Note
On mixed Ramsey numbers

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Abstract

For a graph theoretic parameter $f$, an integer $m$ and a graph $H$, the mixed Ramsey number $r(f; m; H)$ is defined as the least positive integer $p$ such that if $G$ is any graph of order $p$, then either $f(G) \geq m$ or $\overline{G}$ contains a subgraph isomorphic to $H$. Let $\rho$ denote vertex linear arboricity and let $H$ be any connected graph of order $n$. In this note we show that $r(\rho; m; H) = 1 + (n + n_{\rho}(H) - 2)(m - 1)$, where $n_{\rho}(H)$ is the path partition number of $H$. © 1999 Elsevier Science B.V. All rights reserved

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We use Bondy and Murty [2] and Achuthan et al. [1] for terminology and notation not defined here and consider simple graphs only.

Let $G$ be a graph, where $V(G)$ and $E(G)$ denote vertex set and edge set of $G$, respectively. $\overline{G}$ denotes the complement of $G$. $K_n$, $C_n$ and $P_n$ denote the complete graph, cycle and path of order $n$, respectively. For $X \subseteq V(G), G[X]$ denotes the subgraph of $G$ induced by $X$. For two graphs $H$ and $K$, The join $H \oplus K$ is the graph formed from $H \cup K$ by joining every vertex of $H$ to every vertex of $K$. We write $V_1 \oplus F_1 \oplus V_2 \oplus F_2 \oplus \ldots \oplus V_t \oplus F_t$ for the join of the graphs $F_1, F_2, \ldots, F_t$.

A linear forest is a graph whose every component is a path. A partition of $V(G)$ into $t$ subsets such that each subset induces a linear forest is called a $t$-linear forest partition. The vertex linear arboricity of $G$, denoted by $\rho(G)$, is the least positive integer $t$ for which $V(G)$ has a $t$-linear forest partition.

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H \subseteq (m - 1)\overline{P}_{p(H);n-1} = \overline{G}_1. This implies \( v(\rho; m; H) \geq (P(H) - 1)(m - 1) + 1 \). Thus, Claim 1 holds.

**Claim 2.** \( P(H) = |V(H)| + n_p(\overline{H}) - 1 \).

Let \( n_p(\overline{H}) = n \) and \( V(\overline{H}) = \bigcup_{i=1}^{n} V_i \) be an \( n \)-path partition of \( \overline{H} \). Thus, \( \overline{H}[V_i] \) has a spanning path \( P[V_i] \) for each \( i(1 \leq i \leq n) \). Let \( F = \bigcup_{i=1}^{n} P[V_i] \). Clearly, \( F \) is a linear forest of order \( |V(H)| \). Let \( K_{n-1} = \{u_i | 1 \leq i \leq n - 1 \} \). A new path of order \( |V(H)| + n - 1 \) can be formed by adding 2\((n - 1)\) edges to the graph \( F \cup K_{n-1} \), which is written as follows:

\[
P_{|V(H)|+n-1} = (P[V_1] - u_1 - P[V_2] - u_2 - \cdots - P[V_{n-1}] - u_{n-1} - P[V_n]).
\]

Since \( F \subseteq \overline{H} \) and \( |V(F)| = |V(H)| \), we have \( H \subseteq F \subseteq P_{|V(H)|+n-1} \). This establishes the inequality \( P(H) \leq |V(H)| + n_p(\overline{H}) - 1 \).

We next prove \( P(H) \geq |V(H)| + n_p(\overline{H}) - 1 \). Let \( P(H) = t \) and \( n_p(\overline{H}) = n \). Thus \( H \subseteq P_t \) and \( V(H) \subseteq V(P_t) \). Let \( F_1 = P_t \setminus V(H) \) be the subgraph of \( P_t \) induced by \( V(P_t) \setminus V(H) \). Obviously, \( F_1 \) is a linear forest, and \( F_2 = P_t[V(H)] \) is also a linear forest.

If \( K \) is a graph, then \( \omega(K) \) denotes the number of components of the graph \( K \). From the definitions of \( F_1 \) and \( F_2 \), we have \( |\omega(F_1) - \omega(F_2)| \leq 1 \).

To prove \( \omega(F_1) \geq n - 1 \), assume to the contrary that \( \omega(F_1) \leq n - 2 \), and hence \( \omega(F_2) \leq n - 1 \).

All components of \( F_2 \) are written as \( P^{(1)}, P^{(2)}, \ldots, P^{(s)} (s = \omega(F_2) \leq n - 1) \) where \( P^{(i)} \) is a path for every \( i(1 \leq i \leq s) \). Since \( H \subseteq \overline{P}_t \) and \( V(H) = V(F) \), we have \( H \subseteq \overline{P}_t[V(H)] = \overline{P}_t[V(F_2)] = \overline{F}_2 \) and hence \( \overline{F}_2 \subseteq \overline{H} \). This implies that \( \overline{H} \) has an \( s \)-path partition \( V(\overline{H}) = \bigcup_{i=1}^{s} V(P^{(i)}) \), which is impossible because \( n_p(\overline{H}) = n > s \).

Hence \( \omega(F_1) \geq n - 1 \). We have \( t = |V(P_t)| = |V(H)| + |V(F_1)| \geq |V(H)| + \omega(F_1) \geq |V(H)| + n - 1 \). This establishes the inequality \( P(H) \geq |V(H)| + n_p(\overline{H}) - 1 \). Thus Claim 2 holds.

Combining Claims 1 and 2, we have finished the proof of Theorem 1. \( \square \)

**Several remarks.** For any connected graph \( H \), we have determined the mixed Ramsey number \( v(\rho; m; H) \) in terms of the path partition number \( n_p(\overline{H}) \) of \( \overline{H} \). Thus, determining \( v(\rho; m; H) \) is equivalent to determining the value of \( n_p(\overline{H}) \).

Using Theorem 1, we see that Theorems A and B follow immediately from the simple facts that \( n_p(\overline{K}_t) = t, n_p(\overline{K}_{t-1}) = 2 \) and \( n_p(\overline{T}_r) = 1 \) for \( T_r \not\subseteq K_{t-1} \). More generally, we list without proofs the following corollaries, all of which can easily be checked by observing the value of \( n_p(\overline{H}) \) and using Theorem 1.

**Corollary 2.** Let \( H \) be any complete \( t \)-partite graph of order \( n \), and let \( m \geq 1 \) be an integer. Then

\[
v(\rho; m; H) = 1 + (n + t - 2)(m - 1).
\]
Corollary 3. Let $H$ be any connected bipartite graph of order $n$. Then
\[
v(\rho; m; H) = \begin{cases} 
1 + n(m - 1) & \text{when } H \text{ is a complete bipartite graph;} \\
1 + (n - 1)(m - 1) & \text{otherwise.}
\end{cases}
\]

The above two corollaries generalize Theorems A and B, respectively. Furthermore,

Corollary 4. Let $C_n$ denote the cycle of order $n$ and $m \geq 1$ be an integer. Then
\[
v(\rho; m; C_n) = \begin{cases} 
1 + 4(m - 1) & \text{when } n = 3 \text{ or } 4; \\
1 + (n - 1)(m - 1) & \text{when } n \geq 5.
\end{cases}
\]

Corollary 5. Let $H$ be any connected graph of order $n$ with the maximum degree $\Delta(H) \leq \lfloor \frac{1}{2}n \rfloor - 1$. Then $v(\rho; m; H) = 1 + (n - 1)(m - 1)$, where $[x]$ denotes the largest integer not larger than $x$.

Corollary 6. Let $Q_n$ denote $n$-cube and $m \geq 1$ be an integer. Then
\[
v(\rho; m; Q_n) = \begin{cases} 
2m - 1 & \text{when } n = 1; \\
4m - 3 & \text{when } n = 2; \\
1 + (2^n - 1)(m - 1) & \text{when } n \geq 3.
\end{cases}
\]

Although $n_p(H)$ in general is difficult to determine, Theorem 1 gives the relationship between $n_p(H)$ and $v(\rho; m; H)$. Of course, we also can obtain some bounds for $v(\rho; m; H)$.

Corollary 7. For any connected graph $H$ of order $n$ and integer $m \geq 1$, we have
\[v(\rho; m; H) \geq 1 + (n - 1)(m - 1),\]
and equality holds if and only if $\overline{H}$ has a spanning path.

Corollary 8. Let $H$ be any connected graph of order $n(n \geq 2)$, let $\chi(H)$ denote the vertex chromatic number of $H$, and let $m \geq 1$ be an integer. Then
\[v(\rho; m; H) \leq 1 + (n + \chi(H) - 2)(m - 1),\]
and equality holds if and only if $H$ is a complete multipartite graph.

References

