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FURTHER GOSSIP PROBLEMS*

D.J. KLEITMAN and J.B. SHEARER

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

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n people have distinct bits of information. They can communicate via k-party conference calls. How many such calls are needed to inform everyone of everyone else's information? Let $f(n, k)$ be this minimum number. Then we give a simple proof that $f(n, k) = \lceil (n-k)(k-1) \rceil + \lceil n\ k \rceil$ for $1 \leq n \leq k^2$, $f(n, k) = 2\lceil (n-k)(k-1) \rceil$ for $n > k^2$.

In the 2-party case we consider the case in which certain of the calls may permit information flow in only one direction. We show that any $2n$-4 call scheme that conveys everone's information to all must contain a 4-cycle, each of whose calls is "two way", along with some other results. The method follows that of Bumby who first proved the 4-cycle conjecture.

This paper contains several results on "gossip problems" that follow from Bumby's approach to it. These are discussed in the following two sections.

1. Conference calls

Suppose n people have distinct bits of information. They can communicate via k-party conference calls. Let $f(n, k)$ be the minimum number of calls needed to inform everyone of everyone else's information. We prove the following:

Theorem 1.

$$f(n, k) = \left\lceil \frac{n-k}{k-1} \right\rceil + \left\lceil \frac{n}{k} \right\rceil, \quad 1 \leq n \leq k^2,$$

$$f(n, k) = 2\left\lceil \frac{n-k}{k-1} \right\rceil, \quad n \geq k^2.$$

This result was first proved by Lebensold [2] but his argument was very complicated. This paper uses ideas used by Bumby to prove the 4-cycle conjecture for 2-party calls [1].

Lemma 1. $f(n, k) \geq \lceil (n-k)/(k-1) \rceil + \lceil n/k \rceil$.

Proof. We claim no person can know everything until $\lceil (n-1)/(k-1) \rceil$ calls have been made. For suppose after m calls person x knows everything. Then if we

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reverse the order of these m calls we have a scheme in which after m calls everbody knows person x's information. Now clearly $m \geq \lceil (n-1)/(k-1) \rceil$ since each call can only inform $k-1$ additional persons of x's information. Hence after $\lceil (n-1)/(k-1) \rceil - 1 = \lceil (n-k)/(k-1) \rceil$ calls no one knows everything. Hence any scheme achieving $f(n, k)$ must involve at least one additional call for everyone or $\lceil n/k \rceil$ more calls. Hence $f(n, k) \geq \lceil (n-k)/(k-1) \rceil + \lceil n/k \rceil$ as claimed.

Lemma 2. $f(n, k) \geq 2\lceil (n-k)/(k-1) \rceil$.

Proof. Let A be a scheme achieving $f(n, k)$. A will impose a partial order on the $f(n, k)$ calls (where we allow any rearrangement of the calls which does not effect the information exchanged in each call). A call E_1 will precede E_2 in the partial order iff there is a chain of calls occurring after E_1 and before E_2 in any such rearrangement which informs one of the parties of E_2 of the information exchanged in E_1. Let $m = \lceil (n-k)/(k-1) \rceil$. Consider the graph, G, on n points formed by identifying each of the people with a point and replacing each of the first m calls with a tree on the k points involved in that call. At least two components of this graph must be trees T_1, T_2 (as we start with n trees and each call reduces the number of trees by at most $k-1$). Consider those total orders on the calls consistent with the partial order which minimize the size of $T_1 \cup T_2$. Select a particular order which also minimizes T_1. If T_1 consists of a single point P, then after m calls no one else knows P's information and hence $\lceil (n-1)/(k-1) \rceil$ additional calls are needed to inform everyone else. Hence we may assume C_i is the last (among the first m) call in T_i, $i = 1, 2$. Let C_i connect subtrees $T_{i1}, \ldots, T_{ik}$, $i = 1, 2$. Note we can choose a total order in which C_1 is the mth call and C_2 is the $m-1$th. We now claim the remaining calls must be constrained to follow C_1. For suppose not; then there must exist a call D taking place after the first m which is not constrained to follow C_1 or any of the other remaining calls. Then there is a consistent total order in which D replaces C_1 among the first m calls. Consider how this affects G. If D does not connect k of $T_{11}, \ldots, T_{1k}$, T_2 then this contradicts the minimality of $T_1 \cup T_2$. If D connects $k-1$ of $T_{11}, \ldots, T_{1k}$ to T_2 then this contradicts the minimality of T_1. In the remaining case D connects $T_{11}, \ldots, T_{1k}$. But here the order of D and C_2 may be interchanged. Hence there is a consistent total order in which C_2 is replaced by D among the first m calls again contradicting the minimality of $T_1 \cup T_2$. But since the remaining calls must follow C_1 and since no one knows everthing (and hence must participate in at least one more call) after m calls (as G is unconnected), $\lceil (n-k)/(k-1) \rceil$ more calls must occur (since each of the remaining calls can include at most $k-1$ people who have not been informed of the contents of C_1). This proves the lemma.

Lemma 3.

$$f(n,\ k) \geqslant \left\lceil \frac{n-k}{k-1} \right\rceil + \left\lceil \frac{n}{k} \right\rceil, \quad 1 \leqslant n \leqslant k^2,$$

$$f(n,\ k) \geqslant \left\lceil \frac{n-k}{k-1} \right\rceil, \quad k^2 \leqslant n.$$

Proof. Immediate from Lemmas 1 and 2 and the observation that

$$\left\lceil \frac{n}{k} \right\rceil = \left\lceil \frac{nk-n}{k(k-1)} \right\rceil \quad \text{while} \quad \left\lceil \frac{n-k}{k-1} \right\rceil = \left\lceil \frac{nk-k^2}{k(k-1)} \right\rceil.$$

It remains to give constructions showing the above bounds are achievable.

Lemma 4. *If $m \leqslant n$, then $f(n,\ k) \leqslant f(m,\ k) + 2\lceil (n-m)/(k-1) \rceil$.*

Proof. Label the n people $1, \ldots, n$. Let the last $n-m$ inform the first person of their information. This can be done with $\lceil (n-m)/(k-1) \rceil$ calls. Then let the first m people pool their information. This can be done with $f(m,\ k)$ calls. Finally let the first person inform the last $n-m$ of what he now knows. This can be done with $\lceil (n-m)/(k-1) \rceil$ calls. Clearly now everyone knows everthing proving the lemma.

Lemma 5. *If $r \leqslant k$, then $f(rk,\ k) \leqslant 2r$.*

Proof. Label the people $1, \ldots, rk$. Let the first r calls be between the congruence classes mod r. Let the last r be between $\{1, \ldots, k\}$, $\{k+1, \ldots, 2k\}, \ldots \{(r-1)k + 1, \ldots, rk\}$. Clearly everbody now knows everthing as the last calls must include a member of each congruence class mod r since $r \leqslant k$.

Proof of Theorem 1. Lemma 5 shows $f(k^2, k) \leqslant 2k$. Hence by Lemma 4 for $n \geqslant k^2$,

$$f(n,\ k) \leq 2k + 2\left\lceil \frac{n-k^2}{k-1} \right\rceil = 2\left\lceil \frac{n-k}{k-1} \right\rceil.$$

Lemma 5 also implies for $n \leqslant k^2$ that $f(n,\ k) \leqslant 2\ n/k$ holds. Also since $f(k,\ k) = 1$ Lemma 4 implies for $k \leqslant n$ that

$$f(n,\ k) \leqslant 1 + 2\left\lceil \frac{n-k}{k-1} \right\rceil = \left\lceil \frac{n-1}{k-1} \right\rceil + \left\lceil \frac{n-k}{k-1} \right\rceil$$

holds. We claim the last two statements together imply that for $k \leqslant nk^2$ we have

$$f(n,\ k) \leqslant \left\lceil \frac{n-k}{k-1} \right\rceil + \left\lceil \frac{n}{k} \right\rceil.$$

For suppose not. Then $2\lceil n/k\rceil > \lceil (n-k)/(k-1)\rceil + \lceil n/k\rceil$ and hence

$$\left\lceil \frac{n}{k} \right\rceil \geq \left\lceil \frac{n-k}{k-1} \right\rceil + 1$$

and $\lceil (n-k)/(k-1)\rceil + \lceil (n-1)/(k-1)\rceil > \lceil (n-k)/(k-1)\rceil + \lceil n/k\rceil$ and hence

$$\left\lceil \frac{n-1}{k-1} \right\rceil \geq \left\lceil \frac{n}{k} \right\rceil + 1.$$

Summing gives

$$\left\lceil \frac{n}{k} \right\rceil + \left\lceil \frac{n-1}{k-1} \right\rceil \geq \left\lceil \frac{n-k}{k-1} \right\rceil + \left\lceil \frac{n}{k} \right\rceil + 2 \quad \text{or} \quad \left\lceil \frac{n-1}{k-1} \right\rceil \geq \left\lceil \frac{n-1}{k-1} \right\rceil + 1,$$

a contradiction. The result follows from Lemma 3.

2. Four cycles

We consider schemes of 2-party calls achieving $f(n,2) = 2n-4$ (for $n \geq 4$). We assume some of the calls (which we call one-way) convey information in one direction only. As in Section 1 there is associated with any such scheme L, a partial order on the $2n-4$ calls. Also as before we associate a graph with any set of calls.

We prove the following theorem which contains the 4-cycle conjecture. Again the argument uses ideas from [1].

Theorem 2. *Let S be the set of vertices of a tree-like connected component in the graph induced by the first $n-1$ calls in a consistent total order. (Note for any total order at least one such S must exist). Then S contains 4 people a, b, c, d such that L contains calls (a,b), (c,d), (a,c), (b,d) but not (a,d) or (b,c). Moreover these calls must be two-way. Also (a,b) and (c,d) are unrelated in the partial order associated with L and the same is true of (a,c) and (b,d).*

Proof. Let S be a minimal connected component in the graph of the first $(n-1)$ pairs among consistent total orderings of L. Let S have m vertices. S must have at least 2 vertices as otherwise at least $n-1$ additional calls would be required (or $2n-2$ in all) a contradiction. If the $(m-1)$st call in some consistent ordering on S is removed, the vertices in S divide into two connected components S_1 and S_2; choose a consistent reordering of these calls so the S_1 is minimal (that is, so that no other reordering has one of its components properly contained in S_1); let the $(m-1)$st pair here be (x_1, y_1) with x_1 in S_1, y_1 in S_2; we draw the following conclusions, each of which will be proven below.

(1) S_1 contains at least two vertices.

(2) Let $|S_1| = j$. The $(j-1)$ calls in S_1 among the first $(m-2)$ calls in S in this

ordering form a tree T_1. A call in a sequence q of calls is special in q if it must come after every call it overlaps in q. Then there is only one special call in the sequence of calls in T_1. This special call must share an element with (x_1, y_1); let the call be (x_1, x_2).

(3) There is a sequence of calls conveying information from $x_1(x_2)$ to every other element not in the same component of $T_1-(x_1, x_2)$ as $x_1(x_2)$ that involves only the pairs (x_1, x_2) (x_1, y_1) and the $(n-3)$ last pairs in our ordering of L; (x_1, x_2), (x_1, y_1) and these last $n-3$ pairs form a tree. (Remarks one and three in themselves form a proof that $(2n-4)$ calls are the smallest possible number. It obviously requires $(n-1)$ calls among (x_1, x_2), (x_1, y_1) and the pairs after the $(n-1)$st to form a tree, so that there must be at least $(n-3)$ after the first $(n-1)$).

(4) Any sequence of calls conveying information to x_1 or x_2 from any other element can contain at most one call of the final $(n-3)$ calls $\cup\{(x_1, y_1), (x_1, x_2)\}$ and this call must occur last in the sequence.

(5) Let y_2 be defined as the last element before x_2 on some sequence of calls conveying information from y_1 to x_2. Similarly let y_{j+1} be the last element other than x_1 (or x_2) on a sequence from y_j to x_1 if j is even, to x_2 if j is odd. Then for some odd j_1, even $j_2(y_{j_1}, y_{j_2})$ is a two-way call in S_2. Furthermore (y_{j_1}, x_1), (y_{j_2}, x_2) are two-way calls following (y_{j_1}, y_{j_2}) and (x_1, x_2) in the partial order. Finally (x_1, x_2) must be a two-way call.

Proof of (1). If S_1 contains only one element, after the first $n-2$ calls there is no path at all from S_1. To obtain paths from that element to all the others requires at least $n-1$ further calls or $2n-3$ all together, a contradiction.

Proof of (2). Suppose there were a special call (a, b) in S_1 disjoint from (x_1, y_1). Then we could interchange the order of (x_1, y_1) and (a, b) making (a, b) the $(m-1)$st pair in S. The resulting graph of the first $(m-2)$ calls in S would have a connected component properly contained in S_1 violating S_1's minimality.

Proof of (3). Suppose there were no such sequence from x_1 to z_1. By assumption x_1 and z_1 are not in the same component of $T_1-(x_1, x_2)$. There is a sequence from x_1 to z_1 in L; let its first call that is not in T_1 be (a, b), (a to b). (a, b) and the calls beyond (a, b) in the sequence must be among the last $(n-3)\cup\{(x_1, y_1), (x_1, y_2)\}$.

If there is a sequence of calls conveying information from x_1 to a or b that precedes the call (a, b) and only uses the last $(n-3)$ calls and (x_1, x_2) and (x_1, y_1), we could switch and get a sequence of the desired kind from x_1 to z_1. Otherwise it must be possible to reorder the calls consistently so that the call (a, b) comes before (x_1, x_2). Now suppose we reorder the calls in this way so that (x_1, y_1) is the nth, (x_1, x_2) is $(n-1)$st and (a, b) or a call from the last $n-3$ that must precede (a, b) is $(n-2)$nd, and the order of the others is unchanged.

If the new $(n-2)$nd call is disjoint from S_1 or S_2, that set (S_1 or S_2) becomes a connected component in the new ordering, which violates the assumed minimality of S. If it joins S_1 with S_2 then the analogue of S_1 in the new ordering will be properly contained in the old S_1, violating minimality of S_1. This is a contradiction.

Since only $(n-2)$ calls are to be used here to obtain paths from x_1 or x_2 to all $(n-2)$ other elements, these $(n-2)$ pairs along with (x_1, x_2) must form a tree.

Proof of (4). The above tree must contain sequences of calls from x_1 to x_2 to all other elements. Furthermore, if a sequence contains a call from this set all subsequent calls must also be from this set. The only sequences of calls that convey information to x_1 (or x_2) in this tree come from immediate neighbors in it and are one call long. This suffices to prove (4).

Proof of (5). It follows from (4) that there is a sequence of calls from y_j to y_{j+1} among the first $n-1$ calls omitting (x_1, x_2), (x_1, y_1). With respect to these calls S_2 is a tree-like connected component. Hence if $y_j \in S_2$ so is y_{j+1}. Hence all the y_j's are in S_2. Since S_2 is finite there is some j such that $y_j = y_{j+k}$ for some k. k must be even as (x_1, y_j) and (x_2, y_j) cannot both be in the set of the last $n-3$ calls $\cup \{(x_1, x_2), (x_1, y_1)\}$ as these calls form a tree. Since the sequences of calls $y_j \to y_{j+1} \to \cdots \to y_{j+k} = y_j$ lie on a tree every call used in them must be used both ways. Furthermore we claim every point in the above paths must be a y_l for some l $(j \leq l \leq j+k)$. For suppose not. Let a be a point unequal to any y_l. Of all the calls containing a and occurring in $y_j \to y_{j+1} \to \cdots \to y_{j+k}$ let (b, a) be the latest (with respect to the partial order). (b, a) must be used in sequences of calls in both directions. However (b, a) cannot be followed in a sequence by any call involving a because of the way (b, a) was chosen. Hence a must be an endpoint (i.e. a must equal some y_l) a contradiction.

It follows at once that we can find a pair (y_{j_1}, y_{j_2}) where j_1 is odd, j_2 is even. The pairs (y_{j_1}, x_1) and (y_{j_2}, x_2)must be usable towards x_1, x_2 in order to convey information to x_1, x_2. They must be usable in the opposite direction since by (3), (x_1, x_2), (x_2, y_{j_2}) must be a sequence from x_1 to y_{j_2} and similarly (x_2, x_1), (x_1, y_{j_1}) must be a sequence from x_2 to y_{j_1}. This also implies (x_1, x_2) is a two-way call completing the proof of Theorem 2.

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