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Sequential fractional differential equations with three-point boundary conditions

Bashir Ahmad [a], Juan J. Nieto [a,b,*]

[a] Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

[b] Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, 15782, Santiago de Compostela, Spain

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ABSTRACT

This paper studies a nonlinear three-point boundary value problem of sequential fractional differential equations of order $\alpha+1$ with $1<\alpha\leq 2$. The expression for Green's function of the associated problem involving the classical gamma function and the generalized incomplete gamma function is obtained. Some existence results are obtained by means of Banach's contraction mapping principle and Krasnoselskii's fixed point theorem. An illustrative example is also presented. Existence results for a three-point third-order nonlocal boundary value problem of nonlinear ordinary differential equations follow as a special case of our results.

1. Introduction

In this paper, we study the following boundary value problem of sequential fractional differential equations:

$$ {}^cD^{\alpha}(D+\lambda)x(t)=f(t,x(t)),\quad 0<t<1,\ 1<\alpha\leq 2, \tag{1}$$

$$ x(0)=0,\qquad x'(0)=0,\qquad x(1)=\beta x(\eta),\quad 0<\eta<1, \tag{2}$$

where ${}^cD$ is the Caputo fractional derivative, D is the ordinary derivative, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$, λ is a positive real number and β is a real number such that

$$\beta\neq\frac{\lambda+e^{-\lambda}-1}{\lambda\eta+e^{-\lambda\eta}-1}.$$

Initial and boundary value problems of fractional order have extensively been studied by several researchers in recent years. A variety of results ranging from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions have appeared in the literature. Fractional differential equations appear naturally in a number of fields such as physics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, etc. An excellent account in the study of fractional differential equations can be found in [1–4]. For more details and examples, see [5–20] and references therein.

The concept of sequential fractional derivative is given, for example, on page 209 of the monograph [21]. There is a close connection between the sequential fractional derivatives and the non sequential Riemann–Liouville derivatives [22,23]. For some recent work on sequential fractional differential equations, we refer the reader to the papers [24–26].

* Corresponding author at: Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, 15782, Santiago de Compostela, Spain.

E-mail addresses: bashir_qau@yahoo.com (B. Ahmad), juanjose.nieto.roig@usc.es (J.J. Nieto).

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Third-order equations arise in a variety of problems ranging from the study of regularization of the Cauchy problem for one-dimensional hyperbolic conservation laws [27] to nano boundary layer fluid flows [28] or to describe the evolution of physical phenomena in fluctuating environments [29]. Examples include many famous equations in mathematical physics, such as the Korteweg–de Vries equation [30].

2. Preliminaries

Relative to the problem (1)–(2), for $\sigma \in C[0, 1]$, we now introduce the linear equation:

$$ {}^cD^{\alpha}(D+\lambda)x(t) = \sigma(t), \quad 0 < t < 1,\ 1 < \alpha \leq 2. \tag{3}$$

Lemma 2.1. *The unique solution of the Eq.* (3) *subject to the boundary conditions* (2) *is given by*

$$\begin{aligned} x(t) = {} & \int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\sigma(u)du\right)ds + A(t)\left[\beta\int_0^{\eta} e^{-\lambda(\eta-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\sigma(u)du\right)ds\right. \\ & \left. - \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\sigma(u)du\right)ds\right], \end{aligned} \tag{4}$$

where

$$ A(t) = \frac{1}{\Delta}\left(\lambda t + e^{-\lambda t} - 1\right), \quad \Delta = \lambda + e^{-\lambda} - 1 - \beta(\lambda\eta + e^{-\lambda\eta} - 1). \tag{5}$$

Proof. Applying the operator I^{α} on both sides of (3), we get

$$ (D+\lambda)x(t) = c_0 + c_1 t + I^{\alpha}\sigma(t), $$

which can be rewritten as

$$ D(e^{\lambda t}x(t)) = [c_0 + c_1 t + I^{\alpha}\sigma(t)]e^{\lambda t}. $$

Integrating from 0 to t, we get

$$ x(t) = \frac{c_0}{\lambda}(1-e^{-\lambda t}) + \frac{c_1}{\lambda^2}(\lambda t - 1 + e^{-\lambda t}) + c_2 + \int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\sigma(u)du\right)ds. \tag{6}$$

Using the boundary conditions (2), we find that $c_0 = 0$, $c_2 = 0$, and

$$ c_1 = \frac{\lambda^2}{\Delta}\left[\beta\int_0^{\eta} e^{-\lambda(\eta-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\sigma(u)du\right)ds - \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\sigma(u)du\right)ds\right]. \tag{7}$$

Substituting these values of c_0, c_1, c_2 in (6), we get (4). This completes the proof. □

3. Construction of Green's function

In this subsection, we obtain Green's function corresponding to the fractional differential equations (3) of order $\alpha + 1$ with $1 < \alpha \leq 2$ subject to three-point boundary conditions (2). The expression for Green's function involves the classical gamma function and the generalized incomplete gamma function. For some recent applications of the gamma function and its generalizations, see [31,32].

Changing the order of integration, we note that

$$ \int_0^t e^{-\lambda(t-s)}\left(\int_0^s (s-u)^{\alpha-1}\sigma(u)du\right)ds = \int_0^t\left(\int_u^t e^{\lambda s}(s-u)^{\alpha-1}ds\right)e^{-\lambda t}\sigma(u)du. \tag{8}$$

Recall that

$$\begin{aligned} \int_u^t e^{\lambda s}(s-u)^{\alpha-1}ds &= (-1)^{\alpha-1}\lambda^{-\alpha}e^{\lambda u}\left(\Gamma(\alpha, -\lambda(t-u)) - \Gamma(\alpha, 0)\right) \\ &= (-1)^{\alpha-1}\lambda^{-\alpha}e^{\lambda u}\Gamma(\alpha, -\lambda(t-u), 0), \end{aligned} \tag{9}$$

where

$$\Gamma(\alpha, x_0, x_1) = \Gamma(\alpha, x_0) - \Gamma(\alpha, x_1) = \int_{x_0}^{x_1} r^{\alpha-1}e^{-r}dr$$

is the generalized incomplete Gamma function. Using (9) in (8), we get

$$\begin{aligned}\int_0^t e^{-\lambda(t-s)}\left(\int_0^s (s-u)^{\alpha-1}\sigma(u)du\right)ds &= \int_0^t (-1)^{\alpha-1}\lambda^{-\alpha}e^{\lambda(u-t)}\Gamma(\alpha, -\lambda(t-u), 0)\sigma(u)du \\ &= \int_0^1 k^*(t,s)\sigma(s)ds, \end{aligned} \tag{10}$$

where

$$k^*(t,s) = \begin{cases} 0, & \text{if } 0 \le t < s \le 1, \\ (-1)^{\alpha-1}\lambda^{-\alpha}e^{\lambda(s-t)}\Gamma(\alpha, -\lambda(t-s), 0), & \text{if } 0 \le s. \end{cases} \tag{11}$$

Define $k(t,s) = k^*(t,s)/\Gamma(\alpha)$ and note that $k(t,s) = 0$ for $s > t$. Hence, the solution (6) with $c_0 = 0, c_2 = 0$, takes the form

$$x(t) = \frac{c_1}{\lambda^2}(\lambda t - 1 + e^{-\lambda t}) + \int_0^1 k(t,s)\sigma(s)ds. \tag{12}$$

Now, using the boundary condition $x(1) = \beta x(\eta)$, we get

$$c_1 = \frac{\lambda^2\beta}{\lambda + e^{-\lambda} - 1 - \beta(\lambda\eta + e^{-\lambda\eta} - 1)}\int_0^1 k(\eta,s)\sigma(s)ds, \tag{13}$$

provided $\lambda + e^{-\lambda} - 1 - \beta(\lambda\eta + e^{-\lambda\eta} - 1) \neq 0$. Inserting the value of c_1, in (12), we obtain

$$x(t) = \frac{\beta(\lambda t - 1 + e^{-\lambda t})}{\lambda + e^{-\lambda} - 1 - \beta(\lambda\eta + e^{-\lambda\eta} - 1)}\int_0^1 k(\eta,s)\sigma(s)ds + \int_0^1 k(t,s)\sigma(s)ds. \tag{14}$$

Letting

$$\phi(t) = \frac{\beta(\lambda t - 1 + e^{-\lambda t})}{\lambda + e^{-\lambda} - 1 - \beta(\lambda\eta + e^{-\lambda\eta} - 1)},$$

(14) becomes

$$x(t) = \int_0^1 g(t,s)\sigma(s)ds, \tag{15}$$

where

$$g(t,s) = \begin{cases} 0, & \text{if } 0 \le \max\{\eta t\} < s \le 1, \\ k(t,s), & \text{if } 0 \le \eta < s < t \le 1, \\ \phi(t)k(t,s), & \text{if } 0 \le t < s < \eta \le 1, \\ \phi(t)k(t,s) + k(t,s), & \text{if } 0 \le s < \min\{\eta t\} \le 1. \end{cases} \tag{16}$$

We point out that the expression is also valid for the integer case $\alpha = 2$. In this case,

$$\Gamma(2, -\lambda(t-s), 0) = \int_{-\lambda(t-s)}^0 re^{-r}dr.$$

Hence we obtain a new expression for the solutions of a third-order differential equation.

4. Existence of solutions

Let $\mathcal{C} = C([0,1], \mathbb{R})$ denotes the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t \in [0,1]\}$.

For the sake of convenience, we set

$$A_1 = \max_{t\in[0,1]} |A(t)|, \qquad B = \left|\frac{(1+A_1)(1-e^{-\lambda}) + A_1\beta\eta^\alpha}{\lambda\Gamma(\alpha+1)}\right|, \tag{17}$$

where $A(t)$ is given by (5).

In view of Lemma 2.1, we transform problem (1)–(2) as

$$x = \mathcal{F}(x), \tag{18}$$

where $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ is defined by

$$(\mathcal{F}x)(t) = \int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f(u, x(u))du\right)ds + A(t)\left[\beta \int_0^\eta e^{-\lambda(\eta-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f(u, x(u))du\right)ds \right.$$
$$\left. - \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f(u, x(u))du\right)ds\right].$$

Observe that problem (1)–(2) has solutions if the operator Eq. (18) has fixed points.

Theorem 4.1. *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a jointly continuous function satisfying the condition*

$$|f(t, x) - f(t, y)| \le L|x - y|, \quad \forall t \in [0, 1],\ x, y \in \mathbb{R},$$

where L is the Lipschitz constant. Then the boundary value problem (1)–(2) has a unique solution if $B < 1/L$, where B is given by (17).

Proof. As a first step, for $\mathcal{F}$ defined by (18), we show that $\mathcal{F}\mathcal{B}_r \subset \mathcal{B}_r$, where $\mathcal{B}_r = \{x \in \mathcal{C} : \|x\| \le r\}$. For that, set $\sup_{t\in[0,1]} |f(t, 0)| = M$ and choose

$$r \ge \frac{MB}{1 - LB},$$

where B is given by (17). For $x \in B_r$, we have

$$\begin{aligned}
\|(\mathcal{F}x)(t)\| &= \sup_{t\in[0,1]} \left| \int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f(u, x(u))du\right)ds \right. \\
&\quad + A(t)\left[\beta \int_0^\eta e^{-\lambda(\eta-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f(u, x(u))du\right)ds \right. \\
&\quad \left.\left. + \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f(u, x(u))du\right)ds\right]\right| \\
&\le \sup_{t\in[0,1]}\left(\int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} (|f(u, x(u)) - f(u, 0)| + |f(u, 0)|)du\right)ds\right) \\
&\quad + \sup_{t\in[0,1]} |A(t)|\left[\beta \int_0^\eta e^{-\lambda(\eta-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} (|f(u, x(u)) - f(u, 0)| + |f(u, 0)|)du\right)ds \right. \\
&\quad \left. + \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} (|f(u, x(u)) - f(u, 0)| + |f(u, 0)|)du\right)ds\right] \\
&\le \sup_{t\in[0,1]}\left(\int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} (L|x(u)| + |f(u, 0)|)du\right)ds\right) \\
&\quad + \sup_{t\in[0,1]} |A(t)|\left[\beta \int_0^\eta e^{-\lambda(\eta-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} (L|x(u)| + |f(u, 0)|)du\right)ds \right. \\
&\quad \left. + \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} (L|x(u)| + |f(u, 0)|)du\right)ds\right] \\
&\le (Lr + M)\left[\sup_{t\in[0,1]}\left(\int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} du\right)ds\right) \right. \\
&\quad \left. + \sup_{t\in[0,1]} |A(t)|\left\{\beta \int_0^\eta e^{-\lambda(\eta-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} du\right)ds + \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} du\right)ds\right\}\right]
\end{aligned}$$

$$
\begin{aligned}
&\le (Lr+M)\left(\frac{(1+A_1)(1-e^{-\lambda})+A_1\beta\eta^{\alpha}}{\lambda\Gamma(\alpha+1)}\right)\\
&= (Lr+M)B \le r.
\end{aligned}
$$

Now, for $x, y \in \mathcal{C}$ and for each $t \in [0, 1]$, we obtain

$$
\begin{aligned}
\|(\mathcal{F}x)(t)-(\mathcal{F}y)(t)\| &= \sup_{t\in[0,1]} |(\mathcal{F}x)(t)-(\mathcal{F}y)(t)|\\
&\le \sup_{t\in[0,1]}\Bigg[\int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}|f(u,x(u))-f(u,y(u))|du\right)ds\\
&\quad + A(t)\Bigg\{\beta\int_0^{\eta} e^{-\lambda(\eta-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}|f(u,x(u))-f(u,y(u))|du\right)ds\\
&\quad + \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}|f(u,x(u))-f(u,y(u))|du\right)ds\Bigg\}\Bigg]\\
&\le L\|x-y\|\Bigg[\sup_{t\in[0,1]}\left(\int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}du\right)ds\right)\\
&\quad + \sup_{t\in[0,1]}|A(t)|\Bigg\{\beta\int_0^{\eta} e^{-\lambda(\eta-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}du\right)ds\\
&\quad + \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}du\right)ds\Bigg\}\Bigg]\\
&\le L\left|\frac{(1+A_1)(1-e^{-\lambda})+A_1\beta\eta^{\alpha}}{\lambda\Gamma(\alpha+1)}\right|\|x-y\|\\
&= BL\|x-y\|,
\end{aligned}
$$

where B is given by (17). As $B < 1/L$, therefore, $\mathcal{F}$ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle. This completes the proof. □

Now, we state a known result due to Krasnoselskii [33] which is needed to prove the existence of at least one solution of (1)–(2).

Theorem 4.2. *Let M be a closed convex and nonempty subset of a Banach space X. Let $\mathcal{G}_1, \mathcal{G}_2$ be the operators such that:* (i) *$\mathcal{G}_1x+\mathcal{G}_2y \in M$ whenever $x, y \in M$;* (ii) *$\mathcal{G}_1$ is compact and continuous;* (iii) *$\mathcal{G}_2$ is a contraction mapping. Then there exists $z \in M$ such that $z = \mathcal{G}_1z+\mathcal{G}_2z$.*

Theorem 4.3. *Assume that $f : [0, 1]\times\mathbb{R}\to\mathbb{R}$ is a jointly continuous function and the following assumptions hold:*

(H_1) *$|f(t,x)-f(t,y)| \le L|x-y|,\ \forall t\in[0,1],\ x,y\in\mathbb{R}$;*
(H_2) *$|f(t,x)| \le \mu(t),\ \forall (t,x)\in[0,1]\times\mathbb{R}$ with $\mu\in C([0,1],\mathbb{R})$.*

Then the boundary value problem (1)–(2) has at least one solution on $[0, 1]$ if

$$
\left|\frac{A_1(1-e^{-\lambda}+\beta\eta^{\alpha})}{\lambda\Gamma(\alpha+1)}\right| < 1. \tag{19}
$$

Proof. Letting $\sup_{t\in[0,1]}|\mu(t)| = \|\mu\|$, we fix

$$
r \ge \left|\frac{(1+A_1)(1-e^{-\lambda})+A_1\beta\eta^{\alpha}}{\lambda\Gamma(\alpha+1)}\right|\|\mu\|, \tag{20}
$$

and consider $B_r = \{x\in\mathcal{C} : \|x\|\le r\}$. Define the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ on B_r as

$$
\begin{aligned}
(\mathcal{F}_1x)(t) &= \int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}f(u,x(u))du\right)ds,\\
(\mathcal{F}_2x)(t) &= A(t)\left[\beta\int_0^{\eta} e^{-\lambda(\eta-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}f(u,x(u))du\right)ds - \int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}f(u,x(u))du\right)ds\right].
\end{aligned}
$$

For $x, y \in B_r$, it follows from (20) that

$$\| \mathcal{F}_1 x + \mathcal{F}_2 y \| \leq \left| \frac{(1 + A_1)(1 - e^{-\lambda}) + A_1 \beta \eta^\alpha}{\lambda \Gamma(\alpha + 1)} \right| \|\mu\| \leq r.$$

Thus, $\mathcal{F}_1 x + \mathcal{F}_2 y \in B_r$. In view of the condition (19), it can easily be shown that $\mathcal{F}_2$ is a contraction mapping. The continuity of f implies that the operator $\mathcal{F}_1$ is continuous. Also, $\mathcal{F}_1$ is uniformly bounded on B_r as

$$\| \mathcal{F}_1 x \| \leq \frac{|1 - e^{-\lambda}| \|\mu\|}{\lambda \Gamma(\alpha + 1)}.$$

Now we prove the compactness of the operator $\mathcal{F}_1$. Setting $\Omega = [0, 1] \times B_r$, we define $\sup_{(t,x)\in\Omega} |f(t, x)| = M_r$, and consequently we have

$$\begin{aligned} \|(\mathcal{F}_1 x)(t_1) - (\mathcal{F}_1 x)(t_2)\| &= \left\| \int_0^{t_1} e^{-\lambda(t_1 - s)} \left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f(u, x(u)) du \right) ds \right. \\ &\quad \left. - \int_0^{t_2} e^{-\lambda(t_2 - s)} \left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f(u, x(u)) du \right) ds \right\| \\ &\leq \frac{M_r}{\lambda \Gamma(\alpha + 1)} \left(|t_1^\alpha - t_2^\alpha| + |t_1^\alpha e^{-\lambda t_1} - t_2^\alpha e^{-\lambda t_2}| \right), \end{aligned}$$

which is independent of x and tends to zero as $t_2 \to t_1$. Thus, $\mathcal{F}_1$ is relatively compact on B_r. Hence, by the Arzelá–Ascoli Theorem, $\mathcal{F}_1$ is compact on B_r. Thus all the assumptions of Theorem 4.2 are satisfied and the conclusion of Theorem 4.2 implies that the boundary value problem (1)–(2) has at least one solution on [0, 1]. This completes the proof. □

Example 4.1. Consider the problem

$$\begin{cases} {}^c D^{3/2}(D + 4)x(t) = L\Big(t^2 + \cos t + 1 + \tan^{-1} x(t)\Big), & 0 \leq t \leq 1, \\ x(0) = 0, \quad x'(0) = 0, \quad x(1) = x(1/2). \end{cases} \tag{21}$$

Here, $f(t, x(t)) = L\Big(t^2 + \cos t + 1 + \tan^{-1} x(t)\Big)$, $\lambda = 4$, $\beta = 1$, $\eta = 1/2$. Clearly

$$|f(t, x) - f(t, y)| \leq L|\tan^{-1} x - \tan^{-1} y| \leq L|x - y|$$

and

$$A_1 = 4/(2 + e^{-4} - e^{-2}), \qquad B = \frac{(1 + A_1)(1 - e^{-4}) + 2^{-3/2} A_1}{3\sqrt{\pi}}.$$

For $L < \frac{3\sqrt{\pi}}{(1+A_1)(1-e^{-4})+2^{-3/2}A_1} = 1.379430821$, it follows by Theorem 4.1 that problem (21) has a unique solution.

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