Solutions for semilinear elliptic problems with critical Sobolev–Hardy exponents and Hardy potential

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Abstract

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain such that $0 \in \Omega$, $N \geq 5$, $0 \leq s < 2$, $2^*(s) = \frac{2(N-s)}{N-2}$. We prove the existence of nontrivial solutions for the singular critical problem

$$-\Delta u - \mu \frac{u}{|x|^2} = \frac{|u|^{2^*(s)-2}}{|x|^s} u + \lambda u$$

with Dirichlet boundary condition on $\Omega$ for all $\lambda > 0$ and $0 \leq \mu < \left(\frac{N+2}{2}\right)^2 - \left(\frac{N+2}{N}\right)^2$.

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1. Introduction and main results

Consider the following problem:

$$\begin{cases}
-\Delta u - \mu \frac{u}{|x|^2} = \frac{|u|^{2^*(s)-2}}{|x|^s} u + \lambda u, & x \in \Omega, \\
u = 0, & x \in \partial \Omega,
\end{cases}$$

(1.1)
where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N (N \geq 3)$, $0 \in \Omega$, $\lambda > 0$, $0 \leq \mu < \bar{\mu} := (\frac{N-2}{2})^2$, $\bar{\mu}$ is the best constant in the Hardy inequality, $2^* (s) := \frac{2(N-s)}{N-2}$ is the critical Sobolev–Hardy exponent; note that $2^* (0) = 2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent. As a consequence of the Hardy inequality, the linear elliptic operator $L := (-\Delta - \frac{\mu}{|x|^2})$ is positive and has discrete spectrum $\sigma_\mu$ in $H_0^1 (\Omega)$ if $0 \leq \mu < \bar{\mu}$. Let $\lambda = \lambda_1 (\mu)$ be the first eigenvalue of the operator $L$ in $H_0^1 (\Omega)$ and define the energy functional for (1.1) on $H_0^1 (\Omega)$ by

$$ J (u) := \frac{1}{2} \int_\Omega \left( |\nabla u|^2 - \mu \frac{|u|^2}{|x|^2} \right) dx - \frac{1}{2^*(s)} \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx - \frac{\lambda}{2} \int_\Omega |u|^2 dx, $$

due to the invariance of $\int_\Omega |\nabla u|^2 dx$, $\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx$ and $\int_\Omega \frac{|u|^2}{|x|^s} dx$ with respect to the rescaling $u \mapsto u_\varepsilon = \varepsilon^{-\frac{N-2}{2}} u (\varepsilon \cdot)$ and the existence of nontrivial entire solution of the limiting problem (see [1–3])

$$ \begin{cases} 
-\Delta u - \mu \frac{u}{|x|^2} = \frac{|u|^{2^*-2} u}{|x|^s}, & x \in \mathbb{R}^N, \\
|u| < \infty, & |x| \to \infty,
\end{cases} $$

$J (u)$ fails to satisfy the classical Palais–Smale ($PS$ in short) condition in $H_0^1 (\Omega)$. However, a local $PS$ condition can be established. Indeed, define the best constant

$$ A_{\mu,s} := \inf_{u \in H_0^1 (\Omega) \setminus \{0\}} \frac{\int_\Omega \left( |\nabla u|^2 - \mu \frac{|u|^2}{|x|^2} \right) dx}{\left( \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{s}{2^*(s)}}}, $$

suppose $\{u_n\} \subset H_0^1 (\Omega)$ is a sequence such that $J (u_n) \leq c < \frac{2-s}{2(N-s)} (A_{\mu,s})^{\frac{N-s}{2-s}}$, $J' (u_n) \rightarrow 0$ in $H^{-1} (\Omega) = \left( H_0^1 (\Omega) \right)^*$, then $\{u_n\}$ contains a strongly convergent subsequence. Using this local $PS$ condition, Kang and Peng proved in [3] that problem (1.1) has at least one positive solution $u_0 \in H_0^1 (\Omega)$ for $0 < \lambda < \lambda_1$ and suitable parameter $\mu$. For earlier work on (1.1) as $s = 0$, see [4–6]. For the quasi-linear form of (1.1) with $\mu = 0$, see [7]. Note that as $s = 0$, $A_{\mu,s}$ becomes $A_{\mu,0}$, i.e.,

$$ A_{\mu,0} := \inf_{u \in H_0^1 (\Omega) \setminus \{0\}} \frac{\int_\Omega \left( |\nabla u|^2 - \mu \frac{|u|^2}{|x|^2} \right) dx}{\left( \int_\Omega |u|^2 dx \right)^\frac{2}{2^*}}. $$

The best constant $A_{\mu,0}$ is used and plays an important role in the discussion of [4–6].

By the results in [1,2], the authors of [3] found that for $\varepsilon > 0$ and $\beta := \sqrt{\mu} - \mu$, the functions

$$ u_\varepsilon^\beta (x) = \left( \frac{2\beta \varepsilon^2 (N-s)}{\sqrt{\mu}} \right)^{\frac{N-s}{2-s}} / \left( |x|^{\sqrt{\mu} - \beta} \left( \varepsilon + |x|^{\frac{2-s}{\sqrt{\mu}}} \right)^{\frac{N-s}{2-s}} \right) $$

solves the equation

$$ -\Delta u - \mu \frac{u}{|x|^2} = \frac{|u|^{2^*(s)-2}}{|x|^s} u \quad \text{in } \mathbb{R}^N \setminus \{0\} $$
and satisfy
\[ \int_{\mathbb{R}^N} \left( |\nabla u^s_\mu|^2 - \mu \frac{|u^s_\mu|^2}{|x|^s} \right) \, dx = \int_{\mathbb{R}^N} \frac{|u^s_\mu|^2(s)}{|x|^s} \, dx = (A_{\mu,s})^{\frac{N-s}{2}}; \]

$A_{\mu,s}$ is independent of $\Omega$ and is achieved by $U_0$ on $\mathbb{R}^N$.

By Pohozaev’s identity, if $\Omega$ is a star-shaped domain in $\mathbb{R}^N$, then problem (1.1) has no nontrivial solutions for $\lambda \leq 0$. It is easy to verify that as $\lambda \geq \lambda_1$, every solution of (1.1) must change sign. So it is meaningful to study the existence of nontrivial solutions for problem (1.1) as $s \in [0, 2)$ and $\lambda \in (0, +\infty)$. Kang and Peng in [8] discussed the nontrivial solutions to (1.1) and get some existence results for large range $\lambda > 0$.

Recently, Cao and Han in [9] proved affirmatively an open problem in [5], that is:

**Open Problem.** Assume that $\Omega$ is an open bounded domain in $\mathbb{R}^N$, $N \geq 5$, $0 \in \Omega$ and $0 \leq \mu < \bar{\mu} - (N+2)^2/N$. Then for $s = 0$ and for all $\lambda > 0$, problem (1.1) admits a nontrivial solution with critical level in the range $(0, \frac{1}{N}(A_{\mu,0})^{\frac{N}{N-2}})$.

Stimulated by [9], a natural interesting question arises, i.e., whether the above results remain true for (1.1) as $0 < s < 2$, with the critical Sobolev–Hardy growth.

This question is in fact the continuation of the above open problem. In the case of problem (1.1) with $s > 0$, we need to consider not only the effect of parameter $\lambda$ and $\mu$, but also that of parameter $s$. Namely, problem (1.1) becomes more complicated to deal with.

In this paper, by using the techniques in [5,8,9], we obtain the following existence results, which also answer the newly arisen question.

**Theorem 1.1.** Assume that $N \geq 5$, $\Omega$ is an open bounded domain in $\mathbb{R}^N$, $0 \leq s < 2$ and $0 \leq \mu < \bar{\mu} - (N+2)^2/N$. Then for all $\lambda > 0$, problem (1.1) has at least one nontrivial solution $u \in H_0^1(\Omega)$ with energy level in the range $(0, \frac{2-s}{2(N-s)}(A_{\mu,s})^{\frac{N-s}{N-2}})$.

This paper is organized as follows. In Section 2, we establish some asymptotic estimates; in Section 3, we give the proof of our theorem. These ideas are essentially introduced in [5,8,9]. In the following discussion, we denote various positive constants as $C, C_1, C_2, \ldots$, and omit $dx$ in integration for convenience.

2. Some technical asymptotic estimates

We first define the equivalent norm in $H_0^1(\Omega)$ for $0 \leq \mu < \bar{\mu}$:
\[
\|u\| := \left( \int_{\Omega} \left( |\nabla u|^2 - \mu \frac{u^2}{|x|^s} \right) \right)^{\frac{1}{2}}, \quad \forall u \in H_0^1(\Omega).
\]

By Hardy inequality, this norm is equivalent to the usual norm in $H_0^1(\Omega)$. We also denote the norm of $L^p(\Omega)$ space as $|u|_p$.

Fix $k \in \mathbb{N}$ and for all $i \in \mathbb{N}$ denote by $e_i$ an $L^2$ normalized eigenfunction relative to $\lambda_i \in \sigma_{\mu}$, let $H^-$ denote the space spanned by the eigenfunctions corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_k$ and $H^+ := (H^-)^\perp$. Take always $m \in \mathbb{N}$ large enough so that $B_{1/m} \subset \Omega$, where $B_{1/m}$ denotes the ball of
radius \(1/m\) with center at 0. Define
\[
\zeta_m(x) := \begin{cases} 
0, & x \in B_{1/m}, \\
 m|x| - 1, & x \in A_m = B_{2/m} \setminus B_{1/m}, \\
1, & x \in \Omega \setminus B_{2/m}.
\end{cases}
\]

\(e_i := \zeta_i e_i, H_m := \text{span}\{e_i; i = 1, \ldots, k\}\) and \(\Lambda := \{u \in H_m^{-}; |u|_2 = 1\}\).

**Lemma 2.1** ([5,9]). As \(m \to \infty\), we have

(i) \(e^m_i \to e_i\) in \(H_1^0(\Omega), \forall i \in \mathbb{N}\).

(ii) \[
\max_{u \in \Lambda} \|u\|^2 \leq \lambda_k + Cm^{-2\beta}. \quad \Box
\]

Consider the function \(u^*_\varepsilon(x)\) in (1.2). Since \(u^*_\varepsilon\) is a radial function, we can view it also as a function on \(\mathbb{R}^+\). For all \(m \in \mathbb{N}\) and \(\varepsilon > 0\), define the shifted functions
\[
u^m_\varepsilon(x) := \begin{cases} 
 u^*_\varepsilon(x) - u^*_\varepsilon\left(\frac{1}{m}\right), & x \in B_{1/m} \setminus \{0\}, \\
0, & x \in \Omega \setminus B_{1/m},
\end{cases}
\]
then we have the following estimates.

**Lemma 2.2** ([8]). There exist \(C_1, C_2\) and \(K > 0\), such that if \(\varepsilon < 2(N-2)\alpha m^{2\beta} < K\), then
\[
\|u^m_{\varepsilon}\|^2 \leq (A_{\mu,s}) \frac{N-s}{2-s} + C_1 \varepsilon \frac{2(N-2)\alpha m^{2\beta}}{2-s}, \quad (2.1)
\]
\[
\int_{\Omega} |u^m_{\varepsilon}|^{2^*(s)} \geq (A_{\mu,s}) \frac{N-s}{2-s} - C_2 \varepsilon \frac{2(N-2)\alpha m^{2\beta}}{2-s}. \quad (2.2)
\]

3. **Proof of Theorem 1.1**

We recall that a sequence \(\{u_n\} \subset H_1^0(\Omega)\) is called a PS sequence for \(J\) at level \(c\) if \(J(u_n) \to c\) and \(J'(u_n) \rightharpoonup 0\) in \(H^{-1}(\Omega)\). The functional \(J\) is said to satisfy the (PS)\(_c\) condition, if every (PS) sequence of \(J\) at level \(c\) contains a strongly convergent subsequence. The following results concerning the local compactness are already known. The proof of Lemma 3.1 can also be found in Theorem 4.1 of [7] by setting \(p = 2\) and by using \(A_{\mu,s}\) and our equivalent norm in \(H_1^0(\Omega)\).

**Lemma 3.1** ([3]). Suppose \(\lambda > 0\), then \(J(u)\) satisfies the (PS)\(_c\) condition for all \(c < \frac{2-s}{2(N-s)}(A_{\mu,s}) \frac{N-s}{2-s}\).  

For any \(m > 0\) and \(\varepsilon > 0\), we define
\[
c_{\varepsilon} := \inf_{h \in \Gamma} \max_{v \in Q^c_m} J(h(v)),
\]
where
\[
\Gamma := \{h \in C(Q^c_m, H_0^1(\Omega)); h(v) = v, \forall v \in \partial Q^c_m\}
\]
and
\[
Q^c_m := [(B_{R} \cap H_m^{-}) \oplus [0, R][u^m_{\varepsilon}]].
\]
The following result is crucial in our discussion.

**Lemma 3.2.** Assume that \( \lambda > 0, N \geq 5 \) and \( 0 \leq \mu < \tilde{\mu} - \frac{(N+2)^2}{N} \). Then we have

\[
c_{\varepsilon} < \frac{2 - s}{2(N - s)} (A_{\mu, s})^{\frac{N-s}{N}}.
\]

**Proof.** We may assume that \( \lambda_k \leq \lambda < \lambda_{k+1} \). Let

\[
\max_{u \in Q_m} J(u) = J(w^m + t_m u^m) \quad \text{for some } w^m \in H_m^-.
\]

Note that the space \( H_m^- \) is finite dimensional and the convergence of \( \{w^m\} \) can be viewed as in any norm topology; thus from Lemma 2.1 we get

\[
J(w^m) = \frac{1}{2} \|w^m\|^2 - \frac{\lambda}{2} \int_{\Omega} |w^m|^2 - \frac{1}{2^* (s)} \int_{\Omega} |w^m|^{2^* (s)}
\]

\[
\leq \left( \frac{\lambda_k - \lambda}{2} + C_3 m^{-2\beta} \right) |w^m|_2^2 - C_4 |w^m|_2^{2^* (s)}
\]

\[
\leq C_3 m^{-2\beta} |w^m|_2^2 - C_4 |w^m|_2^{2^* (s)}
\]

\[
\leq \max_{t \geq 0} (C_3 m^{-2\beta} t^2 - C_4 t^{2^* (s)})
\]

\[
= C_5 m^{-\frac{2(N-\beta)}{2-s}}.
\]

On the other hand, as in [8], setting \( \varepsilon = m^{-(N+2)(2-s)\beta/(2(N-2))} \), from now on we denote \( u^m \) and \( t_m \) as \( u^m \) and \( t_m \) with the above choice of \( \varepsilon \). Then as \( m \to \infty \), (2.1) and (2.2) become

\[
\|u^m\|^2 \leq (A_{\mu, s})^{\frac{N-s}{2-s}} + C_1 m^{-N\beta}, \tag{3.1}
\]

\[
\int_{\Omega} |u^m|^{2^* (s)} \geq (A_{\mu, s})^{\frac{N-s}{2-s}} - C_2 m^{-\frac{\mu(N-\beta)}{N-2}}. \tag{3.2}
\]

Moreover, we also get that (see [8])

\[
\int_{\Omega} |u^m|^2 \geq C_0 m^{-(N+2)}. \tag{3.3}
\]

Note that \( id \in I \) and \( |\text{supp } w^m \cap \text{supp } u^m| = 0 \), thus

\[
c_{\varepsilon} \leq \max_{u \in Q_m} J(u) = J(w_m + t_m u^m) = J(w_m) + J(t_m u^m) \tag{3.4}
\]

for some \( t_m > 0 \). From (3.1)–(3.3) we have that

\[
J(t_m u^m) = \frac{1}{2} \|t_m u^m\|^2 - \frac{\lambda}{2} \|t_m u^m\|_2^2 - \frac{1}{2^* (s)} \int_{\Omega} |t_m u^m|^{2^* (s)}
\]

\[
= \frac{t_m^2}{2} (\|u^m\|^2 - \lambda |u^m|^2) - \frac{t_m^2}{2^* (s)} \int_{\Omega} |t_m u^m|^{2^* (s)}
\]

\[
\leq \frac{t_m^2}{2} ((A_{\mu, s})^{\frac{N-s}{2-s}} + C_1 m^{-N\beta} - \lambda C_0 m^{-(N+2)}). 
\]
for $m$ large enough, where we have used the fact that

$$\max_{t \geq 0} \left( \frac{t^2}{2} B_1 - \frac{t^{2s}(s)}{2^s(s)} B_2 \right) = \frac{2 - s}{2(N - s)} B_1 \left( \frac{B_1}{B_2} \right)^{\frac{N - s}{2} - 1}, \quad B_1 > 0, B_2 > 0$$

and

$$N + 2 < N \beta < \frac{N(N - s)}{N - 2} \beta \quad \text{for } 0 \leq \mu < \tilde{\mu} - \left( \frac{N + 2}{N} \right)^2.$$ 

By $0 \leq \mu < \tilde{\mu} - (\frac{N + 2}{N})^2$ we also get

$$N + 2 < N \beta < \frac{2(N - s)}{2 - s} \beta.$$

Hence, for $m$ large enough we deduce from (3.4) that

$$c_e \leq \frac{2 - s}{2(N - s)} (A_{\mu,s})^{\frac{N - s}{2 - s}} + C_7 m^{-N \beta} - C_8 m^{-(N + 2)} + C_5 m^{-\frac{2(N - s)}{2 - s} \beta}$$

$$\leq \frac{2 - s}{2(N - s)} (A_{\mu,s})^{\frac{N - s}{2 - s}}. \quad \square$$

**Proof of Theorem 1.1.** The proof is standard and we only give a sketch of it. According to [5], as $m$, $R$ large enough, the functional $J(u)$ satisfies all the assumptions of the linking theorem (see [10]) except for the $(PS)_c$ condition, namely:

(i) There exist $\alpha, \rho > 0$ such that

$$J(u) \geq \alpha \quad \text{for all } u \in \partial B_\rho \cap H^+.$$

(ii) There exists $R > \rho$ such that

$$J|_{\partial Q_m} \leq p(m) \quad \text{with } p(m) \to 0 \text{ as } m \to \infty.$$ 

On the other hand, $\partial B_\rho \cap H^+$ and $\partial Q_m$ link (also see [10]). Then we can get a $PS$ sequence $\{u_n\}$ for $J$ at level $c_e$ with

$$c_e \geq \inf_{u \in \partial B_\rho \cap H^+} J(u) \geq \alpha > 0,$$
see Theorem 2.12 of [11]. From our Lemmas 3.1 and 3.2, there exists a subsequence of \( \{u_n\} \), still denoted by \( \{u_n\} \), such that \( u_n \rightarrow u \) strongly in \( H^1_0(\Omega) \) for some \( u \in H^1_0(\Omega) \). Hence, \( c_\varepsilon \) is a critical value of \( J \) and \( u \) is a nontrivial solution of problem (1.1).

Thus we complete the proof of Theorem 1.1.

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