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Note

On mixed Ramsey numbers [1]

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Abstract

For a graph theoretic parameter f, an integer m and a graph H, the mixed Ramsey number $v(f;m;H)$ is defined as the least positive integer p such that if G is any graph of order p, then either $f(G)\geqslant m$ or $\overline{G}$ contains a subgraph isomorphic to H. Let ρ denote vertex linear arboricity and let H be any connected graph of order n. In this note we show that $v(\rho;m;H)=1+(n+n_p(\overline{H})-2)(m-1)$, where $n_p(\overline{H})$ is the path partition number of $\overline{H}$. © 1999 Elsevier Science B.V. All rights reserved

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We use Bondy and Murty [2] and Achuthan et al. [1] for terminology and notation not defined here and consider simple graphs only.

Let G be a graph, where $V(G)$ and $E(G)$ denote vertex set and edge set of G, respectively. $\overline{G}$ denotes the complement of G. K_n, C_n and P_n denote the complete graph, cycle and path of order n, respectively. For $X\subseteq V(G), G[X]$ denotes the subgraph of G induced by X. For two graphs H and K, *The join* HVK is the graph formed from $H\cup K$ by joining every vertex of H to every vertex of K. We write $V_{i=1}^{t}F_i$ for the join of the graphs $F_1,F_2,\ldots,F_t$.

A linear forest is a graph whose every component is a path. A partition of $V(G)$ into t subsets such that each subset induces a linear forest is called a *t-linear forest partition. The vertex linear arboricity of* G, denoted by $\rho(G)$, is the least positive integer t for which $V(G)$ has a t-linear forest partition.

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$H \nsubseteq (m-1)\overline{P}_{P(H)-1} = \overline{G}_1$. This implies $v(\rho;m;H) \geqslant (P(H)-1)(m-1)+1$. Thus, Claim 1 holds.

Claim 2. $P(H) = |V(H)| + n_P(\overline{H}) - 1$.

Let $n_p(\overline{H}) = n$ and $V(\overline{H}) = \bigcup_{i=1}^{n} V_i$ be an n-path partiton of $\overline{H}$. Thus, $\overline{H}[V_i]$ has a spanning path $P_{|V_i|}$ for each $i(1 \leqslant i \leqslant n)$. Let $F = \bigcup_{i=1}^{n} P_{|V_i|}$. Clearly, F is a linear forest of order $|V(H)|$ and $\overline{K}_{n-1} = \{u_i | 1 \leqslant i \leqslant n-1\}$. A new path of order $|V(H)|+n-1$ can be formed by adding $2(n-1)$ edges to the graph $F \cup \overline{K}_{n-1}$, which is written as follows:

$$P_{|V(H)|-n-1} = (P_{|V_1|} - u_1 - P_{|V_2|} - u_2 - \cdots - P_{|V_{n-1}|} - u_{n-1} - P_{|V_n|}).$$

Since $F \subseteq \overline{H}$ and $|V(F)| = |V(\overline{H})|$, we have $H \subseteq \overline{F} \subseteq \overline{P}_{|V(H)|+n-1}$. This establishes the inequality $P(H) \leqslant |V(H)| + n_P(\overline{H}) - 1$.

We next prove $P(H) \geqslant |V(H)| + n_P(\overline{H}) - 1$. Let $P(H) = t$ and $n_P(\overline{H}) = n$. Thus $H \subseteq \overline{P}_t$ and $V(H) \subseteq V(P_t)$. Let $F_1 = P_t - V(H)$ be the subgraph of P_t induced by $V(P_t) \setminus V(H)$. Obviously, F_1 is a linear forset, and $F_2 = P_t[V(H)]$ is also a linear forest.

If K is a graph, then $\omega(K)$ denotes the number of components of the graph K. From the definitions of F_1 and F_2, we have $|\omega(F_1) - \omega(F_2)| \leqslant 1$.

To prove $\omega(F_1) \geqslant n-1$, assume to the contrary that $\omega(F_1) \leqslant n-2$, and hence $\omega(F_2) \leqslant n-1$.

All components of F_2 are written as $P^{(1)}, P^{(2)}, \ldots, P^{(s)}$ $(s = \omega(F_2) \leqslant n-1)$ where $p^{(i)}$ is a path for every $i(1 \leqslant i \leqslant s)$. Since $H \subseteq \overline{P}_t$ and $V(H) = V(F_2)$, we have $H \subseteq \overline{P}_t[V(H)] = \overline{P}_t[V(F_2)] = \overline{F}_2$ and hence $F_2 \subseteq \overline{H}$. This implies that $\overline{H}$ has an s-path partition $V(\overline{H}) = \cup_{i=1}^{s} V(P^{(i)})$, which is impossible because $n_P(\overline{H}) = n > s$.

Hence $\omega(F_1) \geqslant n-1$. We have $t = |V(P_t)| = |V(H)| + |V(F_1)| \geqslant |V(H)| + \omega(F_1) \geqslant |V(H)| + n - 1$. This establishes the inquality $P(H) \geqslant |V(H)| + n_P(\overline{H}) - 1$. Thus Claim 2 holds.

Combining Claims 1 and 2, we have finished the proof of Theorem 1. □

Several remarks. For any connected graph H, we have determined the mixed Ramsey number $v(\rho;m;H)$ in terms of the path partition number $n_P(\overline{H})$ of $\overline{H}$. Thus, determining $v(\rho;m;H)$ is equivalent to determining the value of $n_P(\overline{H})$.

Using Theorem 1, we see that Theorems A and B follow immediately from the simple facts that $n_P(\overline{K}_t) = t, n_P(\overline{K}_{1,t-1}) = 2$ and $n_P(\overline{T}_t) = 1$ for $T_t \ncong K_{1,t-1}$. More generally, we list without proofs the following corollaries, all of which can easily be checked by observing the value of $n_P(\overline{H})$ and using Theorem 1.

Corollary 2. *Let H be any complete t-partite graph of order n, and let $m \geqslant 1$ be an integer. Then*

$$v(\rho;m;H) = 1 + (n+t-2)(m-1).$$

Corollary 3. *Let H be any connected bipartite graph of order n. Then*

$$v(\rho;m;H)=\begin{cases}1+n(m-1) & \text{when } H \text{ is a complete bipartite graph};\\ 1+(n-1)(m-1) & \text{otherwise.}\end{cases}$$

The above two corollaries generalize Theorems A and B, respectively. Furthermore,

Corollary 4. *Let C_n denote the cycle of order n and $m\geqslant 1$ be an integer. Then*

$$v(\rho;m;C_n)=\begin{cases}1+4(m-1) & \text{when } n=3 \text{ or } 4;\\ 1+(n-1)(m-1) & \text{when } n\geqslant 5.\end{cases}$$

Corollary 5. *Let H be any connected graph of order n with the maximum degree $\Delta(H)\leqslant[\frac{1}{2}n]-1$. Then $v(\rho;m;H)=1+(n-1)(m-1)$, where $[x]$ denotes the largest integer not larger than x.*

Corollary 6. *Let Q_n denote n-cube and $m\geqslant 1$ be an integer. Then*

$$v(\rho;m;Q_n)=\begin{cases}2m-1 & \text{when } n=1;\\ 4m-3 & \text{when } n=2;\\ 1+(2^n-1)(m-1) & \text{when } n\geqslant 3.\end{cases}$$

Although $n_p(\overline{H})$ in general is difficult to determine, Theorem 1 gives the relationship between $n_p(\overline{H})$ and $v(\rho;m;H)$. Of course, we also can obtain some bounds for $v(\rho;m;H)$.

Corollary 7. *For any connected graph H of orden n and integer $m\geqslant 1$, we have*

$$v(\rho;m;H)\geqslant 1+(n-1)(m-1),$$

and equality holds if and only if $\overline{H}$ has a spanning path.

Corollary 8. *Let H be any connected graph of order $n(n\geqslant 2)$, let $\chi(H)$ denote the vertex chromatic number of H, and let $m\geqslant 1$ be an integer. Then*

$$v(\rho;m;H)\leqslant 1+(n+\chi(H)-2)(m-1),$$

and equality holds if and only if H is a complete multipartite graph.

References

[1] N. Achuthan, N.R. Achuthan, L. Caccetta, On mixed Ramsey numbers, Discrete Math. 151 (1996) 3–13.
[2] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan, London, 1977.
[3] E.M. Cleves, M.S. Jacobson, On mixed Ramsey numbers: total chromatic number vs. graphs, Congr. Numer. 39 (1983) 193–201.

[4] J.F. Fink, Mixed Ramsey numbers: total chromatic numbers vs. stars (The diagonal case), J. Combin. Inform. System Sci. 5 (1980) 200–204.

[5] L. Lesniak, A.D. Polimeni, D.W. Vander Jagt, Mixed Ramsey numbers: edge chromatic numbers vs graphs, in: Proc. Internat. Conf. on the Theory and Applications of Graphs, Springer, Berlin, 1976, pp. 330–341.

[6] L. Lesniak-Foster, Mixed Ramsey numbers: vertex arboricity vs. graphs, Bull. Calcutta Math. Soc. 71 (1979) 23–28.