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Advances in Mathematics 226 (2011) 1202–1212
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Obstructions to determinantal representability

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Received 6 May 2010; accepted 4 August 2010

Available online 16 August 2010

Communicated by Ezra Miller

Abstract

There has recently been ample interest in the question of which sets can be represented by linear matrix inequalities (LMIs). A necessary condition is that the set is rigidly convex, and it has been conjectured that rigid convexity is also sufficient. To this end Helton and Vinnikov conjectured that any real zero polynomial admits a determinantal representation with symmetric matrices. We disprove this conjecture. By relating the question of finding LMI representations to the problem of determining whether a polymatroid is representable over the complex numbers, we find a real zero polynomial such that no power of it admits a determinantal representation. The proof uses recent results of Wagner and Wei on matroids with the half-plane property, and the polymatroids associated to hyperbolic polynomials introduced by Gurvits.

MSC: 90C22; 90C25; 52B99

Keywords: Linear matrix inequalities; Determinantal representability; Hyperbolic polynomial; Polymatroid; Subspace arrangements; Half-plane property

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1. Representing sets with linear matrix inequalities

Motivated by powerful techniques commonly used in control theory, there has recently been considerable interest in the following question.

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[1] The author is a Royal Swedish Academy of Sciences Research Fellow supported by a grant from the Knut and Alice Wallenberg Foundation.

doi:10.1016/j.aim.2010.08.003

Question 1. Which subsets of $\mathbb{R}^n$ can be represented by linear matrix inequalities (LMIs)? That is, which sets $\mathcal{Y}$ are of the form

$$\mathcal{Y} = \left\{(x_1, \dots, x_n) \in \mathbb{R}^n\colon\ A_0 + A_1x_1 + \dots + A_nx_n \text{ is positive semidefinite}\right\}, \tag{1}$$

where $A_0, \dots, A_n$ are real symmetric $m \times m$ matrices?

In two variables such sets were characterized by Helton and Vinnikov [7] by so-called rigidly convex sets, thereby answering a question posed by Parrilo and Sturmfels [14]. We will always assume that 0 is in the interior of $\mathcal{Y}$ and then the existence of a LMI representation of $\mathcal{Y}$ is equivalent to the existence of a *monic* LMI representation, i.e., a representation in which A_0 is the identity matrix.

A polynomial $p \in \mathbb{R}[x_1, \dots, x_n]$ is a *real zero polynomial* (RZ polynomial) if for each $x \in \mathbb{R}^n$ and $\mu \in \mathbb{C}$

$$p(\mu x) = 0 \quad \text{implies} \quad \mu \text{ is real.} \tag{2}$$

A set $\mathcal{Y} \subseteq \mathbb{R}^n$ is *rigidly convex* (at the origin) if there is a RZ polynomial p for which $\mathcal{Y}$ is equal to the closure of the connected component of

$$\left\{x \in \mathbb{R}^n\colon\ p(x) > 0\right\}$$

containing the origin.

In what follows I will always denote the identity matrix of appropriate size. It is not hard to see that if $A_1, \dots, A_n$ are symmetric or hermitian matrices of the same size then the polynomial $\det(I + x_1A_1 + \dots + x_nA_n)$ is a RZ polynomial. In two variables Helton and Vinnikov provided a converse to this fact.

Theorem 1.1 *(Helton–Vinnikov). (See [7].) Let $p(x, y)$ be a RZ polynomial of degree d, and suppose that $p(0, 0) = 1$. Then there are symmetric matrices A and B of size $d \times d$ such that*

$$p(x, y) = \det(I + xA + yB).$$

Theorem 1.1 also settles a conjecture of Peter Lax which asserts that any hyperbolic degree d polynomial in three variables can be represented by a determinant, see [10]. By a simple count of parameters one sees that the exact analog of Theorem 1.1 in three or more variables fails. However, the count of parameters does not preclude a determinantal representation with matrices of a size larger than the degree. To this end, Helton and Vinnikov [7, p. 668] made the following conjecture.

Conjecture 1.2 *(Helton–Vinnikov). Let $p(x_1, \dots, x_n)$ be a RZ polynomial, and suppose that $p(0) = 1$. Then there are symmetric matrices $A_1, \dots, A_n$ such that*

$$p(x_1, \dots, x_n) = \det(I + x_1A_1 + \dots + x_nA_n).$$

In Section 2 we will find a family of counterexamples to Conjecture 1.2. The following relaxation of Conjecture 1.2 has also been suggested.

Conjecture 1.3. *Let $p(x_1, \dots, x_n)$ be a RZ polynomial, and suppose that $p(0) = 1$. Then there are symmetric matrices $A_1, \dots, A_n$, and a positive integer N such that*

$$p(x_1, \dots, x_n)^N = \det(I + x_1 A_1 + \cdots + x_n A_n).$$

In Section 3 we find a counterexample to Conjecture 1.3 by relating the problem of finding determinantal representations to the problem of determining whether a given polymatroid is representable (comes from a subspace arrangement). The counterexample arises from the fact that the Vámos cube is a matroid that has the so-called *half-plane property* but is not representable over any field, see [16].

The conjecture that any rigidly convex set can be represented by LMIs still remains open.

2. Counterexamples by a count of parameters

A homogeneous polynomial $h(x_1, \dots, x_n) \in \mathbb{R}[x_1, \dots, x_n]$ is *hyperbolic* with respect to $e \in \mathbb{R}^n$ if $h(e) \neq 0$ and if for each $x \in \mathbb{R}^n$ and $\mu \in \mathbb{C}$

$$h(x + \mu e) = 0 \quad \text{implies} \quad \mu \text{ is real}, \tag{3}$$

see [5,15]. Clearly, if h is hyperbolic with respect to e, then $h(x + e)$ is a RZ polynomial. The *hyperbolicity cone* of h at e is the set of all $x \in \mathbb{R}^n$ for which the univariate polynomial $t \mapsto p(x + te)$ has only negative zeros.

We will use the Cauchy–Binet theorem. Let $[n] = \{1, \dots, n\}$ and let $\binom{[n]}{k}$ denote the set of all k-element subsets of $[n]$. If A is an $n \times m$ matrix and $S \subseteq [n]$, $T \subseteq [m]$ are two sets of the same size we denote by $A(S, T)$ the minor of A with rows indexed by S and columns indexed by T.

Theorem 2.1 *(Cauchy–Binet). Let A be an $m \times n$ matrix and B an $n \times m$ matrix. Then*

$$\det(AB) = \sum_{S \in \binom{[n]}{m}} A\big([m], S\big) B\big(S, [m]\big).$$

Theorem 2.2. *Let $h(x) \in \mathbb{R}[x_1, \dots, x_n]$ be a hyperbolic polynomial with respect to e, and let $p(x)$ be the RZ polynomial defined by $p(x) = h(x + e)$. If p admits a representation*

$$p(x) = \det(I + x_1 A_1 + \cdots + x_n A_n)$$

where A_j is symmetric (hermitian) and of size $N \times N$ for all j, then p admits a representation

$$p(x) = \det(I + x_1 B_1 + \cdots + x_n B_n)$$

where B_j is symmetric (hermitian) and of size $d \times d$ for all j, and d is the degree of h and p.

Proof. By considering a linear change of variables we may, and will, assume that $h(x)$ has hyperbolicity cone containing $\mathbb{R}^d_+$, where $\mathbb{R}_+$ is the set of positive reals, and that $e = (1, \dots, 1)^T =: \mathbf{1}$.

We claim that A_j and $I - \sum_{j=1}^n A_j$ are positive semidefinite (PSD) for all j. The univariate polynomial

$$\det(I + tA_j) = p(0, \dots, t, \dots, 0) = h(\mathbf{1} + t\delta_j),$$

where δ_j is the jth standard bases vector, has only real and non-positive zeros (since δ_j is in the closure of the hyperbolicity cone of h). Hence A_j is PSD. Similarly

$$\begin{aligned}\det\left(tI + I - \sum_{j=1}^{n} A_j\right) &= (1+t)^N p\bigl(-1/(1+t), \dots, -1/(1+t)\bigr)\\ &= (1+t)^N h\bigl(1 - 1/(1+t), \dots, 1 - 1/(1+t)\bigr)\\ &= (1+t)^{N-d} t^d.\end{aligned}$$

Hence $I - \sum_{j=1}^{n} A_j$ is PSD of rank $N-d$. Suppose that A_i has rank r_i. Write A_i as $A_i = \sum_{j=1}^{r_i} A_{ij}$, where $A_{ij} = v_{ij}v_{ij}^*$ is PSD of rank 1, v_{ij}^* is the hermitian adjoint of v_{ij}, and $v_{ij} \in \mathbb{C}^N$. Similarly we may write $I - \sum_{j=1}^{n} A_j$ as a sum $\sum_{j=1}^{N-d} C_j$, where $C_j = u_j u_j^*$ is PSD of rank 1. Let now

$$P(\tilde{x}, \tilde{y}) = \det\left(\sum_{j=1}^{N-d} C_j y_j + \sum_{i,j} A_{ij} x_{ij}\right),$$

where $\tilde{x} = (x_{ij})_{i,j}$ and $\tilde{y} = (y_1, \dots, y_{N-d})$ are new variables. Let B be the matrix with columns $u_1, \dots, u_{N-d}, v_{11}, \dots, v_{1r_1}, \dots, v_{nr_n}$. Rename the columns and variables so that $B = [w_1, \dots, w_M]$, and the corresponding variables are $z = (z_1, \dots, z_M)$. By construction and the Cauchy–Binet theorem

$$P(z) = \det\bigl(BZB^*\bigr) = \sum_{S \in \binom{[M]}{N}} \bigl|B(S)\bigr|^2 \prod_{j \in S} z_j,$$

where $Z = \operatorname{diag}(z_1, \dots, z_M)$, and $B(S) = B([N], S)$ is the $N \times N$ minor of B with columns indexed by S.

We obtain $h(x)$ from $P(z)$ by setting $y_j = 1$ and $x_{ij} = x_i$ for all i and j. Since all coefficients of $P(z)$ are nonnegative and $h(x)$ is homogeneous of degree d we have that for each S with $B(S) \neq 0$ there are precisely d indices that correspond to x-variables and $N-d$ variables that correspond to y-variables. This means that $U \cap V = (0)$, where $U = \operatorname{span}\{u_i\}_{i=1}^{N-d}$ and $V = \operatorname{span}\{v_{ij}\colon 1 \leqslant i \leqslant n \text{ and } 1 \leqslant j \leqslant r_i\}$. Hence we may write B as $B = PM$ where P is invertible and M is a block matrix

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix},$$

where M_1 has $N-d$ columns and M_2 has $M+d-N$ columns. Thus

$$\begin{aligned}P(z) &= \det\bigl(PP^*\bigr)\det\bigl(MZM^*\bigr)\\ &= \det\bigl(PP^*\bigr) \sum_{S_1 \in \binom{[N-d]}{N-d}} \sum_{S_2 \in \binom{[N-d+1, M]}{d}} \bigl|M_1(S_1)\bigr|^2 \bigl|M_2(S_2)\bigr|^2 \prod_{j \in S_1} z_j \prod_{j \in S_2} z_j\end{aligned}$$

$$= \det(PP^*)\left|\det(M_1)\right|^2 y_1 \cdots y_{N-d} \sum_{S_2 \in \binom{[N-d+1,M]}{d}} \left|M_2(S_2)\right|^2 \prod_{j \in S_2} z_j$$

$$= \det(PP^*)\left|\det(M_1)\right|^2 y_1 \cdots y_{N-d} \det\left(\sum_{i=1}^{M-n+d} z_{N-d+i} m_i m_i^*\right),$$

where m_i is the ith column of M_2. Setting $y_i = 1$ and $x_{ij} = x_i$ for all i and j we obtain a representation

$$h(x) = \det(PP^*)\left|\det(M_1)\right|^2 \det\left(\sum_{i=1}^{n} T_i x_i\right),$$

where each T_i is PSD of size $d \times d$, and $\sum_{i=1}^{n} T_i$ is positive definite. It follows that $p(x)$ has a representation of the desired form. □

Nuij [12] proved that the space of all hyperbolic polynomials of degree d that are hyperbolic with respect to $e \in \mathbb{R}^n$ has nonempty interior. Hence so does the space of RZ polynomials considered in Theorem 2.2. Since any such polynomial that admits a determinantal representation also admits a determinantal representation with matrices of size d, a count of parameters provides counterexamples to Conjecture 1.2.

3. Representability of polymatroids

We will see here that Question 1 is closely related to the old problem of determining if a polymatroid is representable over $\mathbb{C}$.

An (integral) *polymatroid* on a finite set E is a function $r : 2^E \to \mathbb{N}$ such that:

(1) $r(\emptyset) = 0$;
(2) If $S \subseteq T \subseteq E$, then $r(S) \leqslant r(T)$;
(3) r is *submodular*, that is,

$$r(S \cup T) + r(S \cap T) \leqslant r(S) + r(T),$$

for all subsets S and T of E.

A natural class of polymatroids arises from *subspace arrangements*. Let E be a finite set and $\mathcal{V} = (V_j)_{j \in E}$ a collection of subspaces of a finite dimensional vector space V over a field K. Then the function $r_{\mathcal{V}} : 2^E \to \mathbb{N}$ defined by

$$r_{\mathcal{V}}(S) = \dim\left(\sum_{i \in S} V_i\right),$$

where $\sum_{i \in S} V_i$ is the smallest subspace containing $\bigcup_{i \in S} V_i$, is a polymatroid. This follows from the dimension formula for subspaces

$$\dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W).$$

We say that a polymatroid, r, is *representable over the field* K if there is a subspace arrangement $\mathcal{V}$ of subspaces of a vector space over K such that $r = r_{\mathcal{V}}$. There are several inequalities known to hold for representable polymatroids, see [4,8,9]. The simplest of these are known as the *Ingleton inequalities*.

Lemma 3.1 *(Ingleton inequalities). (See [8].) Suppose that* $\mathcal{V} = (V_1, \dots, V_n)$ *is a subspace arrangement. Then*

$$\begin{aligned} &r_{\mathcal{V}}(S_1 \cup S_2) + r_{\mathcal{V}}(S_1 \cup S_3 \cup S_4) + r_{\mathcal{V}}(S_3) + r_{\mathcal{V}}(S_4) + r_{\mathcal{V}}(S_2 \cup S_3 \cup S_4) \\ &\quad \leqslant r_{\mathcal{V}}(S_1 \cup S_3) + r_{\mathcal{V}}(S_1 \cup S_4) + r_{\mathcal{V}}(S_2 \cup S_3) + r_{\mathcal{V}}(S_2 \cup S_4) + r_{\mathcal{V}}(S_3 \cup S_4) \end{aligned}$$

for all $S_1, S_2, S_3, S_4 \subseteq [n]$.

To see that subspace arrangements over $\mathbb{C}$ or $\mathbb{R}$ are closely related to determinantal representability we proceed to express the rank function in terms of determinants. Suppose that $A_1, \dots, A_n$ are positive semidefinite matrices of the same size m, and let $\mathcal{V} = (V_1, \dots, V_n)$ be the subspace arrangement in $\mathbb{C}^m$ defined by letting V_i be the image of A_i for all i. Then

$$r_{\mathcal{V}}(S) = \operatorname{rank}\left(\sum_{i \in S} A_i\right) = \deg\left(\det\left(I + t \sum_{i \in S} A_i\right)\right),$$

for all $S \subseteq [n]$. To see this it is enough (by spectral decomposition) to consider the case when all matrices are of rank one and that $S = [n]$. Write A_i as $A_i = v_i v_i^*$ where $v_i \in \mathbb{C}^m$, and let D be the $(m+n) \times (m+n)$ diagonal matrix with the first m entries equal to one and the remaining entries equal to t. Let further B be the $m \times (m+n)$ matrix with columns $\delta_1, \dots, \delta_m, v_1, \dots, v_n$, where δ_i is the ith standard bases vector of $\mathbb{C}^m$. Then by the Cauchy–Binet theorem

$$\det\left(I + t \sum_i A_i\right) = \det\left(BDB^*\right) = \sum_{S \in \binom{[m+n]}{m}} \left|B(S)\right|^2 t^{|S \cap \{m+1, \dots, m+n\}|}.$$

Hence the degree of the above polynomial is the size of a maximal linearly independent subset of $\{v_1, \dots, v_n\}$, that is, the dimension of $V_1 + \cdots + V_n$.

Next we will see how polymatroids arise from hyperbolic polynomials. This connection was observed by Gurvits [6]. If $h(x_1, \dots, x_n)$ is a hyperbolic polynomial with respect to e, we define a rank function $\operatorname{rank}_h : \mathbb{R}^n \to \mathbb{N}$ by

$$\operatorname{rank}_h(x) = \deg\bigl(h(e + xt)\bigr).$$

The rank does not depend on the choice e, but only on the hyperbolicity cone of h, that is, $\deg(h(e + xt)) = \deg(h(e' + xt))$ for all e' in the hyperbolicity cone containing e, see [6] and Section 4.

The next proposition follows from the work of Gurvits [6]. He uses Theorem 1.1. In Section 4 we give a proof that does not rely on the Lax conjecture.

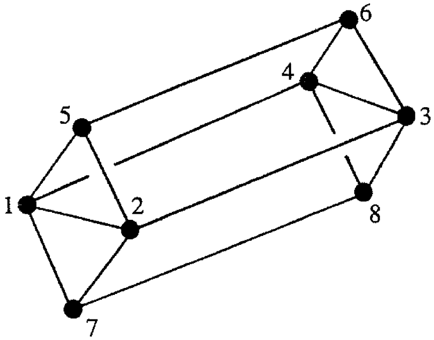

Fig. 1. The Vámos cube.

Proposition 3.2. *Let $h \in \mathbb{R}[x_1, \dots, x_m]$ be a hyperbolic polynomial with respect to $e \in \mathbb{R}^m$, and let $\mathcal{E} = (e_1, \dots, e_n)$ be a tuple of n vectors lying in the closure of the hyperbolicity cone of h containing e. Then the function $r_{\mathcal{E}} : 2^{[n]} \to \mathbb{N}$ defined by*

$$r_{\mathcal{E}}(S) = \mathrm{rank}_h\left(\sum_{i \in S} e_i\right)$$

is a polymatroid.

A *matroid*, $\mathcal{M}$, may be defined as a polymatroid for which the rank function satisfies $r_{\mathcal{M}}(\{i\}) \leqslant 1$ for all $i \in E$. Let $\mathcal{M}$ be a matroid on E. The *set of bases* of $\mathcal{M}$ is

$$\mathcal{B}(\mathcal{M}) = \bigl\{S \subseteq E\colon\ |S| = r_{\mathcal{M}}(S) = r_{\mathcal{M}}(E)\bigr\}.$$

It follows from the equivalent definitions of matroids, see [13], that

$$r_{\mathcal{M}}(S) = \max\bigl\{|S \cap B|\colon\ B \in \mathcal{B}(\mathcal{M})\bigr\}, \tag{4}$$

for all $S \subseteq E$. The *bases generating polynomial* of $\mathcal{M}$ is the polynomial in the variables $(x_i)_{i \in E}$ defined by

$$h_{\mathcal{M}}(x) = \sum_{S \in \mathcal{B}(\mathcal{M})} \prod_{j \in S} x_j.$$

For $i \in E$, let $\delta_i \in \mathbb{R}^E$ be defined by $\delta_j(i) = \delta(i, j)$, where $\delta(i, j)$ is the Kronecker delta. By (4)

$$r_{\mathcal{M}}(S) = \deg\left(h_{\mathcal{M}}\left(\mathbf{1} + t\sum_{i \in S} \delta_i\right)\right), \tag{5}$$

for all $S \subseteq E$.

Let V_8 be the Vámos cube, see [13]. The set of bases of V_8 are all subsets of size four in Fig. 1, that do not lie in an affine plane. The Vámos cube is not representable over any field. However, its bases generating polynomial is hyperbolic with hyperbolicity cone containing $\mathbb{R}_+^8$.

This follows from the fact that V_8 is a so-called *half-plane property matroid* (see [3]) which was proved by Wagner and Wei [16].

We are now in a position to establish the counterexample to Conjecture 1.3.

Theorem 3.3. *Let* $p(x) = h_{V_8}(x_1 + 1, \dots, x_8 + 1)$. *Then:*

(1) $p(x)$ *is a RZ polynomial;*
(2) *There is no positive integer* N *such that* $p(x)^N$ *has a determinantal representation.*

Proof. Wagner and Wei [16] proved that $h_{V_8}(x)$ is a stable polynomial, that is, $h_{V_8}(x)$ is non-zero whenever $\mathrm{Im}(x_i) > 0$ for all i. Hence, if $x \in \mathbb{R}^8$ and $y \in \mathbb{R}^8_+$, then the polynomial $h_{V_8}(x + ty)$ has only real zeros. Thus $h_{V_8}(x)$ is hyperbolic with hyperbolicity cone containing $\mathbb{R}^8_+$. As previously noted it follows that $p(x) = h_{V_8}(x + \mathbf{1})$ is a RZ polynomial.

Suppose that there is an integer $N > 0$ for which

$$p(x)^N = \det(I + x_1 A_1 + \cdots + x_8 A_8),$$

where A_i is hermitian for all i. As in the proof of Theorem 2.2 it follows that

$$h_{V_8}(x)^N = \det(x_1 B_1 + \cdots + x_8 B_8),$$

where B_i is positive semidefinite of size $(8N) \times (8N)$ for all i. Of course $h_{V_8}(x)^N$ is also hyperbolic with the same hyperbolicity cone as $h_{V_8}(x)$. By (5), the rank function of $h_{V_8}(x)^N$ with respect to $\mathcal{E} = \{\delta_1, \dots, \delta_8\}$ satisfies

$$r_{\mathcal{E}}(S) = \deg\left(h_{V_8}^N\left(\mathbf{1} + t\sum_{i \in S} \delta_i\right)\right) = N r_{V_8}(S),$$

for all $S \subseteq [8]$. Hence there is a subspace arrangement $\mathcal{V} = (V_1, \dots, V_8)$ for which

$$r_{\mathcal{V}} = N r_{V_8}.$$

However, it is known that r_{V_8} (and thus also $N r_{V_8}$) fails to satisfy Ingleton's inequalities. This is seen by choosing

$$S_1 = \{5, 6\}, \qquad S_2 = \{7, 8\}, \qquad S_3 = \{1, 4\}, \qquad S_4 = \{2, 3\},$$

in the Ingleton inequalities. □

4. Properties of the rank function of a hyperbolic polynomial

For completeness we give proofs that do not use the Lax conjecture of the properties we use about the rank function associated to a hyperbolic polynomial. We show that these properties are simple consequences of known concavity properties of stable polynomials and discrete convex functions.

A *step* from $\alpha \in \mathbb{Z}^n$ to $\beta \in \mathbb{Z}^n$ is a vector $s \in \mathbb{Z}^n$ of unit length such that

$$|\alpha + s - \beta| < |\alpha - \beta|,$$

where $|\alpha| = \sum_{i=1}^{n} |\alpha_i|$. If s is a step from α to β we write $\alpha \xrightarrow{s} \beta$. A set $\mathcal{J} \subseteq \mathbb{Z}^n$ is called a *jump system* if it respects the following axiom.

(J): If $\alpha, \beta \in \mathcal{J}$, $\alpha \xrightarrow{s} \beta$ and $\alpha + s \notin \mathcal{J}$, then there is a step t such that $\alpha + s \xrightarrow{t} \beta$ and $\alpha + s + t \in \mathcal{J}$.

The support, $\mathrm{supp}(p)$, of a polynomial $p(x) = \sum_{\alpha \in \mathbb{N}^n} a(\alpha) x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is the set $\{\alpha \in \mathbb{N}^n \colon a(\alpha) \neq 0\}$. A polynomial $p \in \mathbb{C}[x_1, \ldots, x_n]$ is *stable* if $p(x) \neq 0$ whenever $\mathrm{Im}(x_j) > 0$ for all j. Let $\leqslant$ be the usual product order on $\mathbb{Z}^n$, i.e., $\alpha \leqslant \beta$ if $\alpha_j \leqslant \beta_j$ for all j.

Theorem 4.1. *(See [2].) The support of a stable polynomial is a jump system.*

Moreover, if all the Taylor coefficients of the stable polynomial p are nonnegative, and $\alpha, \beta \in \mathrm{supp}(p)$ *with* $\alpha \leqslant \beta$*, then* $\gamma \in \mathrm{supp}(p)$ *for all* $\alpha \leqslant \gamma \leqslant \beta$*.*

We need the following simple property of jump systems.

Lemma 4.2. *If* $\mathcal{J} \subset \mathbb{Z}^n$ *is a finite jump system and* $\alpha, \beta \in \mathcal{J}$ *are maximal (or minimal) with respect to* $\leqslant$*, then* $|\alpha| = |\beta|$*.*

Proof. The proof is by contradiction. Let M be the set of maximal elements β of $\mathcal{J}$, with $|\beta| = d$ maximal. Suppose further that $\beta \in M$ is of minimal L^1-distance to the set of all maximal (w.r.t. $\leqslant$) elements α with $|\alpha| < d$. Let α be a maximal element that realizes the above distance to β.

Clearly $\alpha_j > \beta_j$ for some j. Thus δ_j is a step from β to α and $\beta + \delta_j \notin \mathcal{J}$. By (J), $\beta' = \beta + \delta_j + s \in \mathcal{J}$ for some step s from $\beta + \delta_j$ to α. Since β is maximal, the non-zero coordinate in s is negative. Now, $|\beta'| = |\beta|$, so β' is maximal (w.r.t. $\leqslant$). However, $|\beta' - \alpha| < |\beta - \alpha|$ which is the desired contradiction. □

Lemma 4.3. *Suppose that h is hyperbolic with respect to* $e \in \mathbb{R}^n$ *and that* $e_1, \ldots, e_m$ *lie in the hyperbolicity cone of h, and* $e_0 \in \mathbb{R}^n$*. Then the polynomial*

$$p(x_1, \ldots, x_m) = h\left(e_0 + \sum_{j=0}^{m} e_j x_j\right)$$

is stable or identically zero.

Moreover if additionally $h(e) > 0$ *and* e_0 *is in the closure of the hyperbolicity cone of e, then all Taylor coefficients of p are nonnegative.*

Proof. By Hurwitz' theorem we may assume that $e_1, \ldots, e_m$ are in the hyperbolicity cone containing e. Assume that $\alpha \in \mathbb{R}^m$ and $\beta \in \mathbb{R}_+^m$. Then

$$p(\alpha + i\beta) = h\left(e_0 + \sum_{j=1}^{m} \alpha_j e_j + i\left(\sum_{j=1}^{m} \beta_j e_j\right)\right) \neq 0,$$

since the hyperbolicity cone is convex, see [5,15]. Thus p is stable.

To prove the last statement we show that all the Taylor coefficients of

$$q(x_0,\dots,x_m)=h(x_0e_0+\cdots+x_me_m)$$

are nonnegative. Clearly q is hyperbolic (or identically zero) with hyperbolicity cone containing $\mathbb{R}^d_+$, or equivalently, q is homogeneous and stable. It is not hard to prove that such polynomials have nonnegative Taylor coefficients, either using Renegar derivatives [15], or as in [1,3]. Hence, the Taylor coefficients of p are nonnegative. □

Lemma 4.4. *Suppose that h is hyperbolic with respect to $e\in\mathbb{R}^n$ and that e' lies in the hyperbolicity cone containing e. Then*

$$\deg\bigl(h(e+xt)\bigr)=\deg\bigl(h\bigl(e'+xt\bigr)\bigr)$$

for all $x\in\mathbb{R}^n$.

Proof. The polynomial $p(s,t)=h(x+se+te')$ is stable by Lemma 4.3. Let the degree of h be d. By Theorem 4.1, $\mathcal{J}=\operatorname{supp}(p)$ is a jump system and by Lemma 4.2

$$\deg\bigl(h(e+sx)\bigr)=\deg\bigl(s^dp\bigl(s^{-1},0\bigr)\bigr)=d-\min\bigl\{i\colon\ (i,0)\in\mathcal{J}\bigr\}=d-\min\bigl\{|\alpha|\colon\ \alpha\in\mathcal{J}\bigr\},$$

which does not depend on e. □

Corollary 4.5. *Let h be a hyperbolic polynomial with respect to $e\in\mathbb{R}^m$, and let $\mathcal{E}=(e_1,\dots,e_n)$ be a tuple of n vectors lying in the closure of the hyperbolicity cone of h containing e. Let further $\mathcal{J}$ be the support of the stable and homogeneous polynomial $h(x_1e_1+\cdots+x_ne_n)$. Then the rank function associated to $\mathcal{E}$ satisfies*

$$r_{\mathcal{E}}(S)=\max\Biggl\{\sum_{i\in S}\alpha_i\colon\ \alpha\in\mathcal{J}\Biggr\},$$

for all $S\subseteq[n]$.

Proof. Let $p(x_1,\dots,x_n)=h(e+x_1e_1+\cdots+x_ne_n)$. By Lemma 4.3 p is stable and has nonnegative Taylor coefficients. Hence

$$r_{\mathcal{E}}(S)=\deg p\Biggl(t\sum_{i\in S}e_i\Biggr)=\max\Biggl\{\sum_{i\in S}\alpha_i\colon\ \sum_{i\in S}\alpha_i\delta_i\in\operatorname{supp}(p)\Biggr\}.$$

Note that the set of maximal elements (w.r.t. $\leqslant$) of $\operatorname{supp}(p)$ is equal to $\mathcal{J}$. Thus the inequality $r_{\mathcal{E}}(S)\leqslant\max\{\sum_{i\in S}\alpha_i\colon\ \alpha\in\mathcal{J}\}$ follows from Lemma 4.2. Suppose that $\alpha\in\mathcal{J}$ and $S\subseteq[n]$. Since $0\in\operatorname{supp}(p)$ and $0\leqslant\sum_{i\in S}\alpha_i\delta_i\leqslant\alpha$ we have by Theorem 4.1 that $\sum_{i\in S}\alpha_i\delta_i\in\mathcal{J}$. Hence $r_{\mathcal{E}}(S)\geqslant\max\{\sum_{i\in S}\alpha_i\colon\ \alpha\in\mathcal{J}\}$. □

We may now prove Proposition 3.2.

Proof of Proposition 3.2. Keep the notation in the proof of Corollary 4.5. Then $\mathcal{J}$ is a jump system for which all vectors have constant sum. Such jump systems are known to coincide with the set of integer points of integral base polyhedra, see [11]. Clearly $r_{\mathcal{E}}$ satisfies (1) and (2) of the definition of a polymatroid. The submodularity of

$$S \mapsto \max\left\{\sum_{i \in S} \alpha_i \colon \alpha \in \mathcal{J}\right\}$$

holds for every constant sum jump system, see [11]. □

Acknowledgments

It is a pleasure to thank Banff International Research Station for generous support during the workshop "Convex Algebraic Geometry" where this work was initiated. I also thank the participants of the workshop for stimulating discussions on the topic of this paper; especially Helton, Parrilo, Schweighofer, Sturmfels and Vinnikov.

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