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Equiangular tight frames from complex Seidel matrices containing cube roots of unity[☆]

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Abstract

We derive easily verifiable conditions which characterize when complex Seidel matrices containing cube roots of unity have exactly two eigenvalues. The existence of such matrices is equivalent to the existence of equiangular tight frames for which the inner product between any two frame vectors is always a common multiple of the cube roots of unity. We also exhibit a relationship between these equiangular tight frames, complex Seidel matrices, and highly regular, directed graphs. We construct examples of such frames with arbitrarily many vectors.

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1. Introduction

Equiangular tight frames play an important role in several areas of mathematics, ranging from signal processing (see, e.g. [1,4,12,13] and references therein) to quantum computing (see, e.g. [2,7]). In comparison with the host of results on the construction of equiangular tight frames for finite dimensional real Hilbert spaces [21,18,8,3,10], relatively few means are known for constructing equiangular frames in the complex case (see, e.g. [11,17,7]).

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The problem of the existence of equiangular frames is known to be equivalent to the existence of a certain type of matrix called a *Seidel matrix* [15] or *signature matrix* [10] with two eigenvalues. A matrix Q is a Seidel matrix provided that it is self-adjoint, its diagonal entries are 0, and its off-diagonal entries are all of modulus one. In the real case, these off-diagonal entries must all be ± 1 ; such matrices can then be interpreted as (Seidel) adjacency matrices of graphs. The case when these adjacency matrices have two eigenvalues has been characterized in graph-theoretic terms [19,20], see also [16], and a fairly complete catalog of all such graphs, when the number of vertices is small, is known. Moreover, such graphs are known to exist for an arbitrarily large number of vertices [5].

In this paper, we study the existence and construction of Seidel matrices with two eigenvalues and off-diagonal entries that are all cube roots of unity. We derive necessary and sufficient conditions for the existence of such matrices and construct examples. In particular, we find arbitrarily large Seidel matrices of this type. We show that the existence of such Seidel matrices is equivalent to the existence of certain highly regular directed graphs, and hence, show that such directed graphs exist on an arbitrarily large number of vertices.

This paper is organized as follows. We complete the introduction with a more detailed description of equiangular tight frames, Seidel matrices and their relationship. In Section 2, we derive conditions for the existence of such matrices. In Section 3, we refine these conditions and use our relations, in Section 4, to determine the possible sizes of such Seidel matrices for $n \leq 100$. Section 5 discusses the connections with directed graphs. Section 6 contains examples and Section 7 gives a method for constructing examples of arbitrarily large size.

1.1. Equiangular tight frames

Let \mathcal{H} be a real or complex Hilbert space. A finite family of vectors $\{f_1, \dots, f_n\}$ is called a *frame* provided that there exist strictly positive real numbers A and B such that

$$A\|x\|^2 \leq \sum_{i=1}^n |\langle x, f_i \rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in \mathcal{H}.$$

A frame is said to be a *tight frame* if we can choose $A = B$. When $A = B = 1$, then the frame is called a *normalized tight frame* or a *Parseval frame*. Replacing f_i by f_i/\sqrt{A} always normalizes a tight frame.

One can show (see [4, p. 21], for example) that a family $\{f_1, \dots, f_n\}$ is a tight frame with constant A if and only if

$$x = \frac{1}{A} \sum_{i=1}^n \langle x, f_i \rangle f_i \quad \text{for all } x \in \mathcal{H}. \tag{1.1}$$

There is a natural equivalence relation for tight frames, motivated by simple operations on the frame vectors which preserve identity (1.1).

We say that two tight frames $\{f_1, f_2, \dots, f_n\}$ and $\{g_1, g_2, \dots, g_n\}$ for \mathcal{H} are *unitarily equivalent* if there exists a unitary operator U on \mathcal{H} such that for all $i \in \{1, 2, \dots, n\}$, $g_i = Uf_i$. We say that they are *switching equivalent* if there exist a unitary operator U on \mathcal{H} , a permutation π on $\{1, 2, \dots, n\}$ and a family of unimodular constants $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ such that for all $i \in \{1, 2, \dots, n\}$, $g_i = \lambda_i Uf_{\pi(i)}$. If \mathcal{H} is a real Hilbert space, U is understood to be orthogonal, and all $\lambda_i \in \{\pm 1\}$.

In this paper, we shall be concerned only with Parseval frames for the k -dimensional complex Hilbert space \mathbb{C}^k , equipped with the canonical inner product. We use the term (n, k) -frame to mean a Parseval frame of n vectors for \mathbb{C}^k .

Every such Parseval frame gives rise to an isometric embedding of \mathbb{C}^k into \mathbb{C}^n via the map

$$V : \mathbb{C}^k \rightarrow \mathbb{C}^n, \quad (Vx)_j = \langle x, f_j \rangle \quad \text{for all } j \in \{1, 2, \dots, n\},$$

which is called the *analysis operator* of the frame. Because V is linear, we may identify V with an $n \times k$ matrix and the vectors $\{f_1, \dots, f_n\}$ are the respective columns of V^* . Conversely, given any $n \times k$ matrix V that defines an isometry, if we let $\{f_1, \dots, f_n\}$ denote the columns of V^* , then this set is an (n, k) -frame and V is the analysis operator of the frame.

If V is the analysis operator of an (n, k) -frame, then since V is an isometry, we see that $V^*V = I_k$ and the $n \times n$ matrix VV^* is a self-adjoint projection of rank k . Note that VV^* has entries $(VV^*)_{ij} = \langle f_j, f_i \rangle$. Thus, VV^* is the *Grammian matrix* (or *correlation matrix*) of the set of vectors. Conversely, any time we have an $n \times n$ self-adjoint projection P of rank k , we can always factor it as $P = VV^*$ for some $n \times k$ matrix V . In this case, we have $V^*V = I_k$ and hence V is an isometry and the columns of V^* are an (n, k) -frame. Moreover, if $P = WW^*$ is another factorization of P , then there exists a unitary U such that $W^* = UV^*$, and the frame corresponding to W differs from the frame corresponding to V by applying the same unitary to all frame vectors, which is included in our equivalence relation. However, switching equivalence is coarser than just identifying all frames with the same Grammian. In [10], it is shown that it corresponds to identifying frames for which the Grammians can be obtained from each other by conjugation with diagonal unitaries and permutation matrices.

Definition 1.1. An (n, k) -frame $\{f_1, \dots, f_n\}$ is called *uniform* if there is a constant $u > 0$ such that $\|f_i\| = u$ for all i . An (n, k) -frame is called *equiangular* if all of the frame vectors are non-zero and the angle between the lines generated by any pair of frame vectors is a constant, that is, provided that there is a constant b such that $|\langle f_i/\|f_i\|, f_j/\|f_j\| \rangle| = b$ for all $i \neq j$.

Many places in the literature define equiangular to mean that the (n, k) -frame is uniform and that there is a constant c so that $|\langle f_i, f_j \rangle| = c$ for all $i \neq j$. However, the assumption that the frame is uniform is not needed in our definition as the following result shows.

Proposition 1.2. Let $\{f_1, \dots, f_n\}$ be a tight frame for \mathbb{C}^k . If all frame vectors are non-zero and if there is a constant b so that $|\langle f_i/\|f_i\|, f_j/\|f_j\| \rangle| = b$ for all $i \neq j$, then $\|f_i\| = \|f_j\|$ for every i and j .

Proof. Without loss of generality, we may assume that the frame is a Parseval frame, so that $P = (\langle f_j, f_i \rangle)_{i,j=1}^n$ is a projection of rank k . Hence, $P = P^2$ and so upon equating the (i, i) th entry and using the fact that the trace of P is k , we see that $\|f_i\|^2 = \langle f_i, f_i \rangle = \sum_{j=1}^n \langle f_j, f_i \rangle \langle f_i, f_j \rangle = \|f_i\|^4 + \sum_{j \neq i} b^2 \|f_i\|^2 \|f_j\|^2 = \|f_i\|^4 + b^2 \|f_i\|^2 (k - \|f_i\|^2)$, which shows that $\|f_i\|^2$ is a (non-zero) constant independent of i . \square

In [10], a family of (n, k) -frames was introduced that was called *2-uniform frames*. It was then proved that a Parseval frame is 2-uniform if and only if it is equiangular. Thus, these terminologies are interchangeable in the literature, but the equiangular terminology has become more prevalent.

1.2. Seidel matrices and equiangular tight frames

At this point we revisit an approach that has been used to construct equiangular tight frames [15]. The previous section shows that an (n, k) -frame is determined up to unitary equivalence by its Grammian matrix. This reduces the problem of constructing an (n, k) -frame to constructing an $n \times n$ self-adjoint projection P of rank k .

If an (n, k) -frame $\{f_1, f_2, \dots, f_n\}$ is uniform, then it is known that $\|f_i\|^2 = k/n$ for all $i \in \{1, 2, \dots, n\}$. It is shown in [10, Theorem 2.5] that if $\{f_1, \dots, f_n\}$ is an equiangular (n, k) -frame, then for all $i \neq j$, $|\langle f_j, f_i \rangle| = c_{n,k} = \sqrt{\frac{k(n-k)}{n^2(n-1)}}$. Thus we may write

$$VV^* = (k/n)I_n + c_{n,k}Q,$$

where Q is a self-adjoint $n \times n$ matrix satisfying $Q_{ii} = 0$ for all i and $|Q_{ij}| = 1$ for all $i \neq j$. This matrix Q is called the *Seidel matrix* or *signature matrix* associated with the (n, k) -frame.

The following theorem characterizes the signature matrices of equiangular (n, k) -frames.

Theorem 1.3 (Theorem 3.3 of [10]). *Let Q be a self-adjoint $n \times n$ matrix with $Q_{ii} = 0$ and $|Q_{ij}| = 1$ for all $i \neq j$. Then the following are equivalent:*

- (a) Q is the signature matrix of an equiangular (n, k) -frame for some k ;
- (b) $Q^2 = (n - 1)I + \mu Q$ for some necessarily real number μ ; and
- (c) Q has exactly two eigenvalues.

This result reduces the problem of constructing equiangular (n, k) -frames to the problem of constructing Seidel matrices with two eigenvalues. In particular, Condition (b) is particularly useful since it gives an easy-to-check condition to verify that a matrix Q is the signature matrix of an equiangular tight frame.

Furthermore, if Q is a matrix satisfying any of the three equivalent conditions in Theorem 1.3, and if $\lambda_1 < 0 < \lambda_2$ are its two eigenvalues, then the parameters n, k, μ, λ_1 , and λ_2 satisfy the following properties:

$$\begin{aligned} \mu &= (n - 2k)\sqrt{\frac{n - 1}{k(n - k)}} = \lambda_1 + \lambda_2, & k &= \frac{n}{2} - \frac{\mu n}{2\sqrt{4(n - 1) + \mu^2}}, \\ \lambda_1 &= -\sqrt{\frac{k(n - 1)}{n - k}}, & \lambda_2 &= \sqrt{\frac{(n - 1)(n - k)}{k}}, & n &= 1 - \lambda_1\lambda_2. \end{aligned} \tag{1.2}$$

These equations follow from the results in [10, Proposition 3.2, Theorem 3.3] and by solving for λ_1 and λ_2 from the given equations.

In the case when the entries of Q are all real, we have that the diagonal entries of Q are 0 and the off-diagonal entries of Q are ± 1 . These matrices can be seen to be Seidel adjacency matrices of a graph [19] on n vertices, and it has been proven that in the real case there is a one-to-one correspondence between the switching equivalence classes of real equiangular tight frames and regular two-graphs [10, Theorem 3.10].

In a similar vein, we now apply switching equivalence to complex Seidel matrices, in order to derive easily verifiable conditions which characterize when they have two eigenvalues.

Definition 1.4. Two Seidel matrices Q and Q' are *switching equivalent* if they can be obtained from each other by conjugating with a diagonal unitary and a permutation matrix.

If Q is a Seidel matrix, we say that Q is in a *standard form* if its first row and column contains only 1's except on the diagonal, as shown in (2.1) below. We say that it is *trivial* if it has a standard form which has all of its off-diagonal entries equal to 1 and *nontrivial* if at least one off-diagonal entry is not equal to 1.

One can verify by conjugation with an appropriate diagonal unitary that the equivalence class of any Seidel matrix contains a matrix of standard form, so we only need to examine when matrices of this form have two eigenvalues.

Since, in the real case, the off-diagonal entries of Q are in the set $\{-1, 1\}$, it seems promising in the complex situation to harness combinatorial techniques and consider the case when the off-diagonal entries of Q are m th roots of unity for some $m \geq 3$. We conjecture that these cases will give a description of families of complex 2-uniform frames in analogy with the characterization that has been obtained in the real case. The purpose of this paper is to examine the simplest complex case (when $m = 3$), and to construct new frames in this setting.

2. Signature matrices with entries in the cube roots of unity

In this section, we consider nontrivial signature matrices whose off-diagonal entries are cube roots of unity. We obtain a number of necessary and sufficient conditions for such a signature matrix of an equiangular (n, k) -frame to exist. These results are useful because they allow us to rule out many values of n and k in the search for examples of such frames.

Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then the set $\{1, \omega, \omega^2\}$ is the set of cube roots of unity. Note also that $\omega^2 = \bar{\omega}$ and $1 + \omega + \omega^2 = 0$.

Definition 2.1. We call a matrix Q a *cube root Seidel matrix* if it is self-adjoint, has vanishing diagonal entries, and off-diagonal entries which are all cube roots of unity. If Q has exactly two eigenvalues, then we say that it is the *cube root signature matrix* of an equiangular tight frame.

All equivalence classes of cube root signature matrices contain representatives in standard form.

Lemma 2.2. *If Q' is an $n \times n$ cube root Seidel matrix, then it is switching equivalent to a cube root Seidel matrix of the form*

$$Q = \begin{pmatrix} 0 & 1 & \cdots & \cdots & 1 \\ 1 & 0 & * & \cdots & * \\ \vdots & * & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & * \\ 1 & * & \cdots & * & 0 \end{pmatrix}, \tag{2.1}$$

where the *'s are cube roots of unity. Moreover, Q' is the signature matrix of an equiangular (n, k) -frame if and only if Q is the signature matrix of an equiangular (n, k) -frame.

Proof. Suppose that Q' is an $n \times n$ cube root Seidel matrix. Then Q' is self-adjoint with $|Q'_{ij}| = 1$ for $i \neq j$, and by Theorem 1.3 we have that $(Q')^2 = (n - 1)I + \mu Q'$ for some real number μ . If we let U be the diagonal matrix

$$U := \begin{pmatrix} 1 & & & & \\ & Q'_{12} & & & \\ & & Q'_{13} & & \\ & & & \ddots & \\ & & & & Q'_{1n} \end{pmatrix},$$

then U is a unitary matrix (since $|Q'_{ij}| = 1$ when $i \neq j$), and we see that $Q := U^* Q' U$ is a self-adjoint $n \times n$ matrix with $Q_{ii} = 0$ and $|Q_{ij}| = 1$ for $i \neq j$. We see that the off-diagonal elements of Q are cube roots of unity and Q has the form shown in (2.1). (To see that the off-diagonal elements in the first row and column are 1's, recall that $Q'_{ij} = \overline{Q'_{ji}}$.) Thus Q is a cube root Seidel matrix that is unitarily equivalent to Q' . Since Q and Q' have the same eigenvalues, if one of them is the signature matrix of an equiangular (n, k) -frame, then the same holds for the other matrix. \square

Next, we present the characterization of cube root Seidel matrices with exactly two eigenvalues after an elementary insight.

Lemma 2.3. *If $a, b, c \in \mathbb{R}$ and $a + b\omega + c\omega^2 = 0$, then $a = b = c$.*

Proof. If $a + b\omega + c\omega^2 = 0$, then $a + b\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + c\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 0$, and hence $(a - \frac{b}{2} - \frac{c}{2}) + i\frac{\sqrt{3}}{2}(b - c) = 0$. It follows that $a - \frac{b}{2} - \frac{c}{2} = 0$ and $b - c = 0$. Thus $a = b = c$. \square

Proposition 2.4. *Let Q be a cube root Seidel matrix in standard form, and suppose Q satisfies the equation*

$$Q^2 = (n - 1)I + \mu Q.$$

Then $e := \frac{n - \mu - 2}{3}$ is an integer, and for any j with $2 \leq j \leq n$, the j th column of Q (and likewise the j th row) contains e entries equal to ω , contains e entries equal to ω^2 , and contains $e + \mu + 1 = \frac{n + 2\mu + 1}{3}$ entries equal to 1.

Proof. For $2 \leq j \leq n$ define

$$\begin{aligned} x_j &:= \#\{i : Q_{ij} = 1\}, \\ y_j &:= \#\{i : Q_{ij} = \omega\}, \\ z_j &:= \#\{i : Q_{ij} = \omega^2\}. \end{aligned}$$

Since the j th column of Q has $n - 1$ non-zero entries (recall the zero on the diagonal) we have

$$x_j + y_j + z_j = n - 1. \tag{2.2}$$

Also, since $Q^2 = (n - 1)I + \mu Q$ we see that for $j \geq 2$ we have

$$\mu = \mu Q_{1j} = [(n - 1)I + \mu Q]_{1j} = Q_{1j}^2 = (x_j - 1)1 + y_j\omega + z_j\omega^2.$$

Thus $(x_j - \mu - 1)1 + y_j\omega + z_j\omega^2 = 0$, and by Lemma 2.3 we have

$$x_j - \mu - 1 = y_j = z_j. \tag{2.3}$$

Thus (2.2) becomes $x_j + 2(x_j - \mu - 1) = n - 1$ so that

$$x_j = \frac{n + 2\mu + 1}{3} \tag{2.4}$$

and (2.3) gives

$$y_j = z_j = \frac{n - \mu - 2}{3}. \tag{2.5}$$

Since the quantities in (2.4) and (2.5) do not depend on j they are valid for any column. In addition, since $Q = Q^*$ and $\bar{\omega} = \omega^2$, the same equations hold for the rows of Q . \square

To summarize the consequences of Proposition 2.4: In the first column of Q there are $n - 1$ entries equal to 1 and one entry equal to 0. In the j th column (for $j \geq 2$) there are $x_j = \frac{n+2\mu+1}{3}$ entries equal to 1, there are $y_j = \frac{n-\mu-2}{3}$ entries equal to ω , there are $z_j = \frac{n-\mu-2}{3}$ entries equal to ω^2 , and there is one entry (on the diagonal) equal to 0. Note that these values do not depend on the value of j and the same equations hold for the rows. In particular, if Q is trivial, then $k = 1$, and its off-diagonal entries are all 1's, $x_j = n - 1$ and $y_j = z_j = 0$ for $j \geq 2$.

3. Equations for nontrivial cube root signature matrices

Suppose that Q is a nontrivial cube root Seidel matrix. Also suppose that Q is in standard form and that Q satisfies $Q^2 = (n - 1)I + \mu Q$. Since Q is nontrivial we have that $Q_{ij} = \omega$ for some $2 \leq i, j \leq n$ with $i \neq j$. For these values of i and j let

$$\begin{aligned} \alpha &= \#\{k : Q_{ik} = \omega \text{ and } Q_{kj} = \omega^2\}, \\ \beta &= \#\{k : Q_{ik} = \omega \text{ and } Q_{kj} = \omega\}, \\ \gamma &= \#\{k : Q_{ik} = \omega \text{ and } Q_{kj} = 1\}, \\ a &= \#\{k : Q_{ik} = \omega^2 \text{ and } Q_{kj} = \omega^2\}, \\ b &= \#\{k : Q_{ik} = \omega^2 \text{ and } Q_{kj} = \omega\}, \\ c &= \#\{k : Q_{ik} = \omega^2 \text{ and } Q_{kj} = 1\}, \\ A &= \#\{k : Q_{ik} = 1 \text{ and } Q_{kj} = \omega^2\}, \\ B &= \#\{k : Q_{ik} = 1 \text{ and } Q_{kj} = \omega\}, \\ C &= \#\{k : Q_{ik} = 1 \text{ and } Q_{kj} = 1\}. \end{aligned}$$

We shall now establish equations relating these nine values. To begin, we see that the number of ω 's in row i is equal to $\alpha + \beta + \gamma + 1$. (The +1 comes from the term $Q_{ij} = \omega$.) Also, the number of ω 's in row i is equal to $e := \frac{n-\mu-2}{3}$ by Proposition 2.4. Thus

$$\alpha + \beta + \gamma = e - 1. \tag{3.1}$$

In addition, the number of ω^2 's in row i is equal to $a + b + c$, and by Proposition 2.4 the number of ω^2 's in row i is equal to e . Thus

$$a + b + c = e. \tag{3.2}$$

Also, the number of 1's in row i is equal to $A + B + C$, and by Proposition 2.4 the number of 1's in row i is equal to $e + \mu + 1$. Hence

$$A + B + C = e + \mu + 1. \tag{3.3}$$

Next we turn our attention to the j th column. We see that the number of ω^2 's in column j is equal to $\alpha + a + A$, and by Proposition 2.4 the number of ω^2 's in column j is equal to e . Thus

$$\alpha + a + A = e. \tag{3.4}$$

In addition, the number of ω 's in column j is equal to $\beta + b + B + 1$. (The +1 comes from the term $Q_{ij} = \omega$.) Also, by Proposition 2.4 the number of ω 's in column j is equal to e . Thus

$$\beta + b + B = e - 1. \tag{3.5}$$

Furthermore, the number of 1's in column j is equal to $\gamma + c + C$, and by Proposition 2.4 the number of 1's in column j is equal to $e + \mu + 1$. Hence

$$\gamma + c + C = e + \mu + 1. \tag{3.6}$$

Finally, since $Q^2 = (n - 1)I + \mu Q$ we have that

$$\begin{aligned} \mu\omega &= \mu Q_{ij} = [(n - 1)I + \mu Q]_{ij} = Q_{ij}^2 = \sum_{k=1}^n Q_{ik} Q_{kj} \\ &= \alpha(\omega\omega^2) + \beta(\omega\omega) + \gamma(\omega 1) + a(\omega^2\omega^2) + b(\omega^2\omega) + c(\omega^2 1) \\ &\quad + A(1\omega^2) + B(1\omega) + C(1)(1) \\ &= \alpha 1 + \beta\omega^2 + \gamma\omega + a\omega + b1 + c\omega^2 + A\omega^2 + B\omega + C1 \\ &= (\alpha + b + C)1 + (\gamma + a + B)\omega + (\beta + c + A)\omega^2, \end{aligned}$$

so that

$$(\alpha + b + C)1 + (\gamma + a + B - \mu)\omega + (\beta + c + A)\omega^2 = 0.$$

It follows from Lemma 2.3 that $\alpha + b + C = \gamma + a + B - \mu = (\beta + c + A)$ and thus

$$\alpha - \gamma - a + b - B + C = -\mu \tag{3.7}$$

and

$$\alpha - \beta + b - c - A + C = 0. \tag{3.8}$$

By looking at (3.1)–(3.8), we have eight equations in the nine unknowns $\alpha, \beta, \gamma, a, b, c, A, B, C$. These equations are not linearly independent: we see that (3.1) + (3.2) + (3.3) = (3.4) + (3.5) + (3.6). However, this is the only relation, and when we row reduce this system we obtain the seven equations

$$\alpha - B = -\frac{2\mu}{3} - \frac{1}{3}, \tag{Eq. 1}$$

$$\beta - C = -\frac{2\mu}{3} - \frac{4}{3}, \tag{Eq. 2}$$

$$\gamma + B + C = \frac{n}{3} + \mu, \tag{Eq. 3}$$

$$a - C = -\frac{\mu}{3} - \frac{2}{3}, \tag{Eq. 4}$$

$$b + B + C = \frac{n}{3} + \frac{\mu}{3} - \frac{1}{3}, \tag{Eq. 5}$$

$$c - B = -\frac{\mu}{3} + \frac{1}{3}, \tag{Eq. 6}$$

$$A + B + C = \frac{n}{3} + \frac{2\mu}{3} + \frac{1}{3} \tag{Eq. 7}$$

with B and C as the two free variables.

It is important to note that the above variables, $\alpha, \beta, \gamma, a, b, c, A, B, C$, really should carry a subscript (i, j) since their actual values could depend on the particular (i, j) that we choose satisfying $Q_{i,j} = \omega$ and we are only asserting that for each such pair (i, j) these equations must be met, not that their values are independent of the pair that we have chosen.

Similarly, we derive equations for the case that $Q_{ij} = 1$ for some $2 \leq i, j \leq n$ with $i \neq j$. Later, we shall prove that such an entry always exists. For these values of i and j let

$$\begin{aligned} \alpha' &= \#\{k : Q_{ik} = \omega \text{ and } Q_{kj} = \omega^2\}, \\ \beta' &= \#\{k : Q_{ik} = \omega \text{ and } Q_{kj} = \omega\}, \\ \gamma' &= \#\{k : Q_{ik} = \omega \text{ and } Q_{kj} = 1\}, \\ a' &= \#\{k : Q_{ik} = \omega^2 \text{ and } Q_{kj} = \omega^2\}, \\ b' &= \#\{k : Q_{ik} = \omega^2 \text{ and } Q_{kj} = \omega\}, \\ c' &= \#\{k : Q_{ik} = \omega^2 \text{ and } Q_{kj} = 1\}, \\ A' &= \#\{k : Q_{ik} = 1 \text{ and } Q_{kj} = \omega^2\}, \\ B' &= \#\{k : Q_{ik} = 1 \text{ and } Q_{kj} = \omega\}, \\ C' &= \#\{k : Q_{ik} = 1 \text{ and } Q_{kj} = 1\}. \end{aligned}$$

We shall now establish equations relating these nine values. To begin, we see that the number of ω 's in row i is equal to $\alpha' + \beta' + \gamma'$. Also, the number of ω 's in row i is equal to e by Proposition 2.4. Thus

$$\alpha' + \beta' + \gamma' = e. \tag{3.9}$$

In addition, the number of ω^2 's in row i is equal to $a' + b' + c'$, and by Proposition 2.4 the number of ω^2 's in row i is equal to $e := \frac{n-\mu-2}{3}$. Thus

$$a' + b' + c' = e. \tag{3.10}$$

Also, the number of 1's in row i is equal to $A' + B' + C' + 1$. (The +1 comes from the (i, j) -entry.) By Proposition 2.4 the number of 1's in row i is equal to $e + \mu + 1$. Hence

$$A' + B' + C' = e + \mu. \tag{3.11}$$

Next we turn our attention to the j th column. We see that the number of ω^2 's in column j is equal to $\alpha' + a' + A'$, and by Proposition 2.4 the number of ω^2 's in column j is equal to e . Thus

$$\alpha' + a' + A' = e. \tag{3.12}$$

In addition, the number of ω 's in column j is equal to $\beta' + b' + B'$. Also, by Proposition 2.4 the number of ω 's in column j is equal to e . Thus

$$\beta' + b' + B' = e. \tag{3.13}$$

Furthermore, the number of 1's in column j is equal to $\gamma' + c' + C' + 1$, and by Proposition 2.4 the number of 1's in column j is equal to $e + \mu + 1$. Hence

$$\gamma' + c' + C' = e + \mu. \tag{3.14}$$

Finally, since $Q^2 = (n - 1)I + \mu Q$ we have that

$$\begin{aligned} \mu &= \mu Q_{ij} = [(n - 1)I + \mu Q]_{ij} = Q^2_{ij} = \sum_{k=1}^n Q_{ik} Q_{kj} \\ &= \alpha'(\omega\omega^2) + \beta'(\omega\omega) + \gamma'(\omega 1) + a'(\omega^2\omega^2) + b'(\omega^2\omega) + c'(\omega^2 1) \\ &\quad + A'(1\omega^2) + B'(1\omega) + C'(11) \\ &= \alpha'1 + \beta'\omega^2 + \gamma'\omega + a'\omega + b'1 + c'\omega^2 + A'\omega^2 + B'\omega + C'1 \\ &= (\alpha' + b' + C')1 + (\gamma' + a' + B')\omega + (\beta'a' + c' + A')\omega^2, \end{aligned}$$

so that

$$(\alpha' + b' + C' - \mu)1 + (\gamma' + a' + B')\omega + (\beta' + c' + A')\omega^2 = 0.$$

It follows from Lemma 2.3 that $\alpha' + b' + C' - \mu = \gamma' + a' + B' = \beta' + c' + A'$ and thus

$$\alpha' - \gamma' - a' + b' - B' + C' = +\mu \tag{3.15}$$

and

$$\alpha' - \beta' + b' - c' - A' + C' = +\mu. \tag{3.16}$$

When we row reduce this system we obtain the following equations:

$$\alpha' = B', \tag{Eq. 8}$$

$$\beta' = C' - \mu, \tag{Eq. 9}$$

$$\gamma' = e + \mu - B' - C', \tag{Eq. 10}$$

$$a' = C' - \mu, \tag{Eq. 11}$$

$$b' = e + \mu - B' - C', \tag{Eq. 12}$$

$$c' = B', \tag{Eq. 13}$$

$$A' = e + \mu - B' - C' \tag{Eq. 14}$$

with B' and C' as the two free variables. Note that it follows from these equations that, $\alpha' = c' = B'$, $\beta' = a' = C' - \mu$ and $\gamma' = A' = b' = e + \mu - B' - C'$.

The above equations are necessary and sufficient to characterize nontrivial cube root signature matrices of equiangular (n, k) -frames.

Theorem 3.1. *Let Q be a self-adjoint $n \times n$ matrix with $Q_{ii} = 0$, $Q_{i1} = Q_{1i} = 1$ for all i and Q_{ij} a cube root of unity for all $i \neq j$, $2 \leq i, j \leq n$ with at least one entry that is not equal to 1. Then $Q^2 = (n - 1)I + \mu Q$ if and only if for each pair $i \neq j$ such that $Q_{i,j} = \omega$, conditions Eqs. (1)–(7) are satisfied and for each pair $i \neq j$, $2 \leq i, j \leq n$ such that $Q_{i,j} = 1$, conditions Eqs. (8)–(14) are satisfied, where $3e + \mu + 2 = n$.*

Proof. We have that $Q^2 = (n - 1)I + \mu Q$ if and only if for every (i, j) the corresponding entries are equal. Equality of the (i, i) th entries follows from the fact that the off-diagonal entries are all of modulus one.

The proof of Proposition 2.4 shows that the relationships between e , μ and n are equivalent to the condition that the $(i, 1)$ th entries and the $(1, i)$ th entries of Q^2 are equal to μ which are the corresponding $(i, 1)$ -entries and $(1, i)$ -entries of $(n - 1)I + \mu Q$.

Eqs. (1)–(7) are equivalent to requiring that if $Q_{i,j} = \omega$, then the (i, j) th entry of Q^2 is equal to $\mu\omega$ which is the (i, j) th entry of $(n - 1)I + \mu Q$.

If $Q_{i,j} = \omega^2$, then $Q_{j,i} = \omega$ and by the last argument we have that the (j, i) th entry of Q^2 is equal to the (j, i) th entry of $(n - 1)I + \mu Q$. But since both of these matrices are self-adjoint, we have that equality of their (j, i) th entry implies equality of their (i, j) th entry.

Finally, if $Q_{i,j} = 1, i \neq j, 2 \leq i, j \leq n$, then Eqs. (8)–(14), are equivalent to the equality of the (i, j) th entries of Q^2 and $(n - 1)I + \mu Q$. \square

Remark 3.2. It is important to realize that all of our results apply only to nontrivial cube root signature matrices, because we assume that the first row contains all 1’s except for one 0, and the second row contains at least one ω . For example, one can check that $Q = \begin{pmatrix} 0 & \omega & \omega^2 \\ \omega^2 & 0 & \omega \\ \omega & \omega^2 & 0 \end{pmatrix}$ is a 3×3 cube root signature matrix that satisfies $Q^2 = 2I + Q$. However, conjugating Q by the unitary matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$ shows that Q has standard form $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, and therefore, Q is trivial. Thus $e = 0$ and the starting point (3.1) for deriving our conditions does not hold.

Because $\alpha, \beta, \gamma, a, b, c, A, B, C, \alpha', \beta', \gamma', a', b', c', A', B'$ and C' are all non-negative integers, the above equations have a number of consequences for the values n and μ . We now state some results exploring these consequences. Remarkably, these consequences all seem to stem from the unprimed equations.

Proposition 3.3. *Let Q be a nontrivial cube root signature matrix of an equiangular (n, k) -frame, satisfying $Q^2 = (n - 1)I + \mu Q$. Then the following hold:*

- (a) *The value μ is an integer and $\mu \equiv 1 \pmod{3}$.*
- (b) *The integer n satisfies $n \equiv 0 \pmod{3}$.*
- (c) *If $\lambda_1 < 0 < \lambda_2$ are the eigenvalues of Q , then λ_1 and λ_2 are integers with $\lambda_1 \equiv 2 \pmod{3}$ and $\lambda_2 \equiv 2 \pmod{3}$.*
- (d) *The integer $4(n - 1) + \mu^2$ is a perfect square and in addition we have $4(n - 1) + \mu^2 \equiv 0 \pmod{9}$.*

Proof. For (a) note that Eq. (7) shows that $A + B + C = (n + 2\mu + 1)/3$ and hence $A + B + C = e + \mu + 1$, with $e = \frac{n - \mu - 2}{3}$. Since e is an integer by Proposition 2.4, and since $A, B,$ and C are integers, it follows that μ is an integer. In addition Eq. (1) shows that $2\mu = -3(\alpha - B) - 1$ so that $2\mu \equiv 2 \pmod{3}$ and $\mu \equiv 1 \pmod{3}$.

For (b) we see that Eq. (3) implies $n = 3(\gamma + B + C - \mu)$. Since $\gamma, B,$ and C are integers, and since μ is an integer by (a), we have that $n \equiv 0 \pmod{3}$.

For (c) we use (1.2) and (a) to see that $(n - 2k)\sqrt{\frac{n-1}{k(n-k)}} = \mu \in \mathbb{Z}$, and thus $\sqrt{\frac{n-1}{k(n-k)}} \in \mathbb{Q}$ and $\lambda_1 = -\sqrt{\frac{k(n-1)}{(n-k)}} = -k^2\sqrt{\frac{n-1}{k(n-k)}} \in \mathbb{Q}$. In addition, using (1.2) we have that $\lambda_2 = (1 - n)/\lambda_1 \in \mathbb{Q}$. Thus λ_1 and λ_2 are both rational. Because Q satisfies $Q^2 = (n - 1)I + \mu Q$, the polynomial $p(x) = x^2 - \mu x - (n - 1)$ annihilates Q , and hence the minimal polynomial of Q divides $p(x)$ and the eigenvalues λ_1 and λ_2 are rational roots of $p(x)$. Since the coefficients of $p(x)$ are integers and the leading coefficient of $p(x)$ is 1, the Rational Root Theorem tells us that the only rational roots of $p(x)$ are integers. Hence λ_1 and λ_2 are integers. Finally, using (1.2) we have that

$\lambda_1 + \lambda_2 = \mu$, so by (a) we have

$$\lambda_1 + \lambda_2 \equiv 1 \pmod{3}. \tag{3.17}$$

Also, using (1.2) we have that $\lambda_1\lambda_2 = 1 - n$, so by (b) we have

$$\lambda_1\lambda_2 \equiv 1 \pmod{3}. \tag{3.18}$$

There are only three possibilities for the residue of an integer modulo 3 (namely 0, 1, or 2) and a consideration of cases shows that the only situation in which (3.17) and (3.18) are both satisfied is when $\lambda_1 \equiv 2 \pmod{3}$ and $\lambda_2 \equiv 2 \pmod{3}$.

For (d) we use (1.2) to write $k = \frac{n}{2} - \frac{\mu n}{2\sqrt{4(n-1)+\mu^2}}$. Thus $\sqrt{4(n-1)+\mu^2} = \frac{\mu n}{n-2k} \in \mathbb{Q}$ by

(a). Since n and μ are integers, we have that $4(n-1)+\mu^2$ is an integer, and $\sqrt{4(n-1)+\mu^2}$ is rational if and only if $\sqrt{4(n-1)+\mu^2}$ is an integer. Thus $\sqrt{4(n-1)+\mu^2} = m$ for some $m \in \mathbb{Z}$ and $4(n-1)+\mu^2 = m^2$, so that $4(n-1)+\mu^2$ is a perfect square. Furthermore, since $4(n-1)+\mu^2 = m^2$ and we have $\mu \equiv 1 \pmod{3}$ by (a) and $n \equiv 0 \pmod{3}$ by (b), it follows that $m^2 \equiv 0 \pmod{3}$. Thus 3 divides m^2 , and since 3 is prime, we have that 3 divides m . Hence 9 divides $m^2 = 4(n-1)+\mu^2$ and $4(n-1)+\mu^2 \equiv 0 \pmod{9}$. \square

Proposition 3.3 shows that $\mu \equiv 1 \pmod{3}$ and $n \equiv 0 \pmod{3}$. However, we can do slightly better than this, as the following proposition shows.

Proposition 3.4. *Let Q be a nontrivial cube root signature matrix of an equiangular (n, k) -frame, satisfying $Q^2 = (n-1)I + \mu Q$. Then one of the following three cases must hold:*

- $n \equiv 0 \pmod{9}$ and $\mu \equiv 7 \pmod{9}$, or
- $n \equiv 3 \pmod{9}$ and $\mu \equiv 1 \pmod{9}$, or
- $n \equiv 6 \pmod{9}$ and $\mu \equiv 4 \pmod{9}$.

Proof. From Proposition 3.3(b), we have that $n \equiv 0 \pmod{3}$ so that n is congruent to 0, 3, or 6 modulo 9. Also, from Proposition 3.3(a), we have that $\mu \equiv 0 \pmod{3}$ so that μ is congruent to 1, 4, or 7 modulo 9. By Proposition 3.3(d) we have $4(n-1)+\mu^2 \equiv 0 \pmod{9}$ and by considering the possible values of n and μ modulo 9, we see the only way this equation is satisfied is if one of the three cases in the statement of this proposition holds. \square

Proposition 3.5. *Let Q be a nontrivial cube root signature matrix of an equiangular (n, k) -frame, satisfying $Q^2 = (n-1)I + \mu Q$. If we set $e := \frac{n-\mu-2}{3}$, then e is an integer with*

$$e \equiv 0 \pmod{3}$$

and e satisfies

$$\frac{2n}{9} \leq e \leq \frac{4n-9}{9}.$$

Proof. The fact that e is an integer follows from Proposition 2.4. Also since $3e = n - \mu - 2$, by considering the three possibilities of Proposition 3.4, we see that in any of these three cases we have $3e \equiv 0 \pmod{9}$. Hence $e \equiv 0 \pmod{3}$.

Next we look at Eq. (1), which shows that $\alpha - B = -\frac{2\mu}{3} - \frac{1}{3}$. Since α is a non-negative integer we have that $0 \leq \alpha = B - \frac{2\mu}{3} - \frac{1}{3}$ and

$$B \geq \frac{2\mu + 1}{3}. \tag{3.19}$$

Also, by Eq. (2) we have that $\beta - C = -\frac{2\mu}{3} - \frac{4}{3}$. Since β is a non-negative integer, we have that $0 \leq \beta = C - \frac{2\mu}{3} - \frac{4}{3}$ and

$$C \geq \frac{2\mu + 4}{3}. \tag{3.20}$$

In addition, by Eq. (5) we have that $b + B + C = \frac{n}{3} + \frac{\mu}{3} - \frac{1}{3} = e + \frac{2\mu+1}{3}$. Since b is a non-negative integer, we may use (3.19) and (3.20) to obtain

$$e + \frac{2\mu + 1}{3} = b + B + C \geq B + C \geq \frac{2\mu + 1}{3} + \frac{2\mu + 4}{3},$$

so that $e \geq \frac{2\mu+4}{3}$. Thus $3e \geq 2\mu + 4 = 2n - 6e$, and

$$e \geq \frac{2n}{9}.$$

For the upper bound on e , we may assume that Q is in standard form, so that every row of Q contains at least one 1, and consequently, $C \geq 1$. Using Eq. (3) and the fact that γ and B are non-negative integers, we have that

$$1 \leq \gamma + B + C = \frac{n}{3} + \mu = -3e + \frac{4n}{3} - 2,$$

so $3e \leq \frac{4n}{3} - 3$ and

$$e \leq \frac{4n - 9}{9}. \quad \square$$

Remark 3.6. By Proposition 2.4, there are e entries that are equal to ω and e entries that are equal to ω^2 . Thus, excluding the $Q_{i,1}$ entry, there are $n - 2 - 2e$ entries that are equal to 1. Thus, by the above inequalities, for each $i, 2 \leq i \leq n$, there are always at least $n - 2 - 2e \geq n - 2 - 2\frac{4n-9}{9} = \frac{n}{9}$ values of $j, 2 \leq j \leq n$ for which $Q_{i,j} = 1$. Hence, there is always at least one such entry.

Corollary 3.7. Let Q be a nontrivial cube root signature matrix of an equiangular (n, k) -frame satisfying $Q^2 = (n - 1)I + \mu Q$. If we set $e := \frac{n-\mu-2}{3}$, then μ is an integer with

$$\mu \equiv 1 \pmod{3}$$

and μ satisfies

$$1 - \frac{n}{3} \leq \mu \leq \frac{n}{3} - 2.$$

Remark 3.8. The fact that μ is an integer congruent to 1 modulo 3 is shown in Proposition 3.3. Using that $e = \frac{n-\mu-2}{3}$ we may rewrite the inequality in Proposition 3.5 in terms of μ to obtain $1 - \frac{n}{3} \leq \mu \leq \frac{n}{3} - 2$.

4. Narrowing the search for cube root signature matrices

In Section 3, we derived a number of conditions that the parameters (e.g. n, k, μ, e) of a nontrivial cube root Seidel matrix must satisfy to make it the signature matrix of an equiangular

(n, k) -frame. These conditions allow us to rule out a number of possible values for n and k .

In particular, the previous results can be incorporated in an algorithm to determine the possible k values for a given n .

Algorithm for deducing possible (n, k) values

Begin with a value of n that is divisible by 3 (see Proposition 3.3(b)).

Step 1: Find all values of e satisfying $\frac{2n}{9} \leq e \leq \frac{4n-9}{9}$ with $e \equiv 1 \pmod{3}$. (See Proposition 3.5.)

Step 2: For each e from Step 1, calculate the value of $\mu = n - 3e - 2$.

Step 3: For each μ from Step 2, calculate the value of $k = \frac{n}{2} - \frac{\mu n}{2\sqrt{4(n-1)+\mu^2}}$ (see (1.2)).

The only allowable (n, k) -frames with nontrivial signature matrices are those with k equal to an integer greater than one. A necessary condition for k to be an integer is that $\sqrt{4(n-1)+\mu^2}$ is rational.

Remark 4.1. One may wonder why in our algorithm we do not simply use Corollary 3.7 to first find the values of μ satisfying $1 - \frac{n}{3} \leq \mu \leq \frac{n}{3} - 2$ with $\mu \equiv 1 \pmod{3}$, and then proceed directly to Step 3. This would seem to eliminate the need to calculate the value of e , and reduce our algorithm by one step. It turns out, however, that this is less efficient. Since $e = \frac{n-\mu-2}{3}$, there will in general be less values of e found in Step 1 than there will be values of μ satisfying the condition of Corollary 3.7.

To demonstrate our algorithm we go through the calculations of possible values of (n, k) for $2 \leq k < n \leq 50$. To do this we use our algorithm on all values of n that are less than 50 and a multiple of 3.

$n = 3$. Step 1: We need $\frac{2}{3} \leq e$ and $e \leq \frac{1}{3}$, which cannot occur. Thus there are no allowable values of k in this case which lead to a nontrivial signature matrix.

$n = 6$. Step 1: We need $1\frac{1}{3} \leq e \leq 1\frac{2}{3}$ and $e \equiv 0 \pmod{3}$, which cannot occur. Thus there are no allowable values of k in this case.

$n = 9$. Step 1: We need $2 \leq e \leq 3$ and $e \equiv 0 \pmod{3}$. Thus e equals 3. Step 2: For $e = 3$ we have $\mu = -2$. Step 3: For $\mu = -2$ we have $k = \frac{n}{2} - \frac{\mu n}{2\sqrt{4(n-1)+\mu^2}} = 6$. Thus $(9, 6)$ is an allowable (n, k) value.

$n = 12$. Step 1: We need $2\frac{2}{3} \leq e \leq 4\frac{1}{3}$ and $e \equiv 0 \pmod{3}$. Thus e equals 3. Step 2: For $e = 3$ we have $\mu = 1$. Step 3: For $\mu = 1$ we have $\sqrt{4(n-1)+\mu^2} = \sqrt{45} \notin \mathbb{Q}$, which cannot occur. Thus there are no allowable values of k in this case.

$n = 15$. Step 1: We need $3\frac{1}{3} \leq e \leq 5\frac{2}{3}$ and $e \equiv 0 \pmod{3}$, which cannot occur. Thus there are no allowable values of k in this case.

$n = 18$. Step 1: We need $4 \leq e \leq 7$ and $e \equiv 0 \pmod{3}$. Thus e equals 6. Step 2: For $e = 6$ we have $\mu = -2$. Step 3: For $\mu = -2$ we have $\sqrt{4(n-1)+\mu^2} = \sqrt{72} \notin \mathbb{Q}$, which cannot occur. Thus there are no allowable values of k in this case.

$n = 21$. Step 1: We need $4\frac{2}{3} \leq e \leq 8\frac{1}{3}$ and $e \equiv 0 \pmod{3}$. Thus e equals 6. Step 2: For $e = 6$ we have $\mu = 1$. Step 3: For $\mu = 1$ we have $k = \frac{n}{2} - \frac{\mu n}{2\sqrt{4(n-1)+\mu^2}} = \frac{28}{3} \notin \mathbb{Q}$. Thus there are no allowable values of k in this case.

$n = 24$. Step 1: We need $5\frac{1}{3} \leq e \leq 9\frac{2}{3}$ and $e \equiv 0 \pmod{3}$. Thus e equals 6 or 9. Step 2: For $e = 6$ we have $\mu = 4$, and for $e = 9$ we have $\mu = -5$. Step 3: For $\mu = 4$ we have $\sqrt{4(n-1) + \mu^2} = \sqrt{108} \notin \mathbb{Q}$, which cannot occur. For $\mu = -5$ we have $\sqrt{4(n-1) + \mu^2} = \sqrt{117} \notin \mathbb{Q}$, which cannot occur. Thus there are no allowable values of k in this case.

$n = 27$. Step 1: We need $6 \leq e \leq 11$ and $e \equiv 0 \pmod{3}$. Thus e equals 6 or 9. Step 2: For $e = 6$ we have $\mu = 7$, and for $e = 9$ we have $\mu = -2$. Step 3: For $\mu = 7$ we have $\sqrt{4(n-1) + \mu^2} = \sqrt{153} \notin \mathbb{Q}$, which cannot occur. For $\mu = -2$ we have $\sqrt{4(n-1) + \mu^2} = \sqrt{108} \notin \mathbb{Q}$, which cannot occur. Thus there are no allowable values of k in this case.

$n = 30$. Step 1: We need $6\frac{2}{3} \leq e \leq 12\frac{1}{3}$ and $e \equiv 0 \pmod{3}$. Thus e equals 9 or 12. Step 2: For $e = 9$ we have $\mu = 1$, and for $e = 12$ we have $\mu = -8$. Step 3: For $\mu = 1$ we have $\sqrt{4(n-1) + \mu^2} = \sqrt{117} \notin \mathbb{Q}$, which cannot occur. For $\mu = -8$ we have $\sqrt{4(n-1) + \mu^2} = \sqrt{180} \notin \mathbb{Q}$, which cannot occur. Thus there are no allowable values of k in this case.

$n = 33$. Step 1: We need $7\frac{1}{3} \leq e \leq 13\frac{2}{3}$ and $e \equiv 0 \pmod{3}$. Thus e equals 9 or 12. Step 2: For $e = 9$ we have $\mu = 4$, and for $e = 12$ we have $\mu = -5$. Step 3: For $\mu = 4$ we have $k = \frac{n}{2} - \frac{\mu n}{2\sqrt{4(n-1) + \mu^2}} = 11$, which is allowed. For $\mu = -5$ we have $\sqrt{4(n-1) + \mu^2} = \sqrt{153} \notin \mathbb{Q}$, which cannot occur. Thus $(33, 11)$ is the only allowable (n, k) value for $n = 33$.

$n = 36$. Step 1: We need $8 \leq e \leq 15$ and $e \equiv 0 \pmod{3}$. Thus e equals 9 or 12 or 15. Step 2: For $e = 9$ we have $\mu = 7$, for $e = 12$ we have $\mu = -2$, and for $e = 15$ we have $\mu = -11$. Step 3: For $\mu = 7$ we have $\sqrt{4(n-1) + \mu^2} = \sqrt{189} \notin \mathbb{Q}$, which cannot occur. For $\mu = -2$ we have $k = \frac{n}{2} - \frac{\mu n}{2\sqrt{4(n-1) + \mu^2}} = 21$, which is allowed. For $\mu = -11$ we have $\sqrt{4(n-1) + \mu^2} = \sqrt{261} \notin \mathbb{Q}$, which cannot occur. Thus $(36, 21)$ is the only allowable (n, k) value for $n = 36$.

$n = 39$. Step 1: We need $8\frac{2}{3} \leq e \leq 16\frac{1}{3}$ and $e \equiv 0 \pmod{3}$. Thus e equals 9 or 12 or 15. Step 2: For $e = 9$ we have $\mu = 10$, for $e = 12$ we have $\mu = 1$, and for $e = 15$ we have $\mu = -8$. Step 3: For $\mu = 10$ we have $\sqrt{4(n-1) + \mu^2} = \sqrt{251} \notin \mathbb{Q}$, which cannot occur. For $\mu = 1$ we have $\sqrt{4(n-1) + \mu^2} = \sqrt{153} \notin \mathbb{Q}$, which cannot occur. For $\mu = -8$ we have $\sqrt{4(n-1) + \mu^2} = \sqrt{216} \notin \mathbb{Q}$, which cannot occur. Thus there are no allowable values of k in this case.

$n = 42$. Step 1: We need $9\frac{1}{3} \leq e \leq 17\frac{2}{3}$ and $e \equiv 0 \pmod{3}$. Thus e equals 12 or 15. Step 2: For $e = 12$ we have $\mu = 4$, and for $e = 15$ we have $\mu = -5$. Step 3: For $\mu = 4$ we have $\sqrt{4(n-1) + \mu^2} = \sqrt{180} \notin \mathbb{Q}$, which cannot occur. For $\mu = -5$ we have $\sqrt{4(n-1) + \mu^2} = \sqrt{189} \notin \mathbb{Q}$, which cannot occur. Thus there are no allowable values of k in this case.

$n = 45$. Step 1: We need $10 \leq e \leq 19$ and $e \equiv 0 \pmod{3}$. Thus e equals 12 or 15 or 18. Step 2: For $e = 12$ we have $\mu = 7$, for $e = 15$ we have $\mu = -2$, and for $e = 18$ we have $\mu = -11$. Step 3: For $\mu = 7$ we have $k = \frac{n}{2} - \frac{\mu n}{2\sqrt{4(n-1) + \mu^2}} = 12$, which is allowed. For $\mu = -2$ we have $\sqrt{4(n-1) + \mu^2} = \sqrt{180} \notin \mathbb{Q}$, which cannot occur. For $\mu = -11$ we have $\sqrt{4(n-1) + \mu^2} = \sqrt{297} \notin \mathbb{Q}$, which cannot occur. Thus $(45, 12)$ is the only allowable (n, k) value for $n = 45$.

$n = 48$. Step 1: We need $10\frac{2}{3} \leq e \leq 20\frac{1}{3}$ and $e \equiv 0 \pmod{3}$. Thus e equals 12 or 15 or 18. Step 2: For $e = 12$ we have $\mu = 10$, for $e = 15$ we have $\mu = 1$, and for $e = 18$ we have $\mu = -8$. Step 3: For $\mu = 10$ we have $\sqrt{4(n-1) + \mu^2} = \sqrt{288} \notin \mathbb{Q}$, which cannot occur. For $\mu = 1$ we have $\sqrt{4(n-1) + \mu^2} = \sqrt{189} \notin \mathbb{Q}$, which cannot occur. For $\mu = -8$ we have $\sqrt{4(n-1) + \mu^2} = \sqrt{252} \notin \mathbb{Q}$, which cannot occur. Thus there are no allowable values of k in this case.

We list the possible values of (n, k) in Table 1. In addition, although we do not reproduce the calculations here, the authors have computed the possible (n, k) values from 50 to 100 as well, and these are also listed in Table 1.

Table 1
Possible (n, k) values for cube root signature matrices with $2 \leq k < n \leq 100$

n	k	μ	e	λ_1	λ_2	Do they exist?
9	6	-2	3	-7	5	Yes (Theorem 6.1)
33	11	4	9	-4	8	Unknown
36	21	-2	12	-7	5	Unknown
45	12	7	12	-4	11	Unknown
51	34	-5	18	-10	5	Unknown
81	45	-2	27	-10	8	Yes (Theorem 6.3)
96	76	-14	36	-19	5	Unknown
99	33	7	30	-7	14	Unknown

5. Graphs of cube root signature matrices

Arguably, the most successful means to find equiangular tight frames in the real case has been via the correspondence between graphs and signature matrices. Much of the graph-theoretic approach to real signature matrices, due largely to [15], can be repeated in our case by replacing graphs by *directed graphs*. In this section, we make the connection between cube root signature matrices and directed graphs explicit and describe necessary and sufficient conditions that a directed graph must satisfy in order to give rise to a cube root signature matrix in standard form.

We define a one-to-one correspondence between $n \times n$ self-adjoint matrices whose diagonal entries are zero and whose off-diagonal entries are cube roots of unity with directed graphs (with no loops) on n vertices in the following manner. Given such a matrix Q , we associate its rows and columns with vertices, and an entry $Q_{i,j} = \omega$ with a directed edge from the i th to the j th vertex, while $Q_{i,j} = \omega^2$ is associated with a directed edge from the j th to the i th vertex. When $Q_{i,j} = 1 = Q_{j,i}$, then there is no edge connecting the i th and j th vertex. We denote this directed graph by $G(Q)$. Conversely, given any directed graph G on n vertices, and an enumeration of the vertices, we obtain such a matrix which we denote by Q_G and we call this matrix the *Seidel adjacency matrix* of the directed graph.

Given a directed graph G and vertices v and w we write $v \rightarrow w$ or $w \leftarrow v$ to indicate that there is a directed edge from v to w . We say that v *emits* the edge and that w *receives* the edge. When there is no directed edge (in either direction) between v and w , we write $v \wr w$.

A key element of Seidel’s theory [15] was the introduction of his switching equivalence of graphs. We reformulate the equivalence relation for complex Seidel matrices for directed graphs.

Suppose that we have a self-adjoint matrix Q whose off-diagonal entries are cube roots of unity and we associate a directed graph to Q as above. If we let D be the diagonal matrix satisfying $d_{j,j} = 1, j \neq i$ and $d_{i,i} = \omega$, then the directed graph G_1 associated to DQD^* is obtained from the directed graph G associated to Q by

- inserting a directed edge from i to j whenever i and j had no edge,
- replacing a directed edge from i to j by a directed edge from j to i , and
- deleting all directed edges from j to i .

We shall refer to this set of operations as the ω -switching on the i th vertex. Note that unlike Seidel’s switching for undirected graphs [15], when we switch twice on the i th vertex, we do not return to the original graph. Here when we switch three times, we return to the original

directed graph. If we consider cube root signature matrices that are in standard form, then in the corresponding directed graph the first vertex neither emits nor receives any edges. Thus, without loss of generality, we may ignore this vertex and focus on the directed subgraph on $m = n - 1$ vertices.

We now wish to describe explicitly in graph theoretical terms the directed graphs that correspond to cube root signature matrices in standard form.

Definition 5.1. We call a directed graph on m vertices e -regular, provided that each vertex emits exactly e directed edges and receives exactly e directed edges.

Given a directed graph G and vertices v and w we now wish to describe the parameters that are the natural equivalents of the parameters of Section 3.

Given a directed graph G with vertex set V and vertices v and w with $v \rightarrow w$, we set

$$\begin{aligned} \alpha &= \#\{u \in V : v \rightarrow u \text{ and } w \rightarrow u\}, \\ \beta &= \#\{u \in V : v \rightarrow u \text{ and } u \rightarrow w\}, \\ \gamma &= \#\{u \in V : v \rightarrow u \text{ and } u \wr w\}, \\ a &= \#\{u \in V : u \rightarrow v \text{ and } w \rightarrow u\}, \\ b &= \#\{u \in V : u \rightarrow v \text{ and } u \rightarrow w\}, \\ c &= \#\{u \in V : u \rightarrow v \text{ and } u \wr w\}, \\ A &= \#\{u \in V : u \wr v \text{ and } w \rightarrow u\}, \\ B &= \#\{u \in V : u \wr v \text{ and } u \rightarrow w\}, \\ C_1 &= \#\{u \in V : u \wr v \text{ and } u \wr w\}. \end{aligned}$$

We have introduced the slight change of notation, C_1 , from our earlier notation C , because since we have omitted the corresponding first vertex, we will have that $C_1 = C - 1$.

Similarly, given a directed graph G with vertex set V and vertices v and w such that $v \wr w$, we set

$$\begin{aligned} \alpha' &= \#\{u \in V : v \rightarrow u \text{ and } w \rightarrow u\}, \\ \beta' &= \#\{u \in V : v \rightarrow u \text{ and } u \rightarrow w\}, \\ \gamma' &= \#\{u \in V : v \rightarrow u \text{ and } u \wr w\}, \\ a' &= \#\{u \in V : u \rightarrow v \text{ and } w \rightarrow u\}, \\ b' &= \#\{u \in V : u \rightarrow v \text{ and } u \rightarrow w\}, \\ c' &= \#\{u \in V : u \rightarrow v \text{ and } u \wr w\}, \\ A' &= \#\{u \in V : u \wr v \text{ and } w \rightarrow u\}, \\ B' &= \#\{u \in V : u \wr v \text{ and } u \rightarrow w\}, \\ C'_1 &= \#\{u \in V : u \wr v \text{ and } u \wr w\}. \end{aligned}$$

Again we will have that $C'_1 = C' - 1$.

We now reinterpret Theorem 3.1 in the terminology of directed graphs.

Theorem 5.2. *There exists a nontrivial $n \times n$ matrix Q in standard form whose off-diagonal entries are cube roots of unity and such that Q satisfies $Q^2 = (n - 1)I + \mu Q$, for some μ , if and only if for some $e > 0$ there exists a non-empty e -regular directed graph on $m = n - 1$ vertices*

such that each pair of vertices with a directed edge from v to w satisfies

$$\alpha - B = \frac{6e + 1 - 2m}{3}, \tag{Eq. d.1}$$

$$\beta - C_1 = \frac{6e - 2m + 1}{3}, \tag{Eq. d.2}$$

$$\gamma + B + C_1 = \frac{4m - 9e - 5}{3}, \tag{Eq. d.3}$$

$$a - C_1 = \frac{3e - m + 2}{3}, \tag{Eq. d.4}$$

$$b + B + C_1 = \frac{2m - 3e - 4}{3}, \tag{Eq. d.5}$$

$$c - B = \frac{3e - m + 2}{3}, \tag{Eq. d.6}$$

$$A + B + C_1 = m - 2e - 1, \tag{Eq. d.7}$$

while each pair of vertices v and w with no edges between them satisfies

$$\alpha' = B', \tag{Eq. d.8}$$

$$\beta' = C'_1 + 3e + 2 - m, \tag{Eq. d.9}$$

$$\gamma' = m - 2e - 2 - B' - C'_1, \tag{Eq. d.10}$$

$$a' = C'_1 + 3e + 2 - m, \tag{Eq. d.11}$$

$$b' = m - 2e - 2 - B' - C'_1, \tag{Eq. d.12}$$

$$c' = B', \tag{Eq. d.13}$$

$$A' = m - 2e - 2 - B' - C'_1. \tag{Eq. d.14}$$

Moreover, in the above case we have that $\mu = m - 3e - 1$ and we obtain Q from the directed graph on $n - 1$ vertices by adjoining a row and column of $+1$'s to the adjacency matrix of the directed graph.

The results of Section 3 can now be interpreted as necessary conditions on m and e for the existence of such graphs.

If G is the directed graph associated to Q and after a sequence of switchings on various vertices, we obtain a new directed graph G_1 , then it is easy to see that the matrix Q_1 associated to G_1 satisfies $Q_1 = DQD^*$ for some diagonal matrix whose entries are cube roots of unity. In fact D can be taken to be the diagonal matrix whose i th diagonal entry is ω^{s_i} where s_i is the number of times that we switched on the i th vertex.

The proof of Lemma 2.2 shows that given a directed graph G on n vertices and a fixed vertex v , then G is switching equivalent to a unique directed graph where the vertex v is isolated, i.e., such that v neither emits nor receives any directed edges. We let G_v denote the directed graph on $n - 1$ vertices that one obtains by deleting this isolated vertex.

Proposition 5.3. *Let G be a directed graph on n vertices with adjacency matrix Q . Then the following are equivalent:*

- (1) $Q^2 = (n - 1)I + \mu Q$ for some real number μ ,

- (2) for some vertex v , the graph G_v is e -regular and satisfies Eqs. (d.1)–(d.14),
- (3) for every vertex v , the graph G_v is e -regular and satisfies Eqs. (d.1)–(d.14).

Moreover, in this case $\mu = n - 3e - 2$.

Proof. We have that $Q^2 = (n - 1)I + \mu Q$ if and only if $(D^* Q D)^2 = (n - 1)I + \mu(D^* Q D)$ for any diagonal matrix of whose entries consist of cube roots of unity. These matrices represent the adjacency matrices of all directed graphs switching equivalent to G with the same enumeration of vertices. Thus, if the adjacency matrix for G satisfies the above equation, then the adjacency matrix for every directed graph switching equivalent to G satisfies the above equation. The result now follows by applying Theorem 5.2. \square

We conclude that a necessary condition for the existence of cube root signature matrices is the existence of directed graphs without undirected edges for which all vertices have the same in-degree and out-degree e . Moreover, if there is an edge $v \rightarrow w$, then the paths between v to w of length 2 satisfy conditions (Eqs. (d.1)–(d.7)) and if there is no edge between v and w , then they satisfy conditions (Eqs. (d.8)–(d.14)). If there is only one solution for these parameters, then this implies that the graph is a directed strongly regular graph [6].

6. Examples of cube root signature matrices with two eigenvalues

6.1. A first example

In the literature on directed strongly regular graphs, there is a graph on 8 vertices, all with same in-degree and out-degree $e = 3$ [14, Example 4.1]. Indeed, it turns out that the cube root Seidel matrix of this directed graph is the signature matrix of an equiangular $(9, 6)$ -frame. This is, in fact, the unique frame up to switching, which follows from the uniqueness of directed strongly regular graphs on 8 vertices with in-degree and out-degree 3 [9].

Theorem 6.1. *The matrix*

$$Q = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega & \omega & \omega^2 & \omega^2 & \omega^2 \\ 1 & 1 & 0 & \omega^2 & \omega^2 & \omega^2 & \omega & \omega & \omega \\ 1 & \omega^2 & \omega & 0 & \omega & \omega^2 & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega & \omega^2 & 0 & \omega & \omega & \omega^2 & 1 \\ 1 & \omega^2 & \omega & \omega & \omega^2 & 0 & \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 & 1 & \omega^2 & \omega & 0 & \omega^2 & \omega \\ 1 & \omega & \omega^2 & \omega^2 & \omega & 1 & \omega & 0 & \omega^2 \\ 1 & \omega & \omega^2 & \omega & 1 & \omega^2 & \omega^2 & \omega & 0 \end{pmatrix}$$

is a 9×9 nontrivial cube root signature matrix of an equiangular $(9, 6)$ -frame. Furthermore, any 9×9 nontrivial cube root signature matrix belonging to $k = 6$ is switching equivalent to Q .

Proof. To see that Q is a signature matrix, one need only verify that $Q^2 = 8I - 2Q$.

We prove the claimed uniqueness using a graph-theoretic argument [9]. It will be enough to show that up to a renumbering of the vertices there is a unique directed graph on 8 vertices satisfying the conditions, Eqs. (d.1)–(d.14). To this end first note that since $e = 3$, there will be exactly $9 - 2e - 2 = 1$ entry in each row, other than $Q_{i,1}$, that is equal to 1. Thus, in the directed graph on 8 vertices, each vertex will emit 3 directed edges, receive 3 directed edges, and not be connected to exactly one vertex.

However, there is a unique 6-regular graph on 8 vertices. This is because each non-adjacent pair of vertices has 6 common neighbors. Thus, the graph can be partitioned into 4 such pairs, and is seen to be the complete 4-partite graph corresponding to this partition of 8 vertices in sets of size 2.

Next one finds that Eqs. (d.1)–(d.14) have a unique set of solutions, $\alpha = \beta = a = b = c = A = 1, \gamma = B = C_1 = 0$ and $\alpha' = \gamma' = b' = c' = A' = 0, \beta' = a' = 3$. Thus, in the directed graph, if $v \wr w$, then since $\beta' = 3$, the three directed edges emitted by v terminate at the three directed edges received by w . Similarly, since $a' = 3$, the three directed edges received by v are emitted by the three vertices where the three directed edges emitted by w terminate.

One now finds that there is a unique directed graph on 8 vertices satisfying these relations. \square

6.2. Creating new signature matrices from old

In this section we discuss ways to form new signature matrices from existing ones.

Proposition 6.2. *Suppose that Q_1 and Q_2 are $n_1 \times n_1$ and $n_2 \times n_2$ signature matrices satisfying the equation*

$$Q_i^2 = (n_i - 1)I_{n_i} - 2Q_i, \quad i \in \{1, 2\}$$

Then the matrix $Q := (Q_1 + I_{n_1}) \otimes (Q_2 + I_{n_2}) - I_{n_1 n_2}$ is an $n_1 n_2 \times n_1 n_2$ signature matrix satisfying the equation

$$Q^2 = (n_1 n_2 - 1)I_{n_1 n_2} - 2Q.$$

Proof. We first observe that Q is self-adjoint, and since Q_1 and Q_2 contain unimodular off-diagonal entries and the identity matrix I contains 1's along the diagonal, we have that the diagonal entries of Q are zero, and the off-diagonal entries of Q are unimodular. Furthermore

$$\begin{aligned} (Q + I_{n_1 n_2})^2 &= [(Q_1 + I_{n_1}) \otimes (Q_2 + I_{n_2})]^2 \\ &= (Q_1 + I_{n_1})^2 \otimes (Q_2 + I_{n_2})^2 \\ &= (Q_1^2 + 2Q_1 + I_{n_1}) \otimes (Q_2^2 + 2Q_2 + I_{n_2}) \\ &= ((n_1 - 1)I_{n_1} - 2Q_1 + 2Q_1 + I_{n_1}) \otimes ((n_2 - 1)I_{n_2} - 2Q_2 + 2Q_2 + I_{n_2}) \\ &= n_1 I_{n_1} \otimes n_2 I_{n_2} \\ &= n_1 n_2 (I_{n_1} \otimes I_{n_2}) \\ &= n_1 n_2 I_{n_1 n_2}. \end{aligned}$$

Thus $Q^2 + 2Q + I_{n_1 n_2} = n_1 n_2 I_{n_1 n_2}$, and $Q^2 = (n_1 n_2 - 1)I_{n_1 n_2} - 2Q$, so Q is a signature matrix. \square

Theorem 6.3. For each $m \in \mathbb{N}$ there exists a nontrivial $9^m \times 9^m$ cube root signature matrix Q satisfying $Q^2 = (9^m - 1)I_{9^m} - 2Q$.

Proof. When $m = 1$ it follows from Theorem 6.1 that there exists a 9×9 nontrivial cube root signature matrix Q' satisfying $(Q')^2 = 8I_9 - 2Q'$. To obtain matrices of size $9^m \times 9^m$ we iterate the construction of the preceding proposition: We form

$$Q := \underbrace{[(Q' + I_9) \otimes \cdots \otimes (Q' + I_9)]}_{m \text{ times}} - I_{9^m}.$$

By considering the Kronecker product of matrices, we see that the off-diagonal entries of Q are cube roots of unity, and hence Q is a cube root signature matrix. Furthermore, we see that Q is nontrivial, because the 9×9 matrix Q' has a nontrivial entry ω in addition to 1's, so the Kronecker product with itself produces at least one nontrivial entry. \square

Remark 6.4. One can see that Theorem 6.3 shows there is an infinite number of nontrivial cube root signature matrices of arbitrarily large size.

Table 1 contains all possible (n, k) values for nontrivial cube root signature matrices, as determined by the algorithm in Section 4. Thus, if Q is a nontrivial cube root signature matrix of an equiangular (n, k) -frame with $n \leq 100$, then n and k must be the values in one of the rows of this table. It is unknown whether there exist nontrivial signature matrices for all of the rows in the table.

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