F, c_i U_\Psi^{-1}\mathcal{L}_{x_i}G\right\rangle = \langle V_g f, V_g e_i\rangle = \langle f, e_i\rangle.$$

By Proposition 6.2 we have the norm estimate

$$\begin{aligned}\|f \mid \mathrm{Co}Y_{\mathcal{A}}\| &= \left\|\sum_{i\in I}\langle f, e_i\rangle\mathcal{L}_{x_i}G \mid Y_{\mathcal{A}}\right\| \leqslant C\left\|G \mid W_{\mathcal{A}}^R\left(C_0, L_w^1\right)\right\| \left\|\left(\langle f, e_i\rangle\right)_{i\in I} \mid Y_{\mathcal{A}}^d\right\| \\ &\leqslant C\left\|G \mid W_{\mathcal{A}}^R\left(C_0, L_w^1\right)\right\| \left\|\left(\left(U_\Psi^{-1}F\right)(x_i)\right)_{i\in I} \mid Y_{\mathcal{A}}^b\right\| \\ &\leqslant C\left\|U_\Psi^{-1} \mid Y_{\mathcal{A}} * G \to Y_{\mathcal{A}} * G\right\| \left\|G \mid W_{\mathcal{A}}^R\left(C_0, L_w^1\right)\right\|^2 \|f \mid \mathrm{Co}Y_{\mathcal{A}}\|\end{aligned}$$

giving the first equivalence in (7.8). Here, we used $\|(c_i\lambda_i) \mid Y_{\mathcal{A}}^d\| \leqslant \|(\lambda_i)_{i\in I} \mid Y_{\mathcal{A}}^b\|$. The second equivalence of (7.8) follows as in (7.11). □

So with these three theorems we settled the existence of atomic decompositions and Banach frames for coorbit spaces consisting of invariant elements. Moreover, given an element $g \in \mathbb{B}_w^{\mathcal{A}}$, with (7.1), (7.3), and (7.7), we have explicit conditions on the density of the point set $(x_i)_{i\in I}$ such that $(\tilde{\pi}(x_i)g)_{i\in I}$ forms a set of atoms and/or a Banach frame. Here, we have quite some freedom for the choice of $(x_i)_{i\in I}$. We only have to make sure that it is a U-dense and relatively separated set (with respect to $\mathcal{A}$).

As in Example 2.1, we consider $\mathcal{G} = \mathbb{R}^d \rtimes (\mathbb{R}_+^* \times SO(d))$, its representation on $L^2(\mathbb{R}^d)$ (the corresponding transform being the continuous wavelet transform) and the automorphism group $SO(d)$ (see also [17]). Then Theorems 7.1–7.3 yield atomic decompositions and Banach frames for subspaces of the homogeneous Besov spaces $\dot{B}_s^{p,q}(\mathbb{R}^d)$ and of the homogeneous Triebel–Lizorkin spaces $\dot{F}_s^{p,q}(\mathbb{R}^d)$ consisting of radial functions. In particular, if g is contained in $\mathbb{B}_w^{\mathcal{A}}$ (for instance a radial Schwartz function with infinitely many vanishing moments), then Theorem 7.3 implies the existence of constants $a > 0, b > 1$ such that the system $\{\tau_{ab^{-j}k} D_{b^{-j}} g,\ k \in \mathbb{N}_0,\ j \in \mathbb{Z}\}$ forms a Banach frame and an atomic decomposition for $\dot{B}_s^{p,q}(\mathbb{R}^d)$ and $\dot{F}_s^{p,q}(\mathbb{R}^d)$. Here, τ denotes the generalized translation defined in Example 2.1. We emphasize again that each element of this Banach frame is a radial function. Also the atomic decomposition developed in [5] is of the same type as in Theorem 7.1. However, Theorems 7.1–7.3 show that we have much more freedom on the choice of g and on the point set than in [5], where g is supposed to be compactly supported in the Fourier domain and the point set is $(2^j x_n e_1, 2^j)_{j\in\mathbb{Z},\, n\in N}$, where x_n is the nth zero of some Bessel function of the first kind and e_1 the first unit vector.

Taking $\mathcal{G}$ to be the d-dimensional Heisenberg group, $\mathcal{A} = SO(d)$ and the Schrödinger-representation on $L^2(\mathbb{R}^d)$ (see [17] for details) we obtain atomic decompositions and Banach frames for subspaces of the modulation spaces $M_s^{p,q}(\mathbb{R}^d)$ consisting of radial functions. Of course, also here each element of the atomic decomposition and the Banach frame is a radial function [17]. Such atomic decompositions were not known before and will be studied in detail elsewhere, see also [18].

Of course, Hilbert space theory is also contained in our abstract theorems yielding (Hilbert) frames for $\mathcal{H}_{\mathcal{A}}$. However, in order to fit into the classical frame theory, we have to renormalize. If $Y = L^2(\mathcal{G})$, then $Y_{\mathcal{A}}^b = l_\nu^2$, where $\nu(i) = a_i^{1/2} = |\mathcal{A}(x_i U)|^{1/2}$. Theorem 7.2 yields (under the stated conditions)

$$C_1 \| f \mid \mathcal{H}_{\mathcal{A}} \| \leqslant \sum_{i\in I} \left| \left\langle f, \sqrt{a_i}\, \tilde{\pi}(x_i) g \right\rangle \right|^2 \leqslant C_2 \| f \mid \mathcal{H}_{\mathcal{A}} \|.$$

Hence, $\{\sqrt{a_i}\, \tilde{\pi}(x_i) g,\ i \in I\}$ is a frame (in the usual sense) for $\mathcal{H}_{\mathcal{A}}$ with frame constants C_1, C_2.

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