Checkered Hadamard Matrices of Order 16

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In this paper all the so-called checkered Hadamard matrices of order 16 are determined (i.e., Hadamard matrices consisting of 16 square blocks $H_{ij}$ of order 4 such that $H_{ii} = J_4$ and $H_{ij} J_4 = J_4 H_{ij} = 0$ for $i \neq j$ and where $J_4$ is the all-one matrix of order 4). It is shown that the checkered Hadamard matrices of order 16 all belong to one of the Hall’s classes I, II or III. Moreover the so-called block equivalency classes are determined.

1. Preliminaries

Definition 1.1. An Hadamard matrix $H$ of order $g^2$ is called a checkered Hadamard matrix if $H$ has the following form

$$H = \begin{pmatrix} J_g & H_{12} & \cdots & H_{1g} \\ H_{21} & J_g & \cdots & H_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ H_{g1} & H_{g2} & \cdots & J_g \end{pmatrix}$$

in which all matrices $H_{ij}$ are square, $J_g$ is the all-one matrix and $H_{ij} J_g = J_g H_{ij} = 0$.

We consider in this paper the case $g = 4$. For the proofs of most results we refer to [1].

According to Wallis [4], K. A. Bush was the first to raise the question of the existence of what we call checkered Hadamard matrices. In [2] we investigated checkered Hadamard matrices and their connection with 3-class association schemes. For example, let $(X, R)$ be a symmetric 3-class association scheme of type $L_1, s(2s)$, then for an appropriately chosen numbering of the relations, $D_0 + D_1 - D_2 + D_3$ is a symmetric checkered Hadamard matrix of order $4s^2$, where the $D_i$ are the adjacency matrices.

In this context it seems natural to determine all the checkered Hadamard matrices of order 16 and using this result and those from [2] to find, in a later paper, all the symmetric and non-symmetric 3-class association schemes connected with this type of Hadamard matrices.

We use in this paper the terminology of [3]. In particular we use the notion of Hall’s classes [3, p. 420] as well as the notion of invariants [3, p. 410]. Unless otherwise stated, $H$ denotes a checkered Hadamard matrix of order 16 and we assume that the rows and columns of $H$ are numbered from 0 to 15.

$I_t$ denotes the $(t \times t)$ identity matrix and $J_t$ denotes the $(t \times t)$ all-one matrix.

A $16 \times 16$ matrix $K$ can be considered to consist of 16 blocks (i.e., $4 \times 4$ matrices) $K_{ij}$.

We say that the matrices $K_{ij}$ $(1 \leq j \leq 4)$ form the $i$-th block row and that the matrices $K_{ij}$ $(1 \leq i \leq 4)$ form the $j$-th block column.

It is usual to distinguish Hadamard matrices up to Hadamard equivalency (H-equivalency). However, general H-equivalent operations may destroy the block form of a checkered Hadamard matrix. Therefore we introduce and investigate in this paper a more special equivalence relation: block equivalency, which preserves the block structure of a checkered Hadamard matrix.

† The second author, H. L. Claassen died on 26 May 1998.
It is worth noting that there are checkered Hadamard matrices which are Hadamard equivalent but are not block equivalent (compare Theorems 4.2, 4.3 and 4.4).

In what follows a matrix is called monomial if in each row and in each column there is exactly one entry ≠ 0.

**Lemma 1.2.** If $S$ and $T$ are square monomial matrices of order 16 with non-zero entries ±1 such that $S(I_4 \otimes J_4)T = I_4 \otimes J_4$, then $S$ and $T$ have the following block structure.

There is a permutation $\alpha$ of $\{1, 2, 3, 4\}$ such that for each $i \in \{1, 2, 3, 4\}$ the blocks $\epsilon_i S_{i, \alpha(i)}$ and $\epsilon_i T_{\alpha(i), i}$ (of well chosen in $\{+1, -1\}$) are permutation matrices and the matrices $S_{ik}$ and $T_{ki}$ with $k \in \{1, 2, 3, 4\} \setminus \{\alpha(i)\}$ are all-zero matrices.

If also $SHT = H'$, then the $i$-th block row and the $j$-th block column of $H$ are mapped onto the $\alpha(i)$-th block row and the $\alpha(j)$-th block column of $H'$, respectively.

In particular, $H_{ij}$ and $H'_{\alpha(i), \alpha(j)}$ have the same rank.

The proof of the lemma is left to the reader.

Two checkered Hadamard matrices $H_1$ and $H_2$ of order 16 are called block equivalent if there are two monomial matrices $S$ and $T$ with non-zero entries $+1$ and $-1$ such that

$$SH_1 T = H_2 \quad \text{and} \quad S(I_4 \otimes J_4)T = I_4 \otimes J_4.$$  

**Theorem 1.3.** There are Hadamard matrices of order 16 which are not H-equivalent to a checkered Hadamard matrix.

**Proof.** As we shall show in the following, Hadamard matrices belonging to Hall’s classes IV or V cannot be H-equivalent to a checkered Hadamard matrix of order 16, implying the theorem. 

\[ \square \]

2. **On the Blocks of a Checkered Hadamard Matrix of Order 16**

The off-diagonal blocks of a checkered Hadamard matrix of order 16 are $(+1, -1)$ matrices $A$ of order 4 with the additional property $AJ_4 = J_4A = 0$. In this section we shall discuss this type of matrix.

Let $x_i \in \mathbb{R}^4$ be a row vector with entries ±1 and entry sum 0. Clearly the following row vectors are the only possibilities.

$$x_0 = (+1, +1, -1, -1), \quad x_1 = (+1, -1, +1, -1), \quad x_2 = (+1, -1, -1, +1)$$

and $x_i = -x_{i-3}$ for $i = 3, 4, 5$. Here the inner product $\langle x_i, x_j \rangle = 0$ if $i \neq j \pmod{3}$. We keep this notation for the rest of this paper.

**Lemma 2.1.** Let $A$ be a square matrix of order 4 with entries ±1 such that $AJ_4 = J_4A = 0$. Then the following hold.

1. Rank $(A)$ is either 1 or 2.
2. If rank $(A) = 1$ and if $y$ is a row of $A$, then the four rows of $A$ are $y$, $-y$ and $-y$.
3. If rank $(A) = 2$ and if $y$ and $z$ are independent rows of $A$, then the four rows of $A$ are $y$, $z$, $-y$ and $-z$.

By Lemma 2.1 a square matrix $A$ of order 4 with entries ±1 such that $AJ_4 = J_4A = 0$ has one of the following row forms $t$ with $t \in \{0, 1, 2, 3, 4, 5\}$; here $y$ and $z$ are vectors of the form $x_i$ ($i \in \{0, 1, 2, 3, 4, 5\}$), while $(y, z) = 0$. 

\[ \square \]
If $A^T$ has row form $t$, then we say that $A$ has column form $t$.

We introduce the following matrices (here $i, j, k, l \in \{0, 1, 2\}$):

$$T_{ij;kl} = \frac{1}{2}[(x_i + x_j)^T x_k + (x_i - x_j)^T x_l].$$

If $i = j$ or $k = l$, we denote $T_{ij;kl}$ by $O_{ik}$. We assume throughout this paper that the rows and columns of the matrices $T_{ij;kl}$ are labelled by $0, 1, 2, 3$ and when considering matrices $T_{ij;kl}$, we assume that $i, j, k, l \in \{0, 1, 2\}$.

The proof of the following lemma is straightforward and is left to the reader.

**Lemma 2.2.** Let $A$ be a $(+1, -1)$ matrix of order 4 such that $AJ_A = J_A A = 0$.

1. If $A$ has rank 1, row form $i$ and column form $k$, then $A = \eta O_{ik}$ with $\eta \in \{+1, -1\}$.
2. If $A$ has rank 2, row form $m$ and column form $n$, then $A = \eta T_{ij;kl}$ for well-chosen $\eta \in \{+1, -1\}$ and for $i \neq j$ and $k \neq l$ such that $i + j = m - 2$ and $k + l = n - 2$.
3. $A = \eta T_{ij;kl}$ with $\eta \in \{+1, -1\}$ has
   (a) row (column) form 3 if $\{i, j\} = \{0, 1\}$ and $\{k, l\} = \{0, 1\}$,
   (b) row (column) form 4 if $\{i, j\} = \{0, 2\}$ and $\{k, l\} = \{0, 2\}$.
   (c) row (column) form 5 if $\{i, j\} = \{1, 2\}$ and $\{k, l\} = \{1, 2\}$.
4. $O_{ik}^T = O_{ki}$ and $(T_{ij;kl})^T = T_{kl;ij}$.

If in Lemma 2.2, $\eta = +1$, then we say that $A$ has sign + and otherwise that its sign is −.

3. A Discussion of the Structure

We recall the convention that unless otherwise stated, $H$ is, in this paper, a checkered Hadamard matrix of order 16.

**Lemma 3.1.** A block row of $H$ may contain (apart from $J_A$)

1. three blocks of rank 1 of different row forms,
2. one block of rank 1 of row form $t$ ($t \in \{0, 1, 2\}$) and two blocks of rank 2 of row form $5 - t$, or
3. three blocks of rank 2 of different row forms.

An analogous result holds if in the above one replaces ‘row’ by ‘column’.

**Proof.** Let a block row of $H$ contain $k_t$ blocks of the row form $t$ ($t \in \{0, 1, 2, 3, 4, 5\}$). Since the inner product of two different rows of the block row is 0, we find (taking the inner products of the first row with the second, third and fourth row)

$$\begin{align*}
+ k_0 &+ k_1 &- k_2 &+ k_3 & = 1 \\
+ k_0 &- k_1 &+ k_2 &+ k_4 & = 1 \\
- k_0 &+ k_1 &+ k_2 &+ k_5 & = 1
\end{align*}$$

Now since $k_t \in \mathbb{N}$, the results follow easily. \qed
In the next Lemma 3.2 the third possibility of the preceding lemma is excluded.

Let $M$ be a block matrix with blocks $M_{ij}$. If $\text{rank}(M_{ij}) = r_{ij}$, then the matrix with $(i, j)$-entry $r_{ij}$ is called the rank distribution matrix of $M$, denoted by $R(M)$.

A submatrix $T = \begin{pmatrix} H_{ij} & H_{ik} \\ H_{ij} & H_{lk} \end{pmatrix}$ of $H$ with $i, j, k$ and $l$ all different is called a tile. If $T' = \begin{pmatrix} H_{ji} & H_{jl} \\ H_{ki} & H_{kl} \end{pmatrix}$, then $T$ and $T'$ are called complementary tiles.

**Lemma 3.2.** The following hold for $H$.

1. Every block row (column) of $H$ contains either no blocks of rank 2 or exactly two blocks of rank 2 of the same row (column) form.
2. Up to permutations of the rows and columns the rank distribution matrix of a tile can only be
   \[
   \begin{pmatrix}
   1 & 1 \\
   1 & 1 \\
   \end{pmatrix}, \quad \begin{pmatrix}
   2 & 1 \\
   2 & 1 \\
   \end{pmatrix}, \quad \begin{pmatrix}
   2 & 1 \\
   1 & 2 \\
   \end{pmatrix}, \quad \begin{pmatrix}
   2 & 2 \\
   2 & 2 \\
   \end{pmatrix}.
   \]
3. If a tile of $H$ has rank distribution matrix $2J_2$, then its complementary tile has rank distribution matrix either $J_2$ or $2J_2$.

**Theorem 3.3.** The following hold.

1. A checkered Hadamard matrix of order 16 contains either no or one or two tiles of which all the blocks have rank 2.
2. Checkered Hadamard matrices with different numbers of tiles of which all the blocks have rank 2 are not block equivalent.
3. Up to block equivalency the rank distribution matrix of $H$ must have one of the following forms:
   \[
   \begin{pmatrix}
   1 & 1 & 1 & 1 \\
   1 & 1 & 1 & 1 \\
   1 & 1 & 1 & 1 \\
   1 & 1 & 1 & 1 \\
   \end{pmatrix}, \quad \begin{pmatrix}
   1 & 2 & 2 & 1 \\
   1 & 1 & 1 & 1 \\
   2 & 1 & 1 & 2 \\
   1 & 1 & 1 & 1 \\
   \end{pmatrix}, \quad \begin{pmatrix}
   1 & 2 & 2 & 1 \\
   1 & 2 & 2 & 1 \\
   2 & 1 & 1 & 2 \\
   1 & 1 & 1 & 1 \\
   \end{pmatrix}.
   \]

**Definition 3.4.** We say that a checkered Hadamard matrix of order 16 with $t$ ($t \in \{0, 1, 2\}$) tiles of which all blocks have rank 2 has the rank format $t$.

We emphasize that two checkered Hadamard matrices of different rank format are not block equivalent.

**Lemma 3.5.** The following hold.

1. If a block of $H$ has rank 1 and row-space $A$, then the row-space of any block in the same block column is orthogonal to $A$.
2. If two blocks of $H$ in the same block column have rank 2, then they have the same row space.
3. If \( \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix} \) is a tile of $H$ with rank distribution matrix $2J_2$ and $(a|b)$ is a row of $(H_1H_2)$, then $(-a|b)$ or $(a|b)$ is a row of $(H_3H_4)$.

An analogous result holds if we interchange ‘row’ and ‘column’.

The next theorem shows that the properties of the blocks of a checkered Hadamard matrix of order 16 found in this section are also sufficient for a block matrix of order 16 to be a checkered Hadamard matrix.
THEOREM 3.6. A square \((-1, +1)\) matrix \(H\) of order 16 with blocks \(H_{ij}\) of order 4 is a checkered Hadamard matrix if and only if the following conditions are fulfilled.

1. \(H_{ii} = J_4\) and \(H_{ij} J_4 = J_4 H_{ij} = 0\) for any \(i\) and \(j\) such that \(i \neq j\).
2. Each block row contains either
   (a) three blocks \(\neq J_4\) of rank 1 of different row forms or
   (b) one block \(\neq J_4\) of rank 1 of row form \(t\) and two blocks of rank 2 of row form \(5 - t\) \((t \in \{0, 1, 2\})\).
3. Each tile of \(H\) has one of the rank distribution matrices given in item 2 of Lemma 3.2.
4. Two blocks in the same block column have orthogonal row spaces if at least one of them has rank 1.
5. Suppose the blocks of a tile \(T\) all have rank 2.
   (a) Then the blocks in the same ‘block column’ of \(T\) have the same row space.
   (b) If \((a|b)\) is a row in one of the ‘block rows’ of \(T\), then \((-a|b)\) or \((a|-b)\) is a row in the other ‘block row’ of \(T\).

4. THE BLOCK EQUIVALENCY CLASSES

If \(H\) is of rank format 0, by definition all blocks \(H_{ij}\) have rank 1.

LEMMA 4.1. Every checkered Hadamard matrix \(H\) of order 16 of rank format 0 is block equivalent to a matrix of the following form:

\[
\begin{pmatrix}
J_4 & \eta_{12} O_{00} & \eta_{13} O_{11} & \eta_{14} O_{22} \\
\eta_{21} O_{00} & J_4 & \eta_{23} O_{22} & \eta_{24} O_{11} \\
\eta_{31} O_{11} & \eta_{32} O_{22} & J_4 & \eta_{34} O_{00} \\
\eta_{41} O_{22} & \eta_{42} O_{11} & \eta_{43} O_{00} & J_4
\end{pmatrix}
\]

with \(\eta_{ij} \in \{-1, +1\}\).

PROOF. Without loss of generality we can assume that a block row of \(H\) has the following form: \((J_4 O_{0k} O_{0l} O_{cm})\), \([a, b, c] = \{0, 1, 2\}\) (by Lemma 3.1) and \(k, l, m\) chosen in \(\{0, 1, 2\}\). By an appropriate row permutation one can change the row form of each of the blocks of the given block row at will (of course within the constraints given in Lemma 3.2). Since the same applies to the column form, it is easy to complete the proof of the theorem.

The next theorem characterizes the block equivalency classes for the rank format 0.

THEOREM 4.2. The set of checkered Hadamard matrices of rank format 0 consists of two block equivalency classes.

One class consists of the matrices which can, by block equivalent operations, be brought to the form (1) with an even number of \(\eta_{ij}\) equal to \(-1\). Matrices of this class belong to Hall’s class I.

The other class consists of the matrices which can, by block equivalent operations, be brought to the form (1) with an odd number of \(\eta_{ij}\) equal to \(-1\). Matrices of this class belong to Hall’s class II.

The next theorem covers rank format 2.
THEOREM 4.3. Let $H$ be a checkered Hadamard matrix of order 16 and of rank format 2. $H$ is block equivalent to a matrix of the form:

$$
\begin{pmatrix}
J_4 & T_{12;12} & T_{12;12} & O_{00} \\
T_{12;12} & J_4 & O_{00} & T_{12;12} \\
\eta O_{00} & T_{12;12} & -T_{12;12} & J_4 \\
\end{pmatrix}
$$

There are three block equivalency classes of checkered Hadamard matrices of order 16 and of rank format 2.

The first class consists of matrices which can be brought to the form (2) with $\eta = \hat{\eta} = +1$. Matrices of this class belong to Hall’s class I.

The second class consists of matrices which can be brought to the form (2) with $\eta \hat{\eta} = -1$. Matrices of this class belong to Hall’s class II.

The third class consists of matrices which can be brought to the form (2) with $\eta = \hat{\eta} = -1$. Matrices of this class belong to Hall’s class III.

Finally we consider the checkered Hadamard matrices of rank format 1.

Let $H$ be a checkered Hadamard matrix of order 16 and of rank format 1.

Applying the appropriate block equivalent operations it is possible to bring $H$ into the following form, which is block equivalent to $H$:

$$
\begin{pmatrix}
J_4 & T_{12;12} & T_{12;12} & O_{00} \\
\epsilon_1 O_{11} & J_4 & \eta_2 O_{00} & \epsilon_2 O_{22} \\
\epsilon_3 O_{22} & \eta_3 O_{00} & J_4 & \epsilon_4 O_{11} \\
\eta_4 O_{00} & T_{12;12} & -T_{12;12} & J_4 \\
\end{pmatrix}
$$

where $\epsilon_i, \eta_j \in \{-1, +1\}$ for $i, j \in \{1, 2, 3, 4\}$. One can, without loss of generality, suppose that $\eta_i = +1$ for $i \in \{1, 2, 3, 4\}$ (by an appropriately chosen column permutation one can bring the matrix in a form for which $\eta_4 = +1$, etc.). Once that has been achieved it is possible, by applying well-chosen row permutations, to reach a situation where $\epsilon_1 = \epsilon_4 = +1$ and all the matrices $O_{ij}$ still have sign +. So we see that $H$ is block equivalent to

$$
\begin{pmatrix}
J_4 & T_{12;12} & T_{12;12} & O_{00} \\
O_{11} & J_4 & O_{00} & \eta O_{22} \\
\eta O_{22} & O_{00} & J_4 & O_{11} \\
O_{00} & T_{12;12} & -T_{12;12} & J_4 \\
\end{pmatrix}
$$

If $\eta = \hat{\eta} = -1$, then apply on this matrix the row permutation (45)(67) and the column permutation (01)(23), yielding a matrix with $\eta = \hat{\eta} = +1$ and if $\eta = +1$ and $\hat{\eta} = -1$, then the above process applied on this matrix yields one with $\eta = -1$ and $\hat{\eta} = +1$. So we have the following theorem.

THEOREM 4.4. Let $H$ be a checkered Hadamard matrix of order 16 and of rank format 1. $H$ is block equivalent to a matrix of the following form:

$$
\begin{pmatrix}
J_4 & T_{12;12} & T_{12;12} & O_{00} \\
O_{11} & J_4 & O_{00} & \eta O_{22} \\
O_{22} & O_{00} & J_4 & O_{11} \\
O_{00} & T_{12;12} & -T_{12;12} & J_4 \\
\end{pmatrix}
$$

There are two block equivalency classes of checkered Hadamard matrices of order 16 and of rank format 1.
The first class consists of matrices which can be brought to the above form with $\eta = +1$. Matrices of this class belong to Hall’s class II.

The second class consists of matrices which can be brought to the above form with $\eta = -1$. Matrices of this class belong to Hall’s class III.

REFERENCES


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