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Triangles Inscribed in a Semicircle, in Minkowski Planes, and in Normed Spaces

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In this paper we mainly consider triangles inscribed in a semicircle of a normed space; in two-dimensional spaces, their perimeter has connections with the perimeter of the sphere. Moreover, by using the largest values the perimeter of such triangles can have, we define two new, simple parameters in real normed spaces: one of these parameters is strictly connected with the modulus of convexity of the space, while the study of the other one seems to be more complicated. We calculate the value of our two parameters and we bring out a few connections among their values and the geometry of real normed spaces. © 2000 Academic Press

1. INTRODUCTION

Let $(X, \|.\|)$ be a normed space, of dimension at least 2, over the field R. In this paper we define two new, simple parameters in normed spaces; then we want to bring out a few connections among their values and some

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geometrical properties of the space. These parameters measure how big the sum of the distances from a point of the unit sphere to two antipodal points can be; in other terms, their value depends on the perimeter of triangles with the diameter as one side and the third vertex on the sphere. One of the two parameters (which we call A_2) has a two-dimensional character and depends on the modulus of convexity of the unit sphere; the other one, which we denote by A_1, has a different character and the description we give for it is not complete. In any case, these constants give information on the geometry of the space, both in the finite- and in the infinite-dimensional case.

We list the notation we shall use.

$$S_X = \{x \in X; \|x\| = 1\},$$

$$B_X = \{x \in X; \|x\| \leq 1\};$$

we shall simply write S (B) instead of S_X (resp.; B_X) when no confusion can arise.

X^* will denote the dual of X.

We shall denote by $(R^2)_\infty$ and $(R^2)_1$ the two-dimensional plane endowed with the max norm and the sum norm, respectively.

Given X, its modulus of convexity, $\delta(\varepsilon)$, for $\varepsilon \in [0, 2]$, is defined as

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x + y\|}{2}; x, y \in S; \|x - y\| \geq \varepsilon\right\}. \tag{1.1}$$

We recall that δ is nondecreasing and continuous for $\varepsilon < 2$. Moreover, if $\delta(\varepsilon) > 0$, we always have (see e.g. [6, p. 56])

$$1 - \frac{\varepsilon}{2} = \delta(2 - 2\delta(\varepsilon)). \tag{1.2}$$

A space is said to be *uniformly nonsquare* when

$$\lim_{\varepsilon \to 2^-} \delta(\varepsilon) > 0. \tag{1.3}$$

Recall that uniformly nonsquare spaces are reflexive (see e.g. [6, p. 57]).

We shall denote by δ^* the modulus of convexity of X^*.

We make some use of the modulus of smoothness of X, as defined in [8]. For our purposes it will be enough to recall that, given X, its modulus of smoothness $\rho_X(t)$, $t \in (0, 2)$, is defined so that

$$\rho_X(1) = \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1; x, y \in S_X\right\}. \tag{1.4}$$

By a well-known result of Lindenstrauss (see [8]), for every space X the modulus of smoothness satisfies:

$$\rho_X(1) = \sup\left\{\frac{\varepsilon}{2} - \delta^*(\varepsilon); 0 \le \varepsilon \le 2\right\}. \tag{1.5}$$

If X is a two-dimensional space, then we can define the perimeter as the "self length" of the unit sphere S,

$$p(X) = 2\gamma(-x, x), \tag{1.6}$$

where

$$\gamma(-x, x) = \text{the length of the curve joining } -x \text{ and } x \text{ along } S$$
$$(x \text{ an arbitrary point of } S).$$

We recall that $p(X) \in [6, 8]$: the extreme values characterize, respectively, the hexagon ($p(X) = 6$) and the parallelogram ($p(X) = 8$) (see e.g. [10, Section 4]).

2. STUDYING TWO NEW CONSTANTS

We define the following numbers:

$$A_1(X) = \tfrac{1}{2} \inf_{x \in S_X} \sup_{y \in S_X} (\|x - y\| + \|x + y\|); \tag{2.1}$$

$$A_2(X) = \tfrac{1}{2} \sup_{x \in S_X} \sup_{y \in S_X} (\|x - y\| + \|x + y\|). \tag{2.2}$$

The second constant had already been considered in [2], where the following fact was proved (see [2, Section 3]): $A_2(X) = 2$ characterizes spaces which are not uniformly nonsquare.

Note that $1 \le A_1(X) \le A_2(X) \le 2$ always; that $A_1(X) = A_2(X) = \sqrt{2}$ in inner product spaces; and that $A_2(X) = 2 > 3/2 = A_1(X)$ for $X = (R^2)_\infty$ and $X = (R^2)_1$.

According to (1.4), we have

$$A_2(X) = \rho_X(1) + 1. \tag{2.3}$$

Therefore, according to (1.5), we have

$$A_2(X) - 1 = \sup\left\{\frac{\varepsilon}{2} - \delta^*(\varepsilon); 0 \le \varepsilon \le 2\right\}. \tag{2.4}$$

LEMMA 2.1. *For every space X, we have*

$$A_2(X) - 1 \le \sup\left\{\frac{\varepsilon}{2} - \delta(\varepsilon); 0 \le \varepsilon \le 2\right\}. \tag{2.5}$$

Proof. Take $x \in S$; if $y \in S$ and $\|x - y\| = \varepsilon$, then $\|x + y\|/2 \leq 1 - \delta(\varepsilon)$. So we obtain, for any $y \in S$,

$$\|x - y\| + \|x - y\| \leq \varepsilon + 2(1 - \delta(\varepsilon)).$$

By taking the supremum for $x, y \in S$, we obtain

$$\begin{aligned} 2 \cdot A_2(X) &\leq \sup\{\varepsilon + 2(1 - \delta(\varepsilon));\ \varepsilon \in [0,2]\} \\ &= 2 + \sup\{\varepsilon - 2\delta(\varepsilon);\ \varepsilon \in [0,2]\}, \end{aligned}$$

so we have the thesis. ∎

PROPOSITION 2.2. *For every space X, we have*

$$\sup\left\{\frac{\varepsilon}{2} - \delta^*(\varepsilon); 0 \leq \varepsilon \leq 2\right\} = \sup\left\{\frac{\varepsilon}{2} - \delta(\varepsilon); 0 \leq \varepsilon \leq 2\right\}; \quad (2.6)$$

$$A_2(X) = A_2(X^*) = 1 + \sup\left\{\frac{\varepsilon}{2} - \delta(\varepsilon); 0 \leq \varepsilon \leq 2\right\}. \quad (2.7)$$

Proof. If X is not uniformly nonsquare, then also X^* is not uniformly nonsquare (see e.g. [11, p. 12]), and then (2.6) and (2.7) are trivial (in this case $A_2(X) = A_2(X^*) = 2$). Now let X be uniformly nonsquare, and so also reflexive. Then, according to (2.4) and (2.5), we have

$$\begin{aligned} &\sup\left\{\frac{\varepsilon}{2} - \delta^*(\varepsilon); 0 \leq \varepsilon \leq 2\right\} \\ &\quad = A_2(X) - 1 \leq \sup\left\{\frac{\varepsilon}{2} - \delta(\varepsilon); 0 \leq \varepsilon \leq 2\right\} \\ &\quad = \sup\left\{\frac{\varepsilon}{2} - \delta^{**}(\varepsilon); 0 \leq \varepsilon \leq 2\right\} = A_2(X^*) - 1 \\ &\quad \leq \sup\left\{\frac{\varepsilon}{2} - \delta^*(\varepsilon); 0 \leq \varepsilon \leq 2\right\}, \end{aligned}$$

so we have the thesis. ∎

Remark. According to Proposition 2.2 and its proof, we obtain (see (2.3)) that in any space X the following is true (cf. [11, p. 63]):

$$\rho_{X^*}(1) = A_2(X^*) - 1 = \sup\left\{\frac{\varepsilon}{2} - \delta^*(\varepsilon); 0 \leq \varepsilon \leq 2\right\}.$$

PROPOSITION 2.3. *In any space X, we have*

$$A_1(X) \cdot A_2(X) \geq 2. \quad (2.8)$$

In particular, in any space X, we have

$$A_2(X) \geq \sqrt{2}. \qquad (2.9)$$

Proof. We shall prove that for every space X, we have $A_2(X) \geq 2/A_1(X)$. Our definition implies that, for every $\varepsilon > 0$, there exists $x \in S$ such that $\sup_{y \in S}(\|x - y\| + \|x + y\|) < 2(A_1(X) + \varepsilon)$. Take $y_0 \in S$ such that $\|x - y_0\| = \|x + y_0\|$ are equal, say $= \alpha$: Clearly $1 \leq \alpha < A_1(X) + \varepsilon$.

Now set $u = (x + y_0)/\alpha$; $v = (x - y_0)/\alpha$ $(u, v \in S)$; then $\|u + v\| = 2\|x\|/\alpha = 2/\alpha = 2\|y_0\|/\alpha = \|u - v\|$. This means that $2A_2(X) \geq \sup_{y \in S}(\|u - y\| + \|u + y\|) \geq \|u - v\| + \|u + v\| = 4/\alpha > 4/(A_1(X) + \varepsilon)$.

Since $\varepsilon > 0$ can be arbitrarily small, this implies that $A_2(X) \geq 2/A_1(X)$, so we have the thesis. ∎

Remarks. The inequality (2.9), which we have proved directly here, will also follow from results in Section 3.

The inequality $A_2(X) \geq \sqrt{2}$, together with the fact that the modulus of convexity δ is nondecreasing, implies that

$$A_2(X) = 1 + \sup\left\{\frac{\varepsilon}{2} - \delta(\varepsilon); 2(\sqrt{2} - 1) \leq \varepsilon < 2\right\}. \qquad (2.7')$$

But a better result can be indicated.

PROPOSITION 2.4. *For any space X, we have*

$$A_2(X) = 1 + \sup\left\{\frac{\varepsilon}{2} - \delta(\varepsilon); \sqrt{2} \leq \varepsilon < 2\right\}. \qquad (2.10)$$

Proof. Given $\sigma > 0$, we can find pairs x, y in S such that $\|x - y\| + \|x + y\| > 2 \cdot A_2(X) - \sigma$; if $\|x - y\| = \varepsilon$ for such a pair, then $\|x + y\|/2 \leq 1 - \delta(\varepsilon)$, so $\varepsilon + 2(1 - \delta(\varepsilon)) > 2 \cdot A_2(X) - \sigma \geq 2(1 + \varepsilon/2 - \delta(\varepsilon)) - \sigma$. By interchanging the role of $\|x - y\|$ and $\|x + y\|$, since $\sigma > 0$ is arbitrary, this means that to obtain the value of A_2 it is enough to consider only $\varepsilon \leq A_2(X)$, or only $\varepsilon \geq A_2(X)$:

$$1 + \sup\left\{\frac{\varepsilon}{2} - \delta(\varepsilon); A_2 \leq \varepsilon < 2\right\} = 1 + \sup\left\{\frac{\varepsilon}{2} - \delta(\varepsilon); 0 < \varepsilon \leq A_2\right\}$$

$$(= A_2(X)). \qquad (2.10')$$

In particular, this proves (2.10). ∎

It is easy to see that if we have two spaces X, Y with $Y \subset X$, then $A_2(Y) \leq A_2(X)$. Moreover,

$$A_2(X) = \sup\{A_2(Y); Y \text{ is a two-dimensional subspace of } X\} \quad (2.11)$$

(the sup being also a maximum if $\dim(X) < \infty$).

Also, according to Dvoretzki's theorem, given $\varepsilon > 0$, if the dimension of X is large enough (in particular, if $\dim(X) = \infty$), then there exists a subspace Y of X, with $\dim(X) = 2$, such that $|A_2(X) - \sqrt{2}| < \varepsilon$ (this again implies that $A_2(X) \geq \sqrt{2}$ if $\dim(X) = \infty$).

Concerning A_1, set

$$U_1(X) = \{A_1(Y); Y \text{ is a two-dimensional subspace of } X\}. \quad (2.11')$$

Note that if $\dim(X) = \infty$, then according to Dvoretzki's theorem we have

$$\inf(U_1(X)) \leq \sqrt{2} \leq \sup(U_1(X)). \quad (2.11'')$$

Propositions 2.5 and 2.8 below will give general lower and upper bounds for $U_1(X)$.

Given X, there exists $x \in S$ such that

$$\sup_{y \in S_X} (\|x + y\| + \|x - y\|) = 2a \cong 2 \cdot A_1(X).$$

Thus $\|x + y\| + \|x - y\| \cong 2 \cdot A_1(X)$ for some $y \in S_X$ (in finite-dimensional spaces, there are also $x, y \in S_X$ for which equality holds). Therefore, if Y is the two-dimensional subspace of X generated by x and y, $\sup_{y \in S_Y}(\|x + y\| + \|x - y\|) \leq 2a$, and $A_1(Y) \leq A_1(X)$, then

$$A_1(X) \geq \inf(U_1(X)). \quad (2.11''')$$

Strict inequality holds in many cases (e.g., in l_1, where $A_1(X) = 2$).

No relation exists between $A_1(X)$ and $\sup(U_1(X))$: In fact, if $\dim(X) = \infty$, the last number is always at least $\sqrt{2}$ (see (2.11″)), while $A_1(X)$ can be smaller. Similarly, $A_1(X)$ can be 2 while Proposition 2.8 will give a smaller upper bound for $U_1(X)$.

The next proposition gives a general lower bound for $A_1(X)$.

PROPOSITION 2.5. *In any space X, we have*

$$A_1(X) \geq \frac{3 + \sqrt{21}}{6} \ (\cong 1.264). \quad (2.12)$$

Proof. We shall prove that $A_1(X) \geq (3 + \sqrt{21})/6$ for every two-dimensional space X.

Take $h > A_1(X)$. Choose $x \in S$ such that $\|x - v\| + \|x + v\| \leq 2h$ for every $v \in S$, then take $y \in S$ so that $\|x - y\| = \|x + y\|$, say $= k$ ($1 \leq k \leq \min\{h, 2\}$). Now consider the function $f(t) = \|x + ty\|$: we have $f(1) = f(-1) = k$; $f(0) = 1$; the slope of f is always not larger than 1. Set $v = (x + y)/k$ ($v \in S$); if $k = 1$, then $\|x - v\| = \|y\| = 1$. Let $k > 1$; we then obtain

$$\|x + v\| = \left\| x \cdot \left(1 + \frac{1}{k}\right) + \frac{y}{k} \right\| = \left(1 + \frac{1}{k}\right) \left\| x + \frac{y}{k} \cdot \frac{k}{k+1} \right\|.$$

It is not a restriction to assume that $\|x + ty\| \geq 1$ for all $t > 0$ (otherwise we may exchange y and $-y$), so that $\|x + v\| \geq 1 + 1/k$.

Now consider $\|x - v\| = \|x(1 - 1/k) - y/k\| = (1 - 1/k) \cdot \|x - y/(k - 1)\|$. We can estimate $\|x - y/(k - 1)\|$ being $(1/(1 - k) \leq -1)$ in the following way.

We have

$$\left\| x - \frac{y}{k-1} \right\| + \left\| x + \frac{y}{k-1} \right\| \geq \frac{2\|y\|}{k-1};$$

but

$$\left\| x + \frac{y}{k-1} \right\| \leq \|x + y\| + \left(\frac{1}{k-1} - 1 \right) \cdot \|y\| = k + \frac{2-k}{k-1}$$
$$= \frac{k^2 - 2k + 2}{k-1},$$

so

$$\left\| x - \frac{y}{k-1} \right\| \geq \frac{2}{k-1} - \frac{k^2 - 2k + 2}{k-1} = \frac{2k - k^2}{k-1}.$$

Thus, both for $k = 1$ and for $k > 1$,

$$\|x - v\| \geq \left(1 - \frac{1}{k}\right) \cdot \frac{2k - k^2}{k-1} = 2 - k.$$

Therefore, in any case

$$2h \geq \|x + v\| + \|x - v\| \geq 1 + \frac{1}{k} + 2 - k = 3 - k + \frac{1}{k},$$

so also

$$2h \geq 3 - h + \frac{1}{h}.$$

Since we can take h arbitrarily near to $A_1(X)$, we also obtain from here that

$$3[A_1(X)]^2 - 3A_1(X) - 1 \geq 0,$$

which implies that

$$A_1(X) \geq \frac{3+\sqrt{21}}{6},$$

and then the proof is complete. ∎

Remark. We do not know if the estimate (2.12) is sharp; in Section 5 we shall see that in some of the spaces l_p, $p > 2$, the value of A_1 is not much larger than $(3+\sqrt{21})/6$.

Let X be a space where James' orthogonality is symmetric. (See, e.g., [2]: this is interesting only for two-dimensional spaces; otherwise, under such an assumption, X is necessarily an inner product space.) In that case, it is possible to prove that $A_1(X) \geq (1+\sqrt{17})/4$ ($\cong 1.28$). In fact, given x as in the preceding proof, we can choose x and y orthogonal and such that $\|x-y\| \geq \|x+y\|$ (so $\|x+y\| \leq h$, $h > A_1(X)$). Then if we take $v = (x+y)/\|x+y\|$ ($v \in S$), we obtain

$$2h \geq \|x+v\| + \|x-v\| \geq 1 + \frac{1}{\|x+y\|} + \frac{1}{\|x+y\|} \geq 1 + \frac{2}{h},$$

so

$$2[A_1(X)]^2 \geq A_1(X) + 2,$$

and then

$$A_1(X) \geq \frac{1+\sqrt{17}}{4}.$$

By using the perimeter, we shall give a general upper bound concerning $U_1(X)$ (see (2.11′)). But we prove first another simple result concerning two-dimensional spaces.

PROPOSITION 2.6. *If* $\dim(X) = 2$, *then*

$$2 \cdot A_2(X) \leq p(X)/2. \tag{2.13}$$

Also; $A_2(X) = 2 \Leftrightarrow p(X) = 8 \Leftrightarrow$ *the unit sphere is a parallelogram.*

Proof. Take any pair x, y in S. One of the two arcs joining points x and $-x$ must contain y, and then $\gamma(-x, x) \geq \|x - y\| + \|x + y\|$, which implies (2.13). Also, $A_2(X) = 2 \Rightarrow p(X) = 8 \Leftrightarrow$ the unit sphere is a parallelogram $\Rightarrow A_2(X) = 2$, so we have the thesis. ∎

The next lemma indicates a simple result, which will be needed to prove Proposition 2.8.

LEMMA 2.7. *Let x, u, v be three different points on the unit sphere in a two-dimensional normed space, and let v belong to the shortest arc joining x and u. Then we have*

$$\|x + u - v\| \leq 1. \tag{2.14}$$

Proof. By assumption, there are two positive numbers a, b such that $v = ax + bu$. We can assume, without loss of generality, that $a \leq b$. Also, we have $a + b \geq 1$. Moreover, $b \leq a + 1$ $(1 = \|v\| \geq \|bu\| - \|ax\|)$. We then obtain

$$\|x + u - v\| = \|(1 - a)x + (1 - b)u\| \leq |(1 - a)| + |(1 - b)|.$$

If $0 \leq a \leq b \leq 1$, we obtain $\|x + u - v\| \leq 2 - a - b \leq 1$ and we are done.

If $0 \leq a \leq 1 \leq b$, we obtain $\|x + u - v\| \leq 1 - a + b - 1 = b - a \leq 1$ and again we are done.

Now let $1 \leq a \leq b$, so we obtain $\|x + u - v\| \leq a - 1 + b - 1 = a + b - 2$; thus we are done if $a + b \leq 3$.

Now let $a + b > 3$: then we can write

$$\alpha = \frac{1 + a - b}{a + b - 1}, \qquad \beta = \frac{b - a + 1}{a + b - 1}, \qquad \gamma = \frac{a + b - 3}{a + b - 1},$$

with α, β, γ nonnegative and such that $\alpha + \beta + \gamma = 1$; also, $v - x - u = (a - 1)x + (b - 1)u = \alpha x + \beta u + \gamma v$. This implies that $\|v - x - u\| \leq \alpha + \beta + \gamma = 1$, thus concluding the proof. ∎

PROPOSITION 2.8. *In any two-dimensional space X, we have*

$$A_1(X) \leq \frac{1 + \sqrt{1 + 4p}}{4}, \tag{2.15}$$

p denoting the perimeter of X.

Thus, since $p \leq 8$ always,

$$A_1(X) \leq \frac{1 + \sqrt{33}}{4} \; (\cong 1.686). \tag{2.16}$$

Proof. Take $x \in S$; let $\sup_{y \in S}(\|x+y\| + \|x-y\|) = 2a = \|x+u\| + \|x-u\|$ for some $u \in S$. We can also assume that $\|x+u\| \geq a \geq 1$ (if necessary, we can exchange the role of u and $-u$).

Now we want to estimate $\sup_{v \in S}(\|x+u+v\| + \|x+u-v\|)$: the points u, x, $-u$, and $-x$ divide S into four arcs γ_1 (between u and x), γ_2, γ_3, and γ_4. Let v, $-v$ belong to the arcs γ_1 and γ_3: to estimate $\|x+u+v\| + \|x+u-v\|$ it is not a restriction to assume that $v \in \gamma_1$. Then according to Lemma 2.7 we obtain

$$\begin{aligned} \|x+u+v\| + \|x+u-v\| &\leq \|x+u\| + \|v\| + \|x+u-v\| \\ &\leq \|x+u\| + 2. \end{aligned}$$

Now assume that $v \in \gamma_2$ and $-v \in \gamma_4$; then

$$\begin{aligned} &\|x+u-v\| + \|x+u+v\| \\ &\quad\leq \|x+u-x\| + \|x-v\| + \|x+u-u\| + \|u+v\| \\ &\quad= 2 + \|x-v\| + \|u+v\| \leq 2 + \operatorname{length}(\gamma_2) \leq 2 + p/2 - \|x-u\|. \end{aligned}$$

But $\|x+u\| \leq p/2 - \|x-u\|$ always, so we have obtained

$$\|x+u+v\| + \|x+u-v\| \leq 2 + p/2 - \|x-u\|$$

always. Since the function

$$f_v(t) = \|t(x+u) + v\| + \|t(x+u) - v\|$$

is convex and $f_v(0) = 2$, we obtain

$$\begin{aligned} f_v\left(\frac{1}{\|x+u\|}\right) &\leq \left(1 - \frac{1}{\|x+u\|}\right) \cdot f_v(0) + \frac{1}{\|x+u\|} f_v(1) \\ &\leq \left(1 - \frac{1}{\|x+u\|}\right) \cdot 2 + \frac{1}{\|x+u\|}(2 + p/2 - \|x-u\|) \\ &= 2 + \frac{p/2 - \|x-u\|}{\|x+u\|} \\ &= 2 + \frac{p/2 - (2a - \|x+u\|)}{\|x+u\|} = 3 + \frac{p-4a}{2\|x+u\|}. \end{aligned}$$

From $\|x+u\| \geq a$ we thus obtain

$$f_v\left(\frac{1}{\|x+u\|}\right) \leq 3 + \frac{p-4a}{2a} = 1 + \frac{p}{2a}.$$

Therefore we obtain

$$2A_1(X) \leq \sup_{v\in S}\left(\left\|\frac{x+u}{\|x+u\|}+v\right\|+\left\|\frac{x+u}{\|x+u\|}-v\right\|\right) = \sup_{v\in S} f_v\left(\frac{1}{\|x+u\|}\right).$$

and then

$$2A_1(X) \leq \inf\left(2a, 1+\frac{p}{2a}\right).$$

But $2a = 1 + p/2a$ means that $4a^2 - 2a - p = 0$, and this is true for $a = (1+\sqrt{1+4p})/4$. Thus $2A_1(X) \leq 2a \leq (1+\sqrt{1+4p})/2$, so we have the thesis. ∎

Remark 1. The estimate (2.15) is "sharp" in the sense that in the case where the unit ball of X is the hexagon, we have $p = 6$ and it is not difficult to see that $A_1(X) = 3/2$ (see also the next remark). But $A_1(X) = 3/2$ also when the unit ball is a parallelogram ($p = 8$ and, according to Proposition 2.6, $A_2(X) = 2$); so in this case the inequality in (2.15) is strict.

Remark 2. In proving Proposition 2.8, we could choose x so that

$$\sup_{y\in S}(\|x+y\|+\|x-y\|) \cong 2A_2(X).$$

So we also obtain $2A_1(X) \leq 1 + p/2A_2(X)$ or $4A_1(X)\cdot A_2(X) \leq 2A_2(X) + p \leq \frac{3}{2}p$: this implies

$$A_1(X)\cdot A_2(X) \leq \tfrac{3}{8}p \leq 3. \tag{2.17}$$

Also, $2A_2(X) \leq p/(2A_1(X) - 1)$; since $p = 6$ implies $A_1(X) = \frac{3}{2}$, in this case we obtain $A_2(X) \leq \frac{3}{2}$, so also $A_2(X) = \frac{3}{2}$ (see also Example 3.2 below).

3. THE CONSTANT A_2 AND THE MODULUS OF CONVEXITY

Formula (2.7) implies some estimates for A_2. For example, let X be given and write simply A_2 instead of $A_2(X)$. Since

$$\delta(\varepsilon) \geq 1 + \frac{\varepsilon}{2} - A_2 \tag{3.1}$$

always, we have

$$\delta(A_2) \geq 1 - \frac{A_2}{2}. \tag{3.2}$$

Letting $\varepsilon \to 2$, we obtain

$$\lim_{\varepsilon \to 2^-} \delta(\varepsilon) \geq 2 - A_2. \tag{3.3}$$

Moreover,

$$\delta(\varepsilon) > 0 \qquad \text{for all } \varepsilon > 2A_2 - 2, \tag{3.1$'$}$$

$$A_2 \geq 1 + \tfrac{1}{2}\varepsilon_0, \tag{3.1$''$}$$

where

$$\varepsilon_0 = \sup\{\varepsilon \geq 0;\ \delta(\varepsilon) = 0\}.$$

In particular, $A_2 < 3/2$ implies that $\delta(1) > 0$, i.e., $\varepsilon_0 < 1$.

Remark. It is known (see e.g. [6, p. 59]) that for every space we have

$$\delta(\varepsilon) \leq 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}\,. \tag{3.4}$$

So, by using (2.7), we have

$$2 \cdot A_2(X) \geq 2 + 2\left(\frac{\varepsilon}{2} - \delta(\varepsilon)\right) \geq \varepsilon + 2\sqrt{1 - \frac{\varepsilon^2}{4}} \qquad \text{for every } \varepsilon \in (0,2). \tag{3.5}$$

By taking $\varepsilon = \sqrt{2}$ we again obtain (2.9).

PROPOSITION 3.1. *The condition $A_2(X) = \sqrt{2}$ implies that* (3.4) *is an equality for $\varepsilon = \sqrt{2}$.*

Proof. By (3.5), $A_2(X) = \sqrt{2}$ implies that $2\sqrt{2} \geq \varepsilon + 2\sqrt{1 - \varepsilon^2/4}$ for every $\varepsilon \in (0,2)$; so, for $\varepsilon = \sqrt{2}$, we have equality. ∎

Remark. It is known that equality in (3.4) for some $\varepsilon \in (0,2) - D$, $D = \{2\cos(k\pi/2n),\ n = 2,3,\dots;\ k = 1,2,\dots,n-1\}$, characterizes inner product spaces (see [1]). Nevertheless, in [2] the following two-dimensional example was considered: Let the unit ball be a regular octagon; the norm is thus defined by

$$\|(x,y)\| = \max\{|x|, |y|, |x+y|/\sqrt{2}, |x-y|/\sqrt{2}\}. \tag{3.6}$$

This is a non-hilbertian space, but easy computations show that $A_2(X) = \sqrt{2}$ (this implies also, according to (2.8), that $A_1(X) = \sqrt{2}$).

The constant A_2 was also considered in [3], where it was denoted by μ'_2; in particular, Proposition 4.2 there gives the inequality

$$A_2 \leq 2 - \delta(A_2). \tag{3.7}$$

We have proved (see (2.10′)) that given X, in order to calculate $A_2(X)$ it is enough to maximize $\varepsilon/2 - \delta(\varepsilon)$ over one of the intervals $[0, A_2(X)]$ or $[A_2(X), 2]$.

If the function $\varepsilon/2 - \delta(\varepsilon)$ attains the maximum for $\varepsilon = A_2$, then we have $A_2 = 1 + A_2/2 - \delta(A_2)$. This happens, for example, if in the right-hand side of (2.2) the maximum is achieved by pairs x, y satisfying $\|x + y\| = \|x - y\|$, or when the function $\varepsilon/2 - \delta(\varepsilon)$ is first increasing in a part of $(0, 2)$, then decreasing after some point. This does not happen in general: see for example the space described in [5].

EXAMPLE 3.2. Let X be the space R^2 with the norm given by the hexagon (cf. the remarks to Proposition 2.8). We then have

$$\delta(\varepsilon) = \max\{0, (\varepsilon - 1)/2\},$$

so

$$1 + \frac{\varepsilon}{2} - \delta(\varepsilon) = \frac{3}{2} = A_2(X) \qquad \text{for all } \varepsilon \in [1, 2].$$

4. OUR TWO CONSTANTS AND "NEARBY" SPACES

The following fact is evident.

PROPOSITION 4.1. *If Y is a dense subspace of X, then $A_i(Y) = A_i(X)$, $i = 1, 2$.*

Now we want to prove that the constants A_1, A_2 are continuous with respect to the Banach–Mazur distance of spaces, in case X and Y are isomorphic. To this end, we shall state in advance some simple results.

We recall that in general, given x, y in $B - \{\theta\}$, we cannot say which of the quantities $\|x - y\|$ and $\|x/\|x\| - y/\|y\|\,\|$ is larger, but we can prove the following lemma (similar to Lemma 6.3 in [7]).

LEMMA 4.2. *Given two points x, y in X, we have*

$$\|x + ty\| + \|x - ty\| \geq \|x + y\| + \|x - y\| \qquad \text{for all } t \geq 1. \tag{4.1}$$

Moreover, equality in (4.1) for some $y \neq \theta$ and some $t \neq \pm 1$ implies that X is not strictly convex.

Proof. Consider the convex, even function of $t \in R$, $f(t) = \|x + ty\| + \|x - ty\|$; we have $f(0) = 2\|x\|$; $f(1) = f(-1) \geq 2\|x\|$. This implies that $f(t) \geq f(1)$ for all $t \geq 1$ (f attains its minimum at 0). Moreover, let X be strictly convex; assume that we have equality in (4.1) for some $y \neq \theta$ and some $t \neq \pm 1$ (so that $x \neq \theta$). We can assume that $t = 1 + \varepsilon$ with $\varepsilon > -1$, $\varepsilon \neq 0$; this would imply that $2\|x\| = \|x + ty\| + \|x - ty\| = \|x + ty + (x - ty)\|$ for $t \in [-1 - \varepsilon, 1 + \varepsilon]$. Then, since $y \neq \theta$, $x - ty = \alpha_t(x + ty)$ for some $t \neq \pm 1$ and some positive α_t, $\alpha_t \neq 1$. But then $x = (1 + \alpha_t)ty/(1 - \alpha_t) = \beta_t y$ for some $\beta_t \neq 0$; this, together with the equality in (4.1), implies that $|\beta_t + t| + |\beta_t - t| = |\beta_t + 1| + |\beta_t - 1|$ for some $t \neq \pm 1$, an absurdity which completes the proof. ∎

PROPOSITION 4.3. *Given $x \in B_X$, we have*

$$\sup_{y \in S_X} (\|x - y\| + \|x + y\|) = \sup_{y \in B_X} (\|x - y\| + \|x + y\|). \tag{4.2}$$

Moreover,

$$A_1(X) = \inf_{x \in S_X} \sup_{y \in B_X} (\|x - y\| + \|x + y\|); \tag{4.3}$$

$$\begin{aligned} A_2(X) &= \sup_{x \in S_X} \sup_{y \in S_X} (\|x - y\| + \|x + y\|) \\ &= \sup_{x \in B_X} \sup_{y \in B_X} (\|x - y\| + \|x + y\|). \end{aligned} \tag{4.4}$$

Proof. We prove (4.2). Let $x \in B_X$; clearly $2 \leq \sup_{y \in S_X}(\|x - y\| + \|x + y\|) \leq \sup_{y \in B_X}(\|x - y\| + \|x + y\|)$. Now let $h = \sup_{y \in B_X}(\|x - y\| + \|x + y\|)$. If $h = 2$ there is nothing to prove. Otherwise, take any $k \in (2, h)$, then take z in B_X such that $\|x - z\| + \|x + z\| > k$ ($z \neq \theta$); according to Lemma 4.2 we then have $\|x - \frac{z}{\|z\|}\| + \|x + \frac{z}{\|z\|}\| \geq \|x - z\| + \|x + z\| > k$, so $\sup_{y \in S_X}(\|x - y\| + \|x + y\|) > k$, which proves (4.2).

Equation (4.3) follows immediately from (4.2).

Still, by Lemma 4.2, if $\|x\| \leq 1$, for any fixed $y \in B_X - \{\theta\}$ we have $\|\frac{x}{\|x\|} - y\| + \|\frac{x}{\|x\|} + y\| \geq \|x - y\| + \|x + y\|$, so

$$\sup_{y \in B_X} \left(\left\| \frac{x}{\|x\|} - y \right\| + \left\| \frac{x}{\|x\|} + y \right\| \right) \geq \sup_{y \in B_X} (\|x - y\| + \|x + y\|).$$

Therefore,

$$\sup_{x \in S_X} \sup_{y \in B_X} (\|x - y\| + \|x + y\|) \geq \sup_{x \in B_X} \sup_{y \in B_X} (\|x - y\| + \|x + y\|),$$

and then we have equality. This, together with (4.2), implies (4.4). ∎

Remark. Indeed, formulas (4.2) and (4.4) are a consequence of the convexity of the norm (for x and y fixed, the functions $\|x+y\|, \|x-y\|$ are convex). Moreover, given x, to compute $\sup_{y \in B_X}(\|x-y\| + \|x+y\|)$ it is enough to consider those points y which are extreme for the unit ball (a similar remark applies to computing $A_2(X)$).

Let X, Y be two isomorphic spaces; we set

$$\Delta(X,Y) = \inf\left\{\|T\| \cdot \|T\|^{-1}; T\colon X \to Y \text{ is an isomorphism}\right\}. \quad (4.5)$$

We have the following result.

PROPOSITION 4.4. *Let X, X' be isomorphic spaces. Then, for $i = 1, 2$, we have*

$$\left|A_i(X) - A_i(X')\right| \leq (4-i) \cdot (\Delta(X,X') - 1), \quad (4.6)$$

thus

$$\left|A_i(X) - A_i(X')\right| \leq 3(\Delta(X,X') - 1). \quad (4.6')$$

Proof. Let T be an isomorphism between the spaces X and X', satisfying the following condition: There exist two numbers $\alpha \in [0,1)$ and $\beta \geq 0$ such that

$$(1-\alpha) \cdot \|x\| \leq \|Tx\| \leq (1+\beta) \cdot \|x\| \qquad \text{for all } x \in B_X. \quad (4.7)$$

Take x', y' in $S_{X'}$; there exist x, y in X such that $x' = Tx$, $y' = Ty$. Moreover, $(1-\alpha) \cdot \|x\| \leq 1$; $(1-\alpha) \cdot \|y\| \leq 1$. Set $x'' = (1-\alpha)x$, $y'' = (1-\alpha)y$ (x'', y'' are in B_X). We then obtain

$$\begin{aligned}\|x' \pm y'\| = \|Tx \pm Ty\| &= \frac{1}{1-\alpha} \cdot \|T(x'' \pm y'')\| \\ &\leq \frac{1}{1-\alpha} \cdot (1+\beta) \cdot \|x'' \pm y''\|.\end{aligned}$$

Thus

$$\begin{aligned}\|x'+y'\| + \|x'-y'\| &\leq \frac{1+\beta}{1-\alpha}(\|x''+y''\| + \|x''-y''\|) \\ &\leq \frac{1+\beta}{1-\alpha} \sup_{y \in B_X} (\|x''+y\| + \|x''-y\|).\end{aligned}$$

This shows that for every $x' \in S_{X'}$, there is some element x'' in B_X such that

$$\sup_{y' \in B_{X'}} \frac{\|x'-y'\| + \|x'+y'\|}{2} \leq \frac{1+\beta}{1-\alpha} \sup_{y \in B_X} \frac{\|x''-y\| + \|x''+y\|}{2};$$

so, according to (4.4),

$$A_2(X') \leq \frac{1+\beta}{1-\alpha} A_2(X); \tag{4.8}$$

thus

$$A_2(X') - A_2(X) \leq \frac{\alpha+\beta}{1-\alpha} A_2(X). \tag{4.9}$$

Now we reverse the role of X and X'; we have that T^{-1} is an isomorphism and, for $x = T^{-1}x'$ with $x' \in B_{X'}$,

$$\frac{1}{1+\beta}\|x'\| \leq \|T^{-1}x'\| \leq \frac{1}{1-\alpha}\|x'\|. \tag{4.10}$$

So we obtain

$$A_2(X) \leq \frac{1+\beta}{1-\alpha} A_2(X'), \tag{4.11}$$

and then

$$A_2(X) - A_2(X') \leq \frac{\alpha+\beta}{1-\alpha} A_2(X'). \tag{4.12}$$

Finally, (4.9) and (4.12) together imply

$$|A_2(X) - A_2(X')| \leq 2\frac{\alpha+\beta}{1-\alpha}. \tag{4.12'}$$

Now let $x \in S_X$ such that $\sup_{y \in B_X}(\|x-y\| + \|x+y\|) = k$; set $x' = Tx/\|Tx\|$ ($\|x'\| = 1$); for any element $y' \in S_{X'}$ we have $y' = Ty$ for some $y \in X$ with $\|y\| \leq 1/(1-\alpha)$, so

$$\begin{aligned}\|x'-y'\| &= \left\| \frac{Tx}{\|Tx\|} - Ty \right\| \\ &\leq \frac{1}{\|Tx\|}\left(\|Tx - T[(1-\alpha)y]\| + \|T[(1-\alpha)y - y\|Tx\|]\|\right) \\ &\leq \frac{1}{1-\alpha}\left[(1+\beta)\|x-(1-\alpha)y\| + \|Ty\| \cdot |(1-\alpha) - \|Tx\||\right].\end{aligned}$$

Since $\|Ty\| = 1$, $|(1-\alpha) - \|Tx\|| = \|Tx\| - (1-\alpha) \leq \alpha + \beta$, we obtain

$$\|x'-y'\| \leq \frac{1+\beta}{1-\alpha}\|x-(1-\alpha)y\| + \frac{\alpha+\beta}{1-\alpha}.$$

A similar estimate holds for $\|x' + y'\|$, so we obtain ($\|(1-\alpha)y\| \le 1$)

$$\sup_{y \in B_{X'}} \left(\frac{\|x' - y'\| + \|x' + y'\|}{2} \right) \le \frac{1+\beta}{1-\alpha} \cdot k + \frac{\alpha+\beta}{1-\alpha}.$$

Since we can take x so that k is arbitrarily near to $A_1(X)$, we obtain

$$A_1(X') \le \frac{1+\beta}{1-\alpha} A_1(X) + \frac{\alpha+\beta}{1-\alpha},$$

and then ($A_1(X) \le 2$),

$$A_1(X') - A_1(X) \le \frac{3(\alpha+\beta)}{1-\alpha}. \tag{4.13}$$

By reversing the role of X and X', we can also obtain (in this case, replacing α, β with $\frac{\beta}{1+\beta}$ and $\frac{\alpha}{1-\alpha}$, respectively)

$$A_1(X) \le \frac{1+\beta}{1-\alpha} A_1(X') + \frac{\beta+\alpha}{1-\alpha}, \tag{4.14}$$

and then

$$A_1(X) - A_1(X') \le \frac{3(\alpha+\beta)}{1-\alpha}. \tag{4.15}$$

Equations (4.13) and (4.15) together give

$$|A_1(X) - A_1(X')| \le 3\frac{\alpha+\beta}{1-\alpha}. \tag{4.15$'$}$$

We have thus proved the following: given T satisfying (4.7), we have (see (4.12′) and (4.15′))

$$|A_i(X) - A_i(X')| \le \frac{(4-i)\cdot(\alpha+\beta)}{1-\alpha}, \qquad i = 1, 2. \tag{4.16}$$

But we can take an isomorphism T: $X \to X'$ so that $\|T\| \cdot \|T\|^{-1} - \Delta(X, X')$ is arbitrarily small, and so $(1+\beta)/(1-\alpha)$ is very near to $\Delta(X, X')$; i.e., $(\alpha+\beta)/(1-\alpha)$ is very near to $\Delta(X, X') - 1$. Therefore (4.6) follows from (4.16), and this concludes the proof. ∎

Remark. In a sense, the above estimates are sharp; for example, concerning (4.8), if we consider as X the space R^2, endowed with the norms $\|\cdot\|_1, \|\cdot\|_2$, then we have $\|\cdot\|_2 \le \|\cdot\|_1 \le \sqrt{2}\|\cdot\|_2$, while $A_2((R^2)_1) = 2 = \sqrt{2} A_2((R^2)_2)$.

5. THE VALUES OF A_2: AN EXAMPLE

If X is one of the spaces $L_1[0,1]$, $C[0,1]$, $C_0[0,1]$, c_0, c, l_∞, then we have $A_2(X) = 2$ since they are not uniformly nonsquare. Now we shall consider another class of classical Banach spaces.

PROPOSITION 5.1. *If* $X = l_p$, $1 < p < \infty$, *then*

$$A_2(l_p) = \max(2^{1/p}; 2^{1-1/p}). \tag{5.1}$$

Proof. According to (2.7), it is enough to consider the case $p > 2$. Recall that in this case we have

$$\delta(l_p) = 1 - \left[1 - \left(\frac{\varepsilon}{2}\right)^p\right]^{1/p}. \tag{5.2}$$

Set

$$f(\varepsilon) = \frac{\varepsilon}{2} - \left(1 - \left[1 - \left(\frac{\varepsilon}{2}\right)^p\right]^{1/p}\right).$$

Note that $f(0) = f(2) = 0$, so its maximum (for $\varepsilon \in [0,2]$) is attained for some $\varepsilon \in (0,2)$.

We have

$$\begin{aligned} f'(\varepsilon) &= \frac{1}{2} + \frac{1}{p}\left[1 - \left(\frac{\varepsilon}{2}\right)^p\right]^{(1/p)-1} \cdot (-p) \cdot \left(\frac{\varepsilon}{2}\right)^{p-1} \cdot \frac{1}{2} \\ &= \frac{1}{2}\left[1 - \left[1 - \left(\frac{\varepsilon}{2}\right)^p\right]^{(1/p)-1} \cdot \left(\frac{\varepsilon}{2}\right)^{p-1}\right]. \end{aligned}$$

Therefore, $f'(\varepsilon) = 0$ when

$$\begin{aligned} 1 &= \left(\frac{2^p - \varepsilon^p}{2^p}\right)^{(1-p)/p} \cdot \left(\frac{\varepsilon}{2}\right)^{p-1} = \left[\left(\frac{2^p - \varepsilon^p}{2^p}\right)^{1/p} \cdot \frac{2}{\varepsilon}\right]^{1-p} \\ &= \left(\frac{(2^p - \varepsilon^p)^{1/p}}{\varepsilon}\right)^{1-p}, \end{aligned}$$

so also when $2^p - \varepsilon^p = \varepsilon^p$: i.e., when $(\varepsilon/2)^p = \frac{1}{2}$. Thus, by using (5.2), (2.7) gives

$$A_2(l_p) - 1 = f(2^{1-1/p}) = \frac{1}{2^{1/p}} - \left[1 - \left(\frac{1}{2}\right)^{1/p}\right] = \frac{2}{2^{1/p}} - 1,$$

so we have the thesis. ∎

Remark. The estimate (5.1) holds also for $p \in \{1, 2, \infty\}$. Moreover, it is valid also for $L_p[0,1]$, as well as for $l_p^{(n)}$ spaces, since it depends on the modulus of convexity (which is a two-dimensional modulus).

6. THE VALUES OF A_1; SOME EXAMPLES AND THE CASE $A_1 = 2$

Estimating $A_1(X)$ is not always simple.

We indicate a relation between the constant A_1 and the following one:

$$\mu_2(X) = \inf_{F \subset S} \sup_{y \in S} \frac{1}{n} \sum_{i=1}^{n} \|a_i - y\|,$$

$$F = \{a_1, a_2, \dots, a_n\} \text{ is any finite subset of } S. \quad (6.1)$$

This constant has been considered e.g. in [3, 4, 9, 12]. Note that also for μ_2, by considering the sup over all points $y \in B$ (or also, by only considering extreme points of B) we obtain an equivalent definition.

PROPOSITION 6.1. *In any space we have*

$$\mu_2(X) \leq A_1(X). \quad (6.2)$$

Proof. It is enough to note that for any $x \in S$ we have

$$2A_1(X) = \inf_{x \in S} \sup_{y \in S} (\|x - y\| + \|x + y\|) \geq 2\mu_2(X).$$

∎

Concerning (6.2), note that we have inequality e.g. in finite-dimensional euclidean spaces (see [3]).

We have seen that A_2 is the same for X and for its dual. For A_1 the situation is different: for example, both for c_0 and l_∞ the value of A_1 is $3/2$, while (see below) it is 2 for l_1 (so, when we pass to the dual of X, the value of A_1 can both increase and decrease).

The condition $A_1(X) = A_2(X)$ (in this case we have $\sup_{y \in S}(\|x - y\| + \|x + y\|)$ constant with respect to x) does not force a space to be an inner product space: consider, for example, l_1 or the two-dimensional hexagon (see the remarks to Proposition 2.8).

Also, the example of the regular octagon, indicated in Section 3, shows that in two-dimensional spaces the condition $A_1(X) = A_2(X) = \sqrt{2}$ does not imply that X is Euclidean. We do not know what happens when the dimension is larger (concerning $A_1(X) = \sqrt{2}$, or at least under the assumption $A_1(X) = A_2(X) = \sqrt{2}$). In fact, it has been conjectured long ago (see e.g. [11, pp. 70 and 83]) that equality in (3.4) for some $\varepsilon \in (0, 2)$ forces X to be an inner product space if $\dim(X) \geq 3$. If this conjecture is true, this would also have implications concerning our constants.

PROPOSITION 6.2. *If* $1 < p < 2$, *then we have*

$$A_1(l_p) = 2^{1/p}. \tag{6.3}$$

For $2 < p < \infty$,

$$2^{1/p} \leq A_1(l_p) \leq \tfrac{1}{2} \sup_{0 \leq t \leq 1} \left\{ \left(|1 + t|^p + 1 - |t|^p \right)^{1/p} + \left(|1 - t|^p + 1 - |t|^p \right)^{1/p} \right\}. \tag{6.4}$$

Proof. Let $p \in (1, \infty)$; we shall prove that

$$A_1(X) \geq 2^{1/p}. \tag{6.5}$$

Take any $\varepsilon > 0$; given $x = (x_1, x_2, \ldots, x_n \ldots) \in S$, take k so that $|x_k| < \varepsilon$. Let e_k be the kth element of the natural basis of l_p. Elementary computations show that for any $p \geq 1$, since $|x_k| \leq 1$, we have $|1 \pm x_k|^p \geq 1 - p \cdot |x_k|$. Then we obtain $\|x \pm e_k\| = (1 - |x_k|^p + |1 \pm x_k|^p)^{1/p} \geq (2 - p \cdot |x_k| - |x_k|^p)^{1/p} > (2 - p\varepsilon - \varepsilon^p)^{1/p}$.

Since $x \in S$ and $\varepsilon > 0$ are arbitrary, this implies (6.5).

If $1 < p < 2$, then—according to (5.3)—we have $2^{1/p} \leq A_1(X) \leq A_2(X) = 2^{1/p}$, so we obtain (6.3). Concerning (6.4), its left part is (6.5), which has already been proved.

Now let $x = (1, 0, \ldots, 0 \ldots)$; take $y = (y_1, y_2, \ldots, y_n \ldots) \in S$. Then we obtain

$$\|x \pm y\|^p = |1 \pm y_1|^p + \sum_{k=2}^{\infty} |y_k|^p = |1 \pm y_1|^p + 1 - |y_1|^p. \tag{6.6}$$

Therefore, $2A_1 \le \sup_{0 \le t \le 1}\{(|1 + t|^p + 1 - |t|^p)^{1/p} + (|1 - t|^p + 1 - |t|^p)^{1/p}\}$, which is the right part of (6.4). ∎

Remarks. Numerical computations on the right-hand side of (6.4) show that for $p = 3$ we obtain $A_1(X) < 1.327$; also, for p around 2.8 we obtain $A_1(X) < 1.325$. Note that $2^{1/3} \cong 1.26$, which is slightly below the general estimate $A_1(X) \ge (3 + \sqrt{21})/6$.

It is not difficult to see that $A_1(X) = 3/2$ in the cases $X = c_0$, $X = c$, and $X = l_\infty$. We have instead $A_1(X) = 2$ in the cases $X = C[0,1]$, $X = C_0[0,1]$, and $X = L_1[0,1]$. We only prove the last assertion, the other ones being simpler.

PROPOSITION 6.3. *If $X = L_1[0,1]$, then $A_1(X) = 2$.*

Proof. Let $\varepsilon > 0$; take $f \in L_1[0,1]$ and let A be a subset of $[0,1]$ such that $\int_A |f| < \varepsilon$. We can take a function g, with support contained in A and such that $\int_A |g| = 1$. Then we have $\int_{[0\ 1]} |f + g| > 2 - \varepsilon$; $\int_{[0\ 1]} |f - g| > 2 - \varepsilon$. This proves that

$$\inf_{f \in S_X} \sup_{g \in S_X} \|f + g\| + \|f - g\| \ge 4 - 2\varepsilon,$$

so (ε being arbitrary) $A_1(X) = 2$. ∎

Concerning the "extreme" value 2, taking into account (6.2), the following implications hold:

$$\mu_2(X) = 2 \Rightarrow A_1(X) = 2 \Rightarrow A_2(X) = 2.$$

Spaces satisfying $A_2(X) = 2$ have been characterized as spaces which are not uniformly nonsquare, while the condition $\mu_2(X) = 2$ characterizes "octahedral" norms (see [9]). We shall prove some consequences of $A_1(X) = 2$.

PROPOSITION 6.4. *Let a space X satisfy $A_1(X) = 2$; then X is not uniformly nonsquare; moreover*, $\dim(X) = \infty$.

Proof. The first part is a consequence of the inequality $A_1(X) \le A_2(X)$. To prove the second part, we prove first the following.

CLAIM. *Assume that the following is true in X: for any $x \in S$ there exists a $y \in S$ such that $\|x - y\| = \|x + y\| = 2$. Then there exists in S an independent sequence $\{x_n\}$ such that*

$$\left\| \sum_{i=1}^{n} x_i \right\| = n. \tag{6.7}$$

Proof of the claim. We reason by induction: Take $x_1 \in S$, then $x_2 \in S$ such that $\|x_1 - x_2\| = \|x_1 + x_2\| = 2$. Of course, x_1 and x_2 are independent. In general, once $x_1, \dots, x_n$ have been chosen so that $\|x_1 + \cdots + x_{n-1} \pm x_n\| = n$, choose $x_{n+1} \in S$ such that $\|(x_1 + \cdots + x_n)/n \pm x_{n+1}\| = 2$. We then have

$$\begin{aligned} n + 1 &\geq \|x_1 + \cdots + x_n \pm x_{n+1}\| \\ &\geq \|x_1 + \cdots + x_n\| \cdot \left(\left\| \frac{x_1 + \cdots + x_n}{n} \pm x_{n+1} \right\| - \left\| \pm x_{n+1} \pm \frac{x_{n+1}}{n} \right\| \right) \\ &= n\left(2 - \left(1 - \frac{1}{n}\right)\right) = n + 1. \end{aligned}$$

So we have all equalities; this also implies that

$$\|x_1 + \cdots + x_n \pm t x_{n+1}\| = n + |t| \qquad \text{for all } t \in R. \tag{6.8}$$

Now assume that $x_{n+1} = \lambda_1 x_1 + \cdots + \lambda_n x_n$; set $\lambda = |\lambda_1| + \cdots + |\lambda_n|$ ($\lambda \neq 0$) and $y_{n+1} = (\lambda_1 x_1 + \cdots + \lambda_n x_n)/\lambda$. Then, according to (6.8), we obtain

$$\begin{aligned} n + \frac{1}{\lambda} &= \|x_1 + \cdots + x_n \pm y_{n+1}\| \\ &= \left\| \left(1 \pm \frac{\lambda_1}{\lambda}\right) x_1 + \cdots + \left(1 \pm \frac{\lambda_n}{\lambda}\right) x_n \right\| \\ &\leq \left|1 \pm \frac{\lambda_1}{\lambda}\right| + \cdots + \left|1 \pm \frac{\lambda_n}{\lambda}\right|; \end{aligned}$$

this implies

$$n + \frac{1}{\lambda} \leq n + \frac{\lambda_1}{\lambda} + \cdots + \frac{\lambda_n}{\lambda} \qquad \text{and}$$

$$n + \frac{1}{\lambda} \leq n - \left(\frac{\lambda_1}{\lambda} + \cdots + \frac{\lambda_n}{\lambda}\right),$$

a contradiction, proving that λ must be 0, so that x_{n+1} is independent from $x_1, \dots, x_n$. This concludes the proof of the claim.

Now, if $\dim(X) < \infty$, then S is compact. So $A_1(X) = 2$ implies the assumption of the claim, whose thesis contradicts $\dim(X) < \infty$. ∎

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