On Acyclic Orientations and Sequential Dynamical Systems

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Received February 15, 2000; accepted February 1, 2001

We study a class of discrete dynamical systems that consists of the following data:

(a) a finite (labeled) graph \( Y \) with vertex set \( \{1, \ldots, n\} \), where each vertex has a binary state, (b) a vertex labeled multi-set of functions \( \{F_i : \mathbb{F}_2 \rightarrow \mathbb{F}_2\} \), and (c) a permutation \( \pi \in \text{Sn} \). The function \( F_i \) updates the binary state of vertex \( i \) as a function of the states of vertex \( i \) and its \( Y \)-neighbors and leaves the states of all other vertices fixed. The permutation \( \pi \) represents a \( Y \)-vertex ordering according to which the functions \( F_i \) are applied. By composing the functions \( F_i \) in the order given by \( \pi \) we obtain the sequential dynamical system (SDS):

\[
[F_{\pi(1)}, \ldots, F_{\pi(n)}] : \mathbb{F}_2^2 \rightarrow \mathbb{F}_2^2.
\]

In this paper we first establish a sharp, combinatorial upper bound on the number of non-equivalent SDSs for fixed graph \( Y \) and multi-set of functions \( \{F_i\} \). Second, we analyze the structure of a certain class of fixed-point-free SDSs.

Key Words: acyclic orientations; sequential dynamical system; orderings; graph automorphisms.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let \( Y \) be a loop-free, labeled, undirected graph with vertex set \( v[Y] = \{1, \ldots, n\} \) and edge set \( e[Y] \). In particular, let Line\(_n\) be the graph with edge set \( \{\{i, i+1\} \mid i = 1, \ldots, n-1\} \), Circ\(_n\) the graph with edge set \( \{\{1, n\}\} \cup \{\{i, i+1\} \mid i = 1, \ldots, n-1\} \), Wheel\(_n\) the vertex join of Circ\(_n\) and 0, and finally Star\(_n\) the graph with vertex set \( \{1, \ldots, n\} \) and edge set \( \{\{1, i\} \mid i = 2, \ldots n\} \). We denote the set of \( Y \)-vertices adjacent to vertex \( i \) by \( S_i \), \( B_i = S_i \cup \{i\} \) and set \( \delta_i = |S_i| \), \( d(Y) = \max_{1 \leq i \leq n} \delta_i \). To emphasize the underlying base graph we will sometimes refer to \( S_i, B_i \) as \( S_{1,Y}(i), B_{1,Y}(i) \). The increasing sequence of elements of the sets \( S_i \)
and $B_1(i)$ is referred to as

(1.1) \[ \tilde{S}_1(i) = (j_1, \ldots, j_{\delta_i}), \quad \tilde{B}_1(i) = (j_1, \ldots, i, \ldots, j_{\delta_i}). \]

Each vertex $i$ has associated a state $x_i \in \mathbb{F}_2$, and for each $k = 1, \ldots, d + 1$ we have a symmetric function $f_{(k)}: \mathbb{F}_2^k \rightarrow \mathbb{F}_2$. In view of (1.1) we introduce the map

\[ \text{proj}[i]: \mathbb{F}_2^k \rightarrow \mathbb{F}_2^{\delta_i+1}, \quad (x_1, \ldots, x_n) \mapsto (x_{j_1}, \ldots, x_i, \ldots, x_{j_{\delta_i}}), \]

and denote the permutation group over $k$ letters by $S_k$. For each $i$ there exists a $(Y$-local$)$ map $F_{i,Y}$ given by

\[ y_i(x) = f_{(\delta_i+1)} \circ \text{proj}[i](x) \]
\[ F_{i,Y}(x) = (x_1, \ldots, x_{i-1}, y_i(x), x_{i+1}, \ldots, x_n) \]

and we refer to the multi-set $(F_{i,Y})$, as $\tilde{\gamma}_Y$. Clearly, for each $Y < K_n$ the multi-set $(f_{(k)})_{1 \leq k \leq n}$ induces a multi-set $\tilde{\gamma}_Y$.

**Definition 1.** Let $[\tilde{\gamma}_Y, \pi]$ be the mapping

(1.2) \[ [\tilde{\gamma}_Y, \pi]: S_n \rightarrow \mathbb{F}_2^{n^2}, \quad [\tilde{\gamma}_Y, \pi] = \prod_{i=1}^n F_{\pi(i),Y} = F_{\pi(n),Y} \circ \cdots \circ F_{\pi(2),Y} \circ F_{\pi(1),Y}. \]

We call $[\tilde{\gamma}_Y, \pi]$ the sequential dynamical system (SDS) over $Y$ with respect to the ordering $\pi$.

In the following we will study SDSs that are induced by the multi-sets $(\text{nor}_{(k)})$ and $(\text{nand}_{(k)})$, where

(1.3) \[ \text{nor}_{(k)}(x_1, \ldots, x_k) = \begin{cases} 1 & \text{if } (x_1, \ldots, x_k) = (0, \ldots, 0) \\ 0 & \text{else} \end{cases} \]

(1.4) \[ \text{nand}_{(k)}(x_1, \ldots, x_k) = \begin{cases} 0 & \text{if } (x_1, \ldots, x_k) = (1, \ldots, 1) \\ 1 & \text{else}. \end{cases} \]

We will refer to these SDSs as $[\text{Nor}_Y, \pi]$ and $[\text{Nand}_Y, \pi]$, respectively.

Sequential dynamical systems have been studied in [1, 3] in the context of foundations of a theory of computer simulations and in [5] as dynamical systems.

Let the graph $Y$ and the multi-set $\tilde{\gamma}_Y$ be fixed. Obviously, an SDS $[\tilde{\gamma}_Y, \pi]$ induces the labeled digraph, $G[\tilde{\gamma}_Y, \pi]$, with vertex set $\mathbb{F}_2^n$ and edge set \{$(x, [\tilde{\gamma}_Y, \pi](x)) \mid x \in \mathbb{F}_2^n$\}. We will call $G[\tilde{\gamma}_Y, \pi]$ the phase space of $[\tilde{\gamma}_Y, \pi]$, denote its set of vertices contained in cycles by $\text{Per}[\tilde{\gamma}_Y, \pi]$, and call $G[\tilde{\gamma}_Y, \pi]$-cycles periodic orbits. A periodic orbit of size 1 is called a fixed-point. One central question in SDS analysis is that of two SDSs $[\tilde{\gamma}_Y, \pi]$ and $[\tilde{\gamma}_Y, \sigma]$ being equivalent. Equivalence of SDS is defined with
non-equivalent SDSs which is sharp for certain classes of SDS. Let Acyc
In our first result we give a combinatorial upper bound on the number of
(1.5)
\begin{align}
|E[Y, \widehat{\gamma}_Y]| & \leq \frac{1}{|\text{Aut}(Y)|} \sum_{\gamma \in \text{Aut}(Y)} |a(\gamma) \setminus Y| \\
|E[\text{Star}_n, \text{NorStar}_n]| & = \frac{1}{|\text{Aut}(\text{Star}_n)|} \sum_{\gamma \in \text{Aut}(\text{Star}_n)} |a(\gamma) \setminus \text{Star}_n)| = n.
\end{align}

In [2] one can find further analysis on the sharpness of the bound in
(1.5), which can be computed for the graphs Circ\(_n\) and Wheel\(_n\):

**Proposition 1.** Let \(n > 2\), \(\pi \in S_n\), and let \(\phi\) be the Euler \(\phi\)-function. Then the following assertions hold:

(1.7) \[
|E[\text{Circ}_n, \widehat{\gamma}_{\text{Circ}_n}]| \leq \begin{cases}
\frac{1}{2n} \sum_{d|n} \phi(d)(2^{n/d} - 2) + 2^{n/2}/4 & \text{iff } n \equiv 0 \text{ mod } 2 \\
\frac{1}{2n} \sum_{d|n} \phi(d)(2^{n/d} - 2) & \text{iff } n \equiv 1 \text{ mod } 2
\end{cases}
\]
(1.8) \[ |E[\text{Wheel}_n, \tilde{\gamma}_{\text{Wheel}_n}]| \]
\[
\leq \begin{cases} 
\frac{1}{2n} \sum_{d|n} \phi(d)(3^{n/d} - 3) + 3^{n/2}/2 & \text{iff } n \equiv 0 \text{ mod } 2 \\
\frac{1}{2n} \sum_{d|n} \phi(d)(3^{n/d} - 3) & \text{iff } n \equiv 1 \text{ mod } 2.
\end{cases}
\]

A permutation \( \pi = (i_1, \ldots, i_n) \) induces an orientation \( \mathcal{C}(Y)_\pi \) of \( Y \) by setting for \( \{i_k, i_r\} \in e[Y] \) and \( k < r, \pi(\{i_k, i_r\}) = i_k, \text{ and } \pi(t(\{i_k, i_r\})) = i_r. \) By construction \( \mathcal{C}(Y)_\pi \) is acyclic and we have a mapping \( w: S_n \rightarrow \text{Acyc}(Y), \pi \mapsto \mathcal{C}(Y)_\pi, \) which is surjective and for any \( \pi, \sigma \in S_n, \mathcal{C}_\pi = \mathcal{C}_\sigma \) implies \([\tilde{\gamma}_Y, \pi] = [\tilde{\gamma}_Y, \sigma] \). Accordingly, we obtain that

\[(1.9) \quad h: \text{Acyc}(Y) \longrightarrow \{([\tilde{\gamma}_Y, \pi] : \pi \in S_n\}, \quad \mathcal{C}_\pi \mapsto [\tilde{\gamma}_Y, \pi]\]

is well defined. Let \( \mathcal{I}(Y) \) be the set of \( Y \)-independence sets. We will next analyze the structure of SDSs that are induced by a multi-set \((f(k))_k\) such that they are fixed-point-free for any graph \( Y \):

**Theorem 2.** Let \((f(m))_m\) be a family of Boolean, symmetric functions inducing an arbitrary graph \( Y \) the fixed-point-free SDS \( [\tilde{\gamma}_Y, \pi] \). Then \([\tilde{\gamma}_Y, \pi] \) is equivalent to \([\text{Nor}_Y, \pi] \).

Suppose \([\tilde{\gamma}_Y, \pi] \) is equivalent to \([\text{Nor}_Y, \pi] \), then we have:

(a) Each periodic point of \([\tilde{\gamma}_Y, \pi] \) corresponds uniquely to a \( Y \)-independence set; i.e., there exists a bijective mapping \( \iota: \text{Per}[\tilde{\gamma}_Y, \pi] \longrightarrow \mathcal{I}(Y) \).

(b) Each \( G[\tilde{\gamma}_Y, \pi] \)-vertex is either periodic or has in-degree 0. Furthermore, \( (0) \) has maximal in-degree in \( G[\tilde{\gamma}_Y, \pi] \).

(c) Let \( Y = \text{Line}_n \) or \( Y = \text{Circ}_n \). Then \( G[\tilde{\gamma}_Y, \pi] \cong \lambda \cdot G[\tilde{\gamma}_Y, \sigma] \) implies \( \lambda((0)) = (0) \). In particular, the corresponding orbits containing \( (0) \) are isomorphic.

(d) Suppose \( \text{Aut}(Y) \) is transitive and there exist \( \rho, \sigma \in S_n \) such that \([\tilde{\gamma}_{\rho(Y)}, \sigma] = [\tilde{\gamma}_Y, \pi] \) holds. Then we have \( \rho \in \text{Aut}(Y) \) and \( \mathcal{C}(Y)_{\rho^{-1}\sigma} = \mathcal{C}(Y)_\pi \).

**2. SOME GROUP ACTIONS ON SDS**

\( S_n \) acts on the set of \( Y \)-vertices by permutation and thereby induces the natural group action on the set of all mappings \( t: \{1, \ldots, n\} \longrightarrow \mathbb{F}_2 \) given by \( \{\rho \cdot t)(i) = t(\rho^{-1}(i)) \). In particular, we may view \( t \) as an \( n \)-tuple, \((x_1, \ldots, x_n)\) and accordingly obtain the \( S_n \)-action on \( \mathbb{F}_2^n \):

\[(2.1) \quad \cdot: S_n \times \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^n, \quad (\rho, (x_j)) \mapsto \rho \cdot (x_j) = (x_{\rho^{-1}(j)}).\]

Clearly, we have $h g \cdot (x_{j}) = (x_{h^{-1} h^{-1}(j)}) = h \cdot (g \cdot (x_{j}))$. The action $\cdot : S_{n} \times F_{2}^{n} \rightarrow F_{2}^{n}$ induces an $S_{n}$-action on mappings $\Phi : F_{2}^{n} \rightarrow F_{2}^{n}$ given by

$$\{\rho \cdot \Phi \}(x_{j}) = \rho \cdot (\Phi(\rho^{-1} \cdot (x_{j}))).$$

(2.2)

**Proposition 2.** Let $Y$ be an arbitrary graph with vertex set $\{1, \ldots, n\}$ acted upon by the group $G$. Then we have the group-action

$$\cdot : S_{n} \times \{[\bar{\delta}_{\pi(Y), \sigma}] \mid \pi, \sigma \in S_{n}\} \rightarrow \{[\bar{\delta}_{\pi(Y), \sigma}] \mid \pi, \sigma \in S_{n}\}$$

(2.3)

and $\cdot$ induces by restriction the action

$$\cdot : G \times \{[\bar{\delta}_{\pi(Y), \sigma}] \mid \sigma \in S_{n}\} \rightarrow \{[\bar{\delta}_{\pi(Y), \sigma}] \mid \sigma \in S_{n}\}$$

(2.5)

Furthermore, $G$ acts naturally on $\text{Acyc}(Y)$ via $g \circ (\{i, k\}) = \circ (\{g^{-1}(i), g^{-1}(k)\})$ and $h : \text{Acyc}(Y) \rightarrow [\bar{\delta}_{\pi(Y), 1}]$ is a $G$-map.

**Proof.** We first show

$$\forall \rho \in S_{n}, i = 1, \ldots, n, \quad \rho \cdot F_{i, \pi}(\rho^{-1} \cdot (x_{i})) = F_{\rho(i), \rho(Y)}(x_{j}).$$

(2.7)

To prove (2.7) we first note that, for arbitrary $\rho \in S_{n}$, we have $\rho(B_{1, \pi}(i)) = B_{1, \rho(Y)}(\rho(i))$. In view of $(\rho^{-1} \cdot (x_{i}))_{k} = x_{\rho(i)}$ and $(\rho \cdot (y_{i}))_{\rho(i)} = y_{i}$ we derive

$$\rho \cdot F_{i, \pi}(\rho^{-1} \cdot (x_{i})) = (x_{1}, \ldots, y_{\rho(i)} = f_{\rho(B_{1, \pi}(i))}(x_{\rho(i)}), \ldots, x_{n})$$

(2.8)

$$= (x_{1}, \ldots, y_{\rho(i)} = f_{\rho(B_{1, \pi}(\rho(i)))}(x_{\rho(i)}), \ldots, x_{n}).$$

(2.9)

Now (2.7) follows in view of

$$\{x_{\rho(i)} \mid \rho(s) \in B_{1, \rho(Y)}(\rho(i))\} = \{x_{\rho(i)} \mid s \in B_{1, \pi}(i)\}.$$  

(2.10)

Obviously, (2.4) is implied by composing the corresponding local maps and it remains to prove (2.6). Since $G$ acts on $Y$ we have, for all $\rho \in G$, $B_{1, \rho(Y)}(i) = B_{1, \pi}(i)$ and since $F_{i, \pi}$ is a symmetric function we have

$$\forall \rho \in G, \quad F_{i, \rho(Y)} = F_{i, \pi}.$$  

(2.11)

Assertion (2.6) follows immediately from (2.11) and it remains to show that $h$ is a $G$-map. In view of $\circ_{g} = g \circ_{\varpi}$ and (2.6) we derive

$$h(g \circ_{\varpi}) = [\bar{\delta}_{\gamma, g \varpi}] = g \cdot [\bar{\delta}_{\gamma, \varpi}] = g \cdot h(\circ_{\varpi})$$

completing the proof of the proposition.  ■
3. PROOF OF THEOREM 1

Let $\mathcal{C}(Y)$ be an acyclic orientation of $Y$ and let $P(\mathcal{C}(Y))$ be the set of all directed $\mathcal{C}(Y)$-paths, $\pi$. Further let $\omega(\pi)$, $\tau(\pi)$, and $\ell(\pi)$ be its start-vertex, end-vertex, and length of the directed $\mathcal{C}(Y)$-path $\pi$, respectively. We consider the mapping

$$\text{rk}: v[Y] \rightarrow \mathbb{N}, \quad \text{rk}(i) = \max\{\ell(\pi) | \pi \in P(\mathcal{C}(Y))\};$$

$$\omega(\pi) \text{ is an } \mathcal{C}\text{-origin and } \tau(\pi) = i\}.$$

An acyclic orientation $\mathcal{C}$ induces a partial ordering $<_{\mathcal{C}}$, by setting $i <_{\mathcal{C}} k$ if and only if $\text{rk}(i) < \text{rk}(k)$. Since $v[Y] = \{1, \ldots, n\}$ we can consider an acyclic orientation $\mathcal{C}$ as a mapping $\mathcal{C}: e[Y] \rightarrow \mathbb{F}_2$, where

$$\mathcal{C}([i, k]) = \begin{cases} 1 & \text{if either } i >_{\mathcal{C}} k \text{ and } i > k \text{ or } k >_{\mathcal{C}} i \text{ and } k > i \\ 0 & \text{otherwise.} \end{cases}$$

According to Proposition 2 the $G$-action on $Y$ induces a $G$-action on Acyc($Y$) given by

$$g \mathcal{C}([i, k]) = \mathcal{C}([g^{-1}(i), g^{-1}(k)]).$$

We set Acyc($Y)^G = \{\mathcal{C} \in \text{Acyc}(Y) | \forall g \in G; g \mathcal{C} = \mathcal{C}\} \text{ and } \text{Fix}(g) = \text{Acyc}(Y)^{g}$. Moreover, $\pi_G: Y \rightarrow G \setminus Y$ induces the mapping

$$\omega'_G: \text{Acyc}(G \setminus Y) \rightarrow \text{Acyc}(Y), \quad \mathcal{T} \mapsto \mathcal{C},$$

where $\mathcal{C}([i, k]) = \overline{\mathcal{C}}([G(i), G(j)])$. It is immediately clear that $\omega'_G(\text{Acyc}(G \setminus Y)) \subset \text{Acyc}(Y)^G$ holds. Next we prove that $\omega_G: \text{Acyc}(G \setminus Y) \rightarrow \text{Acyc}(Y)^G$ is bijective having the inverse

$$\psi_G: \text{Acyc}(Y)^G \rightarrow \text{Acyc}(G \setminus Y), \quad \mathcal{C} \mapsto \mathcal{C}_G,$$

where $\mathcal{C}_G([G(i), G(k)]) = \mathcal{C}([i, k])$.

**Proposition 3.** Let $Y$ be an undirected graph being acted upon by the group $G$. Then $\psi_G$ is bijective and we have $\psi_G \circ \omega_G = \text{id}$ and $\omega_G \circ \psi_G = \text{id}$. In particular, $\text{Acyc}(Y)^G \neq \emptyset$ if and only if all $G$-vertex orbits are contained in $Y$-independence sets.

**Proof.** Let $\mathcal{C} \in \text{Acyc}(Y)^G$. By construction we have, for $g \in G$,

$$\mathcal{C}([g^{-1}(i), g^{-1}(k)]) = \mathcal{C}([i, k]), \text{ whence } \mathcal{C}: e[Y] \rightarrow \mathbb{F}_2 \text{ is constant on } G\text{-edge orbits.}$$

To define $\mathcal{C}_G$, let $\{G(i), G(k)\}$ be a $G \setminus Y$-edge. We select $\{j, h\} \in \pi_{G}^{-1}([G(i), G(k)])$ and set $\mathcal{C}_G([G(i), G(k)]) = \mathcal{C}([j, h])$. Since $\mathcal{C}([g^{-1}(i), g^{-1}(k)]) = \mathcal{C}([i, k])$ the mapping $\mathcal{C}_G: e(G \setminus Y) \rightarrow \mathbb{F}_2$ is well defined and for $\mathcal{C} \in \text{Acyc}(Y)^G$ the mapping $\mathcal{C} \mapsto \mathcal{C}_G$ is bijective. It remains to prove that $\mathcal{C}_G \in \text{Acyc}(G \setminus Y)$. To prove this let $L$ be a directed
$G \setminus Y$-loop w.r.t. $\mathcal{C}_G$ over the vertices $G(i_1), \ldots, G(i_s)$ and the edges $G(y_1), \ldots, G(y_t)$. Restricting $\mathcal{C}$ to the subgraph $Y' = \pi_{G}^{-1}(L)$ we obtain the acyclic orientation $\mathcal{C}'$.

**Claim.** Each vertex-orbit $G(i_j)$, $j = 1, \ldots, s$, contains only $Y'$ vertices which are not $\mathcal{C}'$-origins.

Suppose $G(i_j)$ contains a $Y'$ vertex, $k$, that is an $\mathcal{C}'$-origin. Since $L$ is an $\mathcal{C}_G$-directed loop there exists a $G \setminus Y$-vertex $G(h)$ that precedes $G(k)$ in $\mathcal{C}_G$. Since $\pi_G$ is locally surjective there exists a $Y$-edge of the form ${k', k} \in \pi^{-1}_G({G(h), G(k)})$ and we obtain $\mathcal{C}({k', k}) = \mathcal{C}({k', k}) = \mathcal{C}({G(h), G(k)})$ contradicting the fact that $k$ is an $\mathcal{C}'$-origin. Consequently, there exists no $Y'$-vertex in a $G(i_j)$-orbit that is an $\mathcal{C}'$-origin, proving the claim.

Obviously, the acyclicity of $\mathcal{C}'$ implies that there exists at least one $Y'$-vertex $i$, that is an $\mathcal{C}'$-origin, which is impossible. Therefore, $\mathcal{C} \in \text{Acyc}(Y)^G$ implies $\mathcal{C}_G \in \text{Acyc}(G \setminus Y)$, whence $\psi_G: \text{Acyc}(Y)^G \longrightarrow \text{Acyc}(G \setminus Y)$ is a well-defined bijection and $\psi_G \circ \omega_G = \text{id}$ and $\omega_G \circ \psi_G = \text{id}$ follow immediately. It is straightforward to show that $\text{Acyc}(Y)^G \neq \emptyset$ holds if and only if $G \setminus Y$ contains no loop of size 1. Obviously, the non-existence of a $G \setminus Y$-loop of size 1 is equivalent to the statement that all $G$-vertex orbits are contained in $Y$-independence sets, completing the proof of the proposition.

In [4] one can find a generalization of Proposition 3 for locally surjective graph morphisms.

An immediate consequence of Propositions 2 and 3 reads

**Corollary 1.** Let $Y$ be an undirected graph with automorphism group $G$. Then we have

$$|E[Y, \tilde{Y}_Y]| \leq \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)| = \frac{1}{|G|} \sum_{g \in G} a((g) \setminus Y).$$

**Proof.** Any $g \in G$ induces the bijective mapping $\lambda_g : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$, $\lambda_g(x_j) = g \cdot (x_j)$ (see (2.1)), and in view of Proposition 2 we have

$$\begin{array}{ccc}
g^{-1} \cdot (x_j) & \xrightarrow{[\tilde{Y}_Y, \pi]} & [\tilde{Y}_Y, \pi](g^{-1} \cdot (x_j)) \\
\lambda_g^{-1} \downarrow \downarrow & \lambda_g & \\
(x_j) & \xrightarrow{g \cdot [\tilde{Y}_Y, \pi]} & g \cdot [\tilde{Y}_Y, \pi](x_j) = g \cdot [\tilde{Y}_Y, \pi](g^{-1} \cdot (x_j)).
\end{array}$$

Accordingly, $\lambda_g : G[\tilde{Y}_Y, \pi] \rightarrow G[\tilde{Y}_Y, g \pi]$ is a digraph-isomorphism. Using Burnside’s lemma and Proposition 3 we derive

$$|E[Y, \tilde{Y}_Y]| \leq \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)| = \frac{1}{|G|} \sum_{g \in G} a((g) \setminus Y),$$

which proves the corollary. □
The second statement of Theorem 1 consists of the following

**Proposition 4.**

\[
|E[\text{Star}_n, \text{Nor}_{\text{Star}_n}]| = \frac{1}{|\text{Aut(Star}_n)|} \sum_{\gamma \in \text{Aut(Star}_n)} |a(\langle \gamma \rangle \setminus \text{Star}_n)| = n.
\]

The proof can be found in [5].

In fact, the RHS of (3.3) can be calculated efficiently for several classes of graphs. As an illustration we give a new proof of the formulas for the graphs \text{Circ}_n and \text{Wheel}_n [5] which were originally proved by a somewhat tedious computation.

**Proof of Proposition 1.** In the following we prove

\[(3.4) \quad \frac{1}{|G|} \sum_{\gamma \in G} a(\langle \gamma \rangle \setminus \text{Circ}_n) = \begin{cases} 
\frac{1}{2n} \sum_{d|n} \phi(d) \left( 2^{n/d} - 2 \right) + 2^{n/2}/4 & \text{iff } n \equiv 0 \mod 2 \\
\frac{1}{2n} \sum_{d|n} \phi(d) \left( 2^{n/d} - 2 \right) & \text{iff } n \equiv 1 \mod 2
\end{cases}
\]

\[(3.5) \quad \frac{1}{|G|} \sum_{\gamma \in G} a(\langle \gamma \rangle \setminus \text{Wheel}_n) = \begin{cases} 
\frac{1}{2n} \sum_{d|n} \phi(d)(3^{n/d} - 3) + 3^{n/2}/2 & \text{iff } n \equiv 0 \mod 2 \\
\frac{1}{2n} \sum_{d|n} \phi(d)(3^{n/d} - 3) & \text{iff } n \equiv 1 \mod 2
\end{cases}
\]

In view of Proposition 3, we have to compute the set \text{Acyc}(\text{Circ}_n) \langle \gamma \rangle for \( \gamma \in \text{Aut(} \text{Circ}_n) \). First we observe that \text{Aut(} \text{Circ}_n) = \langle \sigma \rangle \rtimes \langle \tau \rangle \), where \( \sigma = (2, 3, \ldots, n, 1) \) and \( \tau = \prod_{i=2}^{[n/2]} (i, n - i + 2) \). Furthermore we have \( a(\text{Circ}_n) = 2^n - 2 \) and \( a(\text{Wheel}_n) = 3^n - 3 \). Second, let \((0 \otimes Y)\) be the vertex-join of \(Y\) and 0, then \(\pi_G\) has the property

\[(3.6) \quad \forall Y, d(Y) < |v[Y]|, \quad G \setminus (0 \otimes Y) \cong 0 \otimes (G \setminus Y).
\]

Accordingly, the formula for (3.5) follows by taking the vertex-joins of the graphs \(\langle \gamma \rangle \setminus \text{Circ}_n\). Thus it remains to compute \(\langle \gamma \rangle \setminus \text{Circ}_n\). Since \text{Aut(} \text{Circ}_n) \) is a dihedral group we have either \( \gamma = \sigma^k \) or \( \gamma = \tau \sigma^k \). Suppose \( d|n \) then \(\langle \sigma^{n/d} \rangle \setminus \text{Circ}_n \cong \text{Circ}_{n/d} \) and the automorphisms of the form \( \sigma^k \) contribute \( \sum_{d|n} \phi(d)(2^{n/d} - 2) \). For \( n \equiv 1 \mod 2 \) we immediately observe that \(\langle \tau \sigma^k \rangle\) contains at least one loop of size 1 and we are done. In case of \( n \equiv 0 \)
mod 2, \( \langle \tau \sigma^k \rangle \) has for \( k \equiv 1 \mod 2 \) a vertex that corresponds to a \( \langle \tau \sigma^k \rangle \)-orbit which contains two adjacent vertices, whence \( \text{Acyc}(Y)^{\langle \tau \sigma^k \rangle} = \emptyset \). For \( k \equiv 0 \mod 2 \) we conclude that \( \langle \tau \sigma^k \rangle \setminus \text{Circ}_n \cong \text{Line}_{n/2} \), which has \( 2^{n/2} \) acyclic orientations and (3.4) follows.

In view of (3.6) it remains to take the vertex-joins of the graphs \( \langle Y \rangle \setminus \text{Circ}_n \) that have no loops of size 1 and the second formula follows in view of \( 0 \otimes \text{Circ}_{n/4} \cong \text{Wheel}_{n/4} \) and \( a(0 \otimes \text{Line}_{n/2}) = 2 \cdot 3^{n/2} \), whence Proposition 1.

4. PROOF OF THEOREM 2

Let us begin by showing

**Lemma 1.** Let \( (f_{(m)})_m \) be a family of Boolean symmetric functions that induces a fixed-point-free SDS \( [\vec{\nu}_Y, \pi] \) for arbitrary graphs \( Y \). Then \( [\vec{\nu}_Y, \pi] \) and \( [\text{Nor}_Y, \pi] \) are equivalent.

**Proof.** Claim 1. For any \( m \in \mathbb{N} \) we have either \( f_{(m)} = \text{nor}_{(m)} \) or \( f_{(m)} = \text{nand}_{(m)} \).

Let us first consider the case \( m = 2 \). It is clear that a fixed-point-free symmetric function \( f_{(2)} : \mathbb{F}_2^2 \rightarrow \mathbb{F}_2 \) has the properties \( f_{(2)}(0, 0) = 1, f_{(2)}(1, 1) = 0 \). We have either \( f_{(2)}(0, 1) = f_{(2)}(1, 0) = 1 \) in which case \( f_{(2)} = \text{nand}_{(2)} \) or \( f_{(2)}(0, 1) = f_{(2)}(1, 0) = 0 \), that is, \( f_{(2)} = \text{nor}_{(2)} \). Let now \( m > 2 \). Suppose \( f_{(m)} \neq \text{nor}_{(m)} \) and \( f_{(m)} \neq \text{nand}_{(m)} \); then there exist two \( m \)-tuples \( a = (a_1, \ldots, a_m), b = (b_1, \ldots, b_m) \) with \( |\{ i \mid a_i = 1 \}| = \ell \) and \( |\{ i \mid b_i = 1 \}| = \ell' \) such that \( 0 < \ell, \ell' < m \) and \( f_{(m)}(a) = 1, f_{(m)}(b) = 0 \). We consider the graph \( K_2 \). Accordingly, we have either (i) \( f_{(2)}(0, 1) = 0 \) or (ii) \( f_{(2)}(0, 1) = 1 \).

In case (i) we take \( Y(\ell, m - 1) \) to be the graph over \( \ell (m - \ell) \) vertices and \( \ell (m - \ell) \) edges having \( K_{\ell} \) as a subgraph such that each \( K_{\ell} \)-vertex has degree \( m - 1 \) and 1 otherwise. In view of \( f_{(2)}(0, 1) = 0 \) and \( f_{(m)}(a) = 1 \) we obtain a fixed-point by assigning to any \( Y(\ell, m - 1) \)-vertex with degree \( m - 1 \) the state 1 and state 0 otherwise.

In case (ii), we consider \( Y(m - \ell, m - 1) \) defined as above. We assign to each \( Y(m - \ell, m - 1) \)-vertex with degree \( m - 1 \) the state 0 and state 1 otherwise and obtain, in view of \( f_{(2)}(0, 1) = f_{(2)}(1, 0) = 1 \) and \( f_{(m)}(b) = 0 \), a fixed-point, and the claim follows.

**Claim 2.** We have either, for all \( m \in \mathbb{N}, f_{(m)} = \text{nor}_{(m)} \) or, for all \( m \in \mathbb{N}, f_{(m)} = \text{nand}_{(m)} \) holds.

Suppose there exist \( \ell, \ell' \in \mathbb{N} \) such that \( f_{(\ell)} = \text{nor}_{(\ell)} \) and \( f_{(\ell')} = \text{nand}_{(\ell')} \). We consider the bipartite graph \( K_{\ell - 1, \ell' - 1} \) having the vertex set \( A \cup B \), where each \( a \in A \) has degree \( \ell - 1 \) and each \( b \in B \) degree \( \ell' - 1 \). We assign to
each $a \in A$ the state 0 and to each $b \in B$ the state 1 and obtain a fixed-point. This proves Claim 2.

In view of $[\text{Nor}_Y, \pi] = \text{inv} \circ [\text{Nand}_Y, \pi] \circ \text{inv}$ and Observation 1 of the Introduction, $[\text{Nor}_Y, \pi]$ and $[\text{Nand}_Y, \pi]$ are equivalent, whence the lemma. □

We will proceed by proving assertion (a) of Theorem 2.

**Lemma 2.** Let $Y$ be a graph, $\pi = (i_1, \ldots, i_n)$, $\pi^* = (i_n, \ldots, i_1) \in S_n$, and

$$\Psi_Y = \{(\xi_j) \in \mathbb{F}_2^n | \forall j \in \mathbb{N}_n; \xi_j = 1 \Rightarrow \forall i \in S_i(j): \xi_i = 0\}.$$  

Then we have

$$\Psi_Y = \text{Per}[\text{Nor}_Y, \pi] = [\text{Nor}, \pi](\mathbb{F}_2^n).$$

**Proof.** First we observe that $\text{Per}[\text{Nor}_Y, \pi] \subset [\text{Nor}, \pi](\mathbb{F}_2^n) \subset \Psi_Y$ and it remains to show $\Psi_Y \subset \text{Per}[\text{Nor}_Y, \pi]$. To prove this, we first note that $[\text{Nor}_Y, \pi]^* = \text{res}_{\Psi_Y}[\text{Nor}_Y, \pi]$: $\Psi_Y \rightarrow \Psi_Y$ is a well-defined mapping. We will show that $[\text{Nor}_Y, \pi]^*$ is invertible with inverse $[\text{Nor}_Y, \pi^*] = \text{res}_{\Psi_Y}[\text{Nor}_Y, \pi^*]$. To prove invertibility, it suffices, in view of

$$[\text{Nor}_Y, \pi^*] \circ [\text{Nor}_Y, \pi] = \prod_{j=1}^n \text{Nor}_{i_{n+1-j}, Y} \circ \prod_{j=1}^n \text{Nor}_{i_j, Y}$$

$$[\text{Nor}_Y, \pi] \circ [\text{Nor}_Y, \pi^*] = \prod_{j=1}^n \text{Nor}_{i_j, Y} \circ \prod_{j=1}^n \text{Nor}_{i_{n+1-j}, Y}$$

to show

$$\forall (\xi_j) \in \Psi_Y, i \in \mathbb{N}, \quad \text{Nor}_{i, Y} \circ \text{Nor}_{i, Y}((\xi_j)) = (\xi_j).$$

**Case** (a). $\text{Nor}_{i, Y}((\xi_j)) = (\xi_1, \ldots, 1, \ldots, \xi_n)$. Then, by definition of $\text{Nor}_{i, Y}$, all coordinates $\xi_k, k \in B_1(i)$, have the property $\xi_k = 0$ and, clearly,

$$\text{Nor}_{i, Y} \circ \text{Nor}_{i, Y}((\xi_j)) = \text{Nor}_{i, Y}((\xi_1, \ldots, 1, \ldots, \xi_n)) = (\xi_j).$$

**Case** (b). $\text{Nor}_{i, Y}((\xi_j)) = (\xi_1, \ldots, \xi_{i-1}, 0, \xi_{i+1}, \ldots, \xi_n)$. By definition of $\text{Nor}_{i, Y}$, we have either $\xi_i = 1$ or there exists at least one $i$-neighbor, $k$, such that $\xi_k = 1$. We conclude from $(\xi_j) \in \Psi_Y$ that, in case of $\xi_i = 1$, $i$ is the unique vertex in $B_1(i)$ with this property. Therefore we derive

$$\text{Nor}_{i, Y}((\xi_1, \ldots, \xi_{i-1}, 0, \xi_{i+1}, \ldots, \xi_n)) = \begin{cases} (\xi_1, \ldots, \xi_{i-1}, 1, \xi_{i+1}, \ldots, \xi_n) & \text{if } k = i \\ (\xi_1, \ldots, \xi_{i-1}, 0, \xi_{i+1}, \ldots, \xi_n) & \text{otherwise} \end{cases}$$

whence $\text{Nor}_{i, Y} \circ \text{Nor}_{i, Y}((\xi_j)) = (\xi_j)$ and (4.1) follows. We immediately obtain from (4.1) that $[\text{Nor}_Y, \pi]^* \circ [\text{Nor}_Y, \pi^*] = [\text{Nor}_Y, \pi^*] \circ [\text{Nor}_Y, \pi]^*$ is invertible with inverse $[\text{Nor}_Y, \pi^*] = \text{id}$ holds, whence $\Psi_Y \subset \text{Per}[\text{Nor}_Y, \pi]$ and the proof of the lemma is complete. □
In view of $\text{Per}[\mathcal{N}_Y, \pi] = \{(\xi_j) \in \mathbb{F}_2^n \mid \forall j \in \mathbb{N}_n: \xi_j = 1 \Rightarrow \forall i \in S_1(j): \xi_i = 0\}$ we immediately observe that the mapping

$$
i: \text{Per}[\mathcal{N}_Y, \pi] \rightarrow \mathcal{N}(Y), \quad (\xi_j) \mapsto \{j \mid \xi_j = 1\},$$

is a bijection and assertion (a) follows. Obviously, $\text{Per}[\mathcal{N}_Y, \pi] = [\mathcal{N}_Y, \pi(\mathbb{F}_2^n)]$ implies that each $\mathcal{G}[\mathcal{N}_Y, \pi]$-vertex is either contained in a cycle or has in-degree 0. To complete the proof of assertion (b) it remains to show that (0) has maximal $\mathcal{G}[\mathcal{N}_Y, \pi]$ in-degree.

**Lemma 3.** For $x \neq 0$ let $M(x) = \{h \mid x_h = 1\}$ and for $S \subset M(x)$ let $x^S$ be the $n$-tuple with $x^S_j = x_j$ for $j \notin S$ and $x^S_j = 0$ for $j \in S$. Then we have

$$\forall x \in \mathbb{F}_2^n, S \subset M(x), \quad ||[\mathcal{N}_Y, \sigma]^{-1}(x)|| \leq ||[\mathcal{N}_Y, \sigma]^{-1}(x^S)||$$

and in particular $||[\mathcal{N}_Y, \sigma]^{-1}(x)|| \leq ||[\mathcal{N}_Y, \sigma]^{-1}(0)||$ holds.

**Proof.** Obviously, (4.2) holds for any $x$ with the property $||[\mathcal{N}_Y, \sigma]^{-1}(x)|| = 0$. Thus we can w.l.o.g. assume that $||[\mathcal{N}_Y, \sigma]^{-1}(x)|| > 0$ holds. Let $(0) \neq (\xi_j) \in \mathbb{F}_2^n$ with $1 \in [\mathcal{N}_Y, \sigma]^{-1}(\xi_j)$ and $\xi_i = 1$. Writing $j <_\sigma k$ iff $\sigma^{-1}(j) < \sigma^{-1}(k)$, we can w.l.o.g. assume that $i$ is maximal w.r.t. $<_\sigma$.

Let $S_i^\sigma(h) = \{j \in S_i(h) \mid j >_\sigma h\}$ and $S_i^\sigma(h, \xi) = \{j \in S_i^\sigma(h) \mid \xi_h = 1\}$.

By definition of $\mathcal{N}_Y, \xi_i = 1$ implies, for $j \in S_i^\sigma(i)$, $\eta_j = 0$. We set $\Sigma = \Sigma(Y)_\sigma$ and consider the mapping

$$r^{\xi,i}_C: \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n, \quad r^{\xi,i}_C(\eta)_k = \begin{cases} 1 & \text{for } k = i \lor k \in S_i^\sigma(i) \setminus (\bigcup_h S_i^\sigma(h, \xi)) \\ \eta_k & \text{else.} \end{cases}$$

For $(\chi_k)$ given by $\chi_i = 0$ and $\chi_k = \xi_k$ otherwise, $r^{\xi,i}_C$ induces by restriction an injective mapping

$$\text{res}(r^{\xi,i}_C): [\mathcal{N}_Y, \sigma]^{-1}(\xi_k) \rightarrow [\mathcal{N}_Y, \sigma]^{-1}(\chi_k),$$

since, for $k \in S_i^\sigma(i)$, $\eta_k = 0$ holds. The rest is obvious. In particular we have

$$||[\mathcal{N}_Y, \sigma]^{-1}(\xi_k)|| \leq ||[\mathcal{N}_Y, \sigma]^{-1}(\chi_k)||$$

and (4.2) follows by induction on $||[\xi_j \mid \xi_j = 1||$ successively replacing the coordinates $\xi_i = 1$ by 0. Clearly, (4.2) implies $||[\mathcal{N}_Y, \sigma]^{-1}(x)|| \leq ||[\mathcal{N}_Y, \sigma]^{-1}(0)||$.

Finally we prove assertion (c) of Theorem 2. For this purpose we introduce

$$M(Y, \sigma) = \{x \mid x \text{ has maximal } \mathcal{G}[\mathcal{N}_Y, \sigma] \text{ in degree} \}
\wedge [\mathcal{N}_Y, \sigma]^{-1}([\mathcal{N}_Y, \sigma](x)) = \{x\}.$$
Lemma 4. Let $[\text{Nor}_Y, \sigma]$ be a SDS and let $M(Y, \sigma)$ be given by (4.4). Then

(i) for any connected graph $Y$, $(0) \in M(Y, \sigma)$ holds;

(ii) for $Y = \text{Line}_n$ or $Y = \text{Circ}_n$ we have $M(Y, \sigma) = \{(0)\};$

(iii) there exist graphs with the property $|M(Y, \sigma)| > 1.$

Proof. Ad (i): Lemma 3 guarantees that $(0)$ has maximal $\mathbb{G}[\text{Nor}_Y, \sigma]$ indegree for arbitrary $\sigma \in S_\alpha$. Thus it suffices to prove $[\text{Nor}_Y, \sigma]^{-1}([\text{Nor}_Y, \sigma](0)) = \{(0)\}.$ Suppose there exists some $\eta \neq 0$ such that $[\text{Nor}_Y, \sigma](\eta) = [\text{Nor}_Y, \sigma](0).$ Since $\eta \neq 0$ there exists some vertex $i$ with $\eta_i = 1$ and hence $[\text{Nor}_Y, \sigma](0)_i = 0$. By assumption we have, for any vertex $k,$ $([\text{Nor}_Y, \sigma^*] \circ [\text{Nor}_Y, \sigma](0))_k = 0$, from which we can conclude that there exists a vertex $j \in S_i^\alpha(i)$ such that $[\text{Nor}_Y, \sigma](0)_j = 1$. Now we have the following situation: there exists a vertex $j \in S_i^\alpha(i)$ with $[\text{Nor}_Y, \sigma](\eta)_i = [\text{Nor}_Y, \sigma](0)_i = 1$ and $\eta_i = 1$, which is impossible and thus $[\text{Nor}_Y, \sigma]^{-1}(([\text{Nor}_Y, \sigma](0)) = \{(0)\}$ and (i) follows.

Suppose $(0) \neq x = (x_i) \in M$ and let $i$ be a vertex such that $x_i \neq 0$. We can w.l.o.g. assume that the vertex $i$ with $x_i = 1$ is minimal w.r.t. $<_{\sigma}$. To show assertions (ii) and (iii) we prove two claims:

Claim 1. For all $j \in S_i(i)$ we have $j <_{\sigma} i$.

We will prove the claim by contradiction. Suppose there exists some $j \in S_i(i)$ such that $j >_{\sigma} i$ holds and let $x^{(i)}$ be the $n$-tuple defined by $x^{(i)}_r = 0$ for $i \neq r$ and $x^{(i)}_r = 1$. Lemma 3 guarantees (a) $[\text{Nor}_Y, \sigma]^{-1}(x^{(i)})] = [[\text{Nor}_Y, \sigma]^{-1}(0)]$ and (b) that the preimages of $(0)$ correspond uniquely to preimages $\eta'$ of $x^{(i)}$ having the property $\eta'_i = 1$ (see (4.3)). We now consider $\eta = (\eta_i)$ with $\eta_i = 0$ and $\eta_r = 1$, otherwise. Since there exists some $j >_{\sigma} i$ we have $[\text{Nor}_Y, \sigma](\eta) = (0)$, with $\eta_i \neq 1$, contradicting Claim 1 in view of Lemma 3, since $[\text{Nor}_Y, \sigma]^{-1}(x^{(i)})] = [[\text{Nor}_Y, \sigma]^{-1}(0)]$.

Since $Y$ is connected there exists some $j$ adjacent to $i$ with $j <_{\sigma} i$.

Claim 2. $\exists k \in S_i(j); k <_{\sigma} j$.

Let us assume that, $\forall k \in S_i(j), j <_{\sigma} k$. Then we define $x' = (x'_i)$, where

\begin{equation}
(4.5)
x'_r = \begin{cases} 
1 & r = j \\
x_r & r \neq j.
\end{cases}
\end{equation}

Clearly, we have $x \neq x'$ and since $x_i = 1, x_j = 0$ holds. By assumption $\forall k \in S_i(j)$ we have $j <_{\sigma} k$, from which we can conclude $[\text{Nor}_Y, \sigma](x') = [\text{Nor}_Y, \sigma](x)$, which is impossible, and Claim 2 follows.

Since $i$ is minimal w.r.t. $<_{\sigma}$ with the property $x_i = 1$ we have $x_k = 0$ and there exists no $s <_{\sigma} k$ with the property $x_s = 1$. 

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Ad (ii): Let \((0) \neq x \in M\). For \(Y = \text{Line}_n\) or \(\text{Circ}_n\) we can conclude from \(x_k = 0\) that, for any \(\eta \in [\text{Nor}_Y, \sigma]^{-1}(x)\), \(\eta_j = 1\) holds. Again, 
\[\|[\text{Nor}_Y, \sigma]^{-1}(x)\| = \|[\text{Nor}_Y, \sigma]^{-1}(0)\|\]
implies that
\[
\text{res}(r_{\mathcal{C}}): [\text{Nor}_Y, \sigma]^{-1}(x) \rightarrow [\text{Nor}_Y, \sigma]^{-1}(0)
\]
is a bijection having the property \(\text{res}(r_{\mathcal{C}})(\eta)_j = 0\). We now derive a contradiction by showing that there exists a preimage \(\eta' = (\eta'_i)\) of \((0)\) with the property \(\eta'_j = 0\). For this purpose we define \(\eta'\) by
\[
\eta'_r = \begin{cases} 
0 & r = j \\
1 & \text{otherwise}.
\end{cases}
\]
Clearly, we have \([\text{Nor}_Y, \sigma](\eta') = (0)\), whence (ii).

Ad (iii): Let
\[
Y = \begin{array}{ccc}
i & t & \circ(Y) = \begin{array}{ccc}
i & t \\
j & k & j & k
\end{array}
\end{array}
\]
We consider \(x = (x_i, x_j, x_k, x_j, x_k)\), where \(x_i = 1\), and \(x_h = 0\), otherwise and \(\sigma \in S_n\) such that \(\circ(Y)_{\sigma} = \circ(Y)\). Then \([\text{Nor}_Y, \sigma](x)_i = [\text{Nor}_Y, \sigma](x)_k = 1\) and \([\text{Nor}_Y, \sigma](x)_h = 0\), otherwise. For any \(\eta \in [\text{Nor}_Y, \sigma]^{-1}(x)\) we have \(\eta_i = \eta_j = 1\), \(\eta_i = 0\) and conclude that
\[
\text{res}_{\mathcal{C}}: [\text{Nor}_Y, \sigma]^{-1}(x) \rightarrow [\text{Nor}_Y, \sigma]^{-1}(0), \quad \text{res}_{\mathcal{C}}(\eta_h) = \begin{cases} 
\eta_h & h \neq i \\
1 & h = i
\end{cases}
\]
is a bijection. Now let \(\eta \in [\text{Nor}_Y, \sigma]^{-1}([\text{Nor}_Y, \sigma](x))\). Clearly we have \(\eta_k = \eta_i = 0\) and, in view of \([\text{Nor}_Y, \sigma](x)_k = 1\), \(\eta_r = \eta_j = 0\). Finally, \([\text{Nor}_Y, \sigma](x)_i = 0\) implies \(\eta_i = 1\); i.e.,
\[
[\text{Nor}_Y, \sigma]^{-1}([\text{Nor}_Y, \sigma](x)) = x,
\]
proving (iii).

It is clear that assertion (c) of Theorem 2 follows immediately from the above lemma since a digraph isomorphism preserves in-degrees.

Finally, to prove (d), let us assume that there exist \(\lambda, \sigma, \pi \in S_n\) such that
\[
[\text{Nor}_{\lambda(Y)}, \lambda \sigma] = [\text{Nor}_Y, \pi]
\]
holds. Clearly, \(\lambda \notin \text{Aut}(Y)\) implies \(Y \ncong \lambda(Y)\) and there exists some \(Y\)-vertex \(i\) with the property \(S_{1,\lambda(Y)}(i) \neq S_{1,Y}(i)\). Since \(\text{Aut}(Y)\) acts transitively, \(Y\) is regular and in particular we have \([S_{1,\lambda(Y)}(i)] = [S_{1,Y}(i)]\). Consequently, there exist vertices \(k \in S_{1,Y}(i)\) \(S_{1,\lambda(Y)}(i)\) and \(k' \in S_{1,\lambda(Y)}(i) \setminus S_{1,Y}(i)\).
Claim. We can w.l.o.g. assume that $i$ is an $\Sigma(Y)_\pi$-origin.

By Proposition 2, (4.8) is equivalent to
\begin{equation}
\forall \gamma \in \text{Aut}(Y), \quad [\text{Nor}_{\gamma M(Y)}, \gamma \lambda \sigma] = [\text{Nor}_Y, \gamma \pi].
\end{equation}

Furthermore for any vertex $i$ with the property $S_{1, M(Y)}(i) \neq S_{1, Y}(i)$ we have
\[
\gamma(S_{1, M(Y)}(i)) = S_{1, \gamma M(Y)}(\gamma(i)) \neq S_{1, Y}(\gamma(i)) = \gamma(S_{1, Y}(i))
\]
and can therefore conclude
\[
\forall i \in \nu[Y], \quad S_{1, M(Y)}(i) \neq S_{1, Y}(i) \implies \forall \gamma \in \text{Aut}(Y), \quad \gamma(i),
\]
\[
S_{1, \gamma M(Y)}(\gamma(i)) \neq S_{1, Y}(\gamma(i)).
\]

To prove the lemma it then suffices to show $\gamma \lambda \in \text{Aut}(Y)$. By assumption, $\text{Aut}(Y)$ acts transitively and we can choose $\gamma \in \text{Aut}(Y)$ such that $\gamma(i)$ is an $\Sigma(Y)_\pi$-origin, proving the claim.

For an index set $M$ we set
\[
(e_M)_j = \begin{cases} 1 & \text{if } j \in M \\ 0 & \text{otherwise.} \end{cases}
\]

If $i$ is an $\Sigma(\lambda(Y))_\lambda\sigma$-origin, we obtain the contradiction:
\[
0 = ([\text{Nor}_Y, \pi](e_k))_i \neq ([\text{Nor}_{\lambda M(Y)}, \lambda \sigma](e_k))_i = 1.
\]

Thus we may assume that $i$ is not an $\Sigma(\lambda(Y))_\lambda\sigma$-origin. We distinguish the two cases $\exists k' >_{\lambda \sigma} i$ and $\exists k' <_{\lambda \sigma} i$. In the first case we derive
\[
1 = ([\text{Nor}_Y, \pi](e_k))_i \neq ([\text{Nor}_{\lambda M(Y)}, \lambda \sigma](e_{k'}))_i = 0,
\]
which is impossible. For $k' <_{\lambda \sigma} i$ we consider the index set
\[
M = \{h \mid h <_{\lambda \sigma} k' \land h \in S_{1, \lambda M(Y)}(k') \setminus S_{1, Y}(i)\}.
\]

Since $i$ is an $\Sigma(Y)_\pi$-origin we have $([\text{Nor}_Y, \pi](e_M))_i = 1$ and
\[
\forall h \in S_{1, Y}(i), \quad ([\text{Nor}_Y, \pi](e_M))_h = 0 = ([\text{Nor}_{\lambda M(Y)}, \lambda \sigma](e_M))_h.
\]

Therefore, $([\text{Nor}_{\lambda M(Y)}, \lambda \sigma](e_M))_{k'} = 1$ and since $k' \notin S_{1, Y}(i)$,
\[
1 = ([\text{Nor}_Y, \pi](e_M))_i \neq ([\text{Nor}_{\lambda M(Y)}, \lambda \sigma](e_M))_i = 0
\]
holds. We finally prove $\Sigma(Y)_{\lambda \sigma} = \Sigma(Y)_\pi$. In view of (4.8) we have $[\text{Nor}_{\lambda M(Y)}, \lambda \sigma] = [\text{Nor}_Y, \pi]$ and since $\lambda \in \text{Aut}(Y)$ (2.5) guarantees
\begin{equation}
[\text{Nor}_Y, \lambda \sigma] = [\text{Nor}_Y, \pi].
\end{equation}

We immediately observe that $h: \text{Acyc}(Y) \to \{[\text{Nor}_Y, \pi] \mid \pi \in S_\lambda\}$, $\Sigma_\pi \mapsto [\text{Nor}_Y, \pi]$ is bijective. Accordingly, (4.10) implies $\Sigma(Y)_{\lambda \sigma} = \Sigma(Y)_\pi$, whence (d) and the proof of Theorem 2 is complete.
ACKNOWLEDGMENTS

We thank C. L. Barrett, W. Y. C. Chen, Q. H. Hou, and H. S. Mortveit for stimulating discussions. Special thanks and gratitude to D. Morgeson for his continuous support. This research is supported by Laboratory Directed Research and Development under DOE contract W-7405-ENG-36 to the University of California for the operation of the Los Alamos National Laboratory.

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