Zeta functional equation on Jordan algebras of type II

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Abstract

Using the Jordan algebras methods, specially the properties of Peirce decomposition and the Frobenius transformation, we compute the coefficients of the zeta functional equation, in the case of Jordan algebras of type II. As particular cases of our result, we can cite the case of $V = M(n, \mathbb{R})$ studied by Gelbart [Mem. Amer. Math. Soc. 108 (1971)] and Godement and Jacquet [Zeta functions of simple algebras, Lecture Notes in Math., vol. 260, Springer-Verlag, Berlin, 1972], and the case of $V = \text{Herm}(3, \mathbb{O}_s)$ studied by Muro [Adv. Stud. Pure Math. 15 (1989) 429]. Let us also mention, that recently, Bopp and Rubenthaler have obtained a more general result on the zeta functional equation by using methods based on the algebraic properties of regular graded algebras which are in one-to-one correspondence with simple Jordan algebras [Local Zeta Functions Attached to the Minimal Spherical Series for a Class of Symmetric Spaces, IRMA, Strasbourg, 2003]. The method used in this paper is a direct application of specific properties of Jordan algebras of type II.

Keywords: Jordan

1. Introduction

Zeta distributions are used to define positive measure on some symmetric spaces [3], and their functional equation play a fundamental role in the theory of automorphic...
L-functions [1]. In our work, this functional equation is studied in the case of Jordan algebras. For that, we define a real simple Jordan algebra $V$ of rank $r$ and dimension $n$. We denote by $\text{tr}$ and $\Delta$ the trace and the determinant in $V$. If $\Omega$ is the set of invertible elements of $V$, then
$$\Omega = \{ x \in V : \Delta(x) \neq 0 \}.$$ 
For a function $f$ of the Schwartz space $\mathcal{S}(V)$ of $V$ and for a connected component $\Omega_j$ of $\Omega$, the zeta distribution is defined by
$$Z_j(f,s) = \int_{\Omega_j} f(x) \left| \Delta(x) \right|^s dx, \quad s \in \mathbb{C}.$$
This integral is convergent for $\text{Re}(s) > 0$, has an analytic continuation with respect to $s$, and verify the Sato–Shintani functional equation [6]:
$$Z(\hat{f}, s - n/r) = \sum_{j=1}^{l} C_{jk}(s) Z_k(f, -s),$$
where $\hat{f}$ is the Fourier transform of $f$ and $l$ is the number of connected components of $\Omega$.

The aim of this paper is to compute the coefficients $C_{jk}(s)$ of the Sato–Shintani equation if $V$ is a Jordan algebras of type II by using the Peirce decomposition properties and the Frobenius transformation.

In order to give the definition of a Jordan algebras of type II, let us at first recall the principle [4] of the classification of simple real Jordan algebras which are real forms of simple complex Jordan algebras, or simple complex Jordan algebras considered as a real ones. Each simple complex Jordan algebra $V^C$ has an Euclidean real form $W$, and all Euclidean real forms of $V^C$ are isomorphic to $W$. Up to isomorphism, the classification of simple complex Jordan algebras and their Euclidean real form are given by this table:

<table>
<thead>
<tr>
<th>$V^C$</th>
<th>$\mathbb{C} \times \mathbb{C}^{n+1}$</th>
<th>$\text{Sym}(r, \mathbb{C})$</th>
<th>$\text{M}(r, \mathbb{C})$</th>
<th>$\text{Herm}(r, \mathbb{H}) \otimes \mathbb{C}$</th>
<th>$\text{Herm}(3, \mathbb{O}) \otimes \mathbb{C}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W$</td>
<td>$\mathbb{R} \times \mathbb{R}^{n-1}$</td>
<td>$\text{Sym}(r, \mathbb{R})$</td>
<td>$\text{Herm}(r, \mathbb{C})$</td>
<td>$\text{Herm}(r, \mathbb{H})$</td>
<td>$\text{Herm}(3, \mathbb{O})$.</td>
</tr>
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</table>

Where $\text{Herm}(r, \mathbb{A})$, $\text{Sym}(r, \mathbb{A})$, and $\text{M}(r, \mathbb{A})$ are hermitian matrix, symmetric matrix, and matrix of order $r(> 2)$ with coefficients in $\mathbb{A}$. Let $V^C$ be a simple complex Jordan algebra and fix an Euclidean real form $W$ of $V^C$. Let $\alpha$ be an involutive automorphism of $W$ and define

$$W_+ = \{ x \in W : \alpha(x) = x \}; \quad W_- = \{ x \in W : \alpha(x) = -x \}.$$ 

Then $V = W_+ + i W_-$ is a real form of $V^C$ and every real form is obtained in this way. The Jordan algebras of type II are real forms of $V^C$ obtained if $W_+$ is simple and of rank equal to the rank $r$ of $W$. If $r > 2$, the corresponding involutions are given by

$$\alpha(x) = (a_0(x_{ij})), \quad \forall x \in \text{Herm}(r, \mathbb{A}),$$

where $a_0$ is an involution of $\mathbb{A}$. So, $a_0$ is no-trivial for $\mathbb{A} = \mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. If we pose,

$$\mathbb{A}_+ = \{ u \in \mathbb{A} : a_0(u) = u \}; \quad \mathbb{A}_- = \{ u \in \mathbb{A} : a_0(u) = -u \},$$

then $\mathbb{A}_+ = W_+$ and $\mathbb{A}_- = W_-$. If $\mathbb{A} = \mathbb{R}$, then

$$\mathbb{A}_+ = \{ x \in \mathbb{R} : x_0 = 0 \}; \quad \mathbb{A}_- = \{ x \in \mathbb{R} : x_0 = 0 \}.$$
and $\mathfrak{h}^+ = \mathfrak{h}_+ + i \mathfrak{h}_-$, then the Jordan algebras of type II are isomorphic to the Jordan algebras $\text{Herm}(r, \mathfrak{h}^+)$.
One can prove that $\text{Herm}(r, \mathbb{C})$ is isomorphic to the algebras $M(r, \mathbb{R})$ and $\text{Herm}(r, \mathbb{H})$ is isomorphic to the Jordan algebras of skew-symmetric matrix skew$(2r, \mathbb{R})$.

If $r = 2$, the involution of $W = \mathbb{R} \times \mathbb{R}^{n-1}$ is given by
\[
\alpha \left( (\lambda, x) \right) = (\lambda, I_{pq}(x)), \quad q = 1, \ldots, n-3 \ (p + q = n - 1),
\]
where $I_{pq}$ is the diagonal matrix with $p$ elements equal to 1 and $q$ elements equal to $-1$.

After the definition of the Jordan algebras of type II, let us give how our paper is organised. In the second section, we introduce the triangular subgroup and give useful integral formulae. In the third section, we determine the connected components of invertible elements $\Omega$ and in the last section, we compute the coefficients of the functional equation when $V$ is a Jordan algebra of type II.

### 2. Triangular group and integral formulae

Let $V$ be a simple Jordan algebra of type II. For $x, y \in V$, recall the following endomorphisms of $V$, denoted by $L(x), P(x)$, and $x \, \Box \, y \ldots$:
\[
L(x)y = xy, \quad P(x) = 2L(x)^2 - L(x^2),
\]
\[
x \, \Box \, y = L(xy) + \left[ L(x), L(y) \right],
\]
and denote $b(x, y)$ the bilinear form on $V$ defined by
\[
b(x, y) = \text{tr}(xy). \quad (2.1)
\]
If $c$ is an idempotent of $V$, the eigenvalues of $L(c)$ are $1, 1/2, 0$. So the Peirce decomposition corresponding to $c$ is written as
\[
V = V(c, 1) \oplus V(c, 0) \oplus V(c, 1/2), \quad (2.2)
\]
where $V(c, \alpha)$ is the eigen subspace of $V$ corresponding to $\alpha$. One can easily verify that $V(c, 1)$ and $V(c, 0)$ are Jordan subalgebras of $V$. For every element $z$ in $V(c, 1/2)$, the Frobenius transformation is defined by
\[
\tau(z) = \exp(2z \, \Box \, c).
\]
By using the Peirce decomposition, the Frobenius transformation of the element $z$ can be written as
\[
\begin{pmatrix}
y_1 \\
y_{1/2} \\
y_0
\end{pmatrix}
= \tau(z)
\begin{pmatrix}
x_1 \\
x_{1/2} \\
x_0
\end{pmatrix}, \quad (2.3)
\]
where
\[
\tau(z) = \begin{pmatrix}
I & 0 & 0 \\
2L(z) & I & 0 \\
2L(e - c)L(z)^2 & L(e - c)L(z) & I
\end{pmatrix}. \quad (2.4)
\]
If \( \{c_1, \ldots, c_r\} \) is a Jordan frame of \( V \), we can generalize the Peirce decomposition and the Frobenius transformation with respect to this frame. The Peirce decomposition is defined by

\[
V = \sum_{i=1}^{r} V_i \oplus \sum_{i<j}^{r} V_{ij},
\]

(2.5)

where

\[
V_i = V(c_i, 1) \quad \text{and} \quad V_{ij} = V(c_i, 1/2) \cap V(c_j, 1/2).
\]

If we consider the subalgebras \( V(j) = V(\sum_{i=1}^{j} c_i, 1) \) of \( V \) and denote \( \Delta_j(x) \) the determinant of the projection of \( x \) on \( V(j) \), we can generalize the Frobenius transformation by the following lemma [5].

**Lemma 2.1.** If \( x \) is an element of \( V \) such that \( \Delta_j(x) \neq 0 \) for \( j = 1, \ldots, r \), there exist nonzero elements \( a_j \in V(c_j, 1/2) \) and \( z(j) \in V(j) = \sum_{k=j+1}^{r} V_{jk} \) such that

\[
x = \tau(z)\tau(1) \cdots \tau(z^{(r-1)}) (a_1 + \cdots + a_r),
\]

and the elements \( a_j \) and \( z(j) \) being uniquely determined.

**Proof.** The proof is done by recurrence on the rank \( r \) of \( V \). We fix a Jordan frame \( \{c_1, \ldots, c_r\} \) of \( V \), and we consider the Peirce decomposition of \( x \) corresponding to \( c_1 \): \( x = x_1 + x_{1/2} + x_0 \). Since \( \Delta_1(x) \neq 0 \), we have that \( x_1 \neq 0 \) and we can pose \( z = x_1^{-1} x_{1/2} \). From Eq. (2.4) we deduce that

\[
x = \tau(z)(x_1 + x_0).
\]

Moreover, \( z \) is uniquely determined by the preceding equation. Or by recurrence hypothesis, we have

\[
x_0 = \tau(z^{(2)}) \cdots \tau(z^{(r-1)}) (a_2 + \cdots + a_r),
\]

and since \( \tau(z^{(j)})x_1 = x_1 \) for \( j = 2, \ldots, r \), we obtain that

\[
x = \tau(z^{(1)}) \cdots \tau(z^{(r-1)}) (a_1 + \cdots + a_r), \quad \text{where} \quad a_1 = x_1.
\]

Let \( u = \sum_{j=1}^{r} u_j c_j \oplus \sum_{i<j}^{r} u_{ij} \in V(c_i, 1/2) \cap V(c_j, 1/2) \) be the Peirce decomposition of an element \( u \) of \( V \), and

\[
R^+ = \{ u \in V : u_j > 0 \}.
\]

If \( u \) is an element of \( R^+ \), we define the triangular transformation \( t(u) \) as

\[
t(u) = P(b_1)\tau(u^{(1)}) P(b_2)\tau(u^{(2)}) \cdots P(b_{r-1})\tau(u^{(r-1)}) P(b_r),
\]
where
\[ b_j = c_1 + \cdots + c_{j-1} + u_j c_j + c_{j+1} + \cdots + c_r, \quad u^{(j)} = \sum_{k=j+1}^r \tilde{u}_{jk}, \tilde{u}_{jk} = u_j u_{jk}. \]

The triangular group corresponding to \( R^+ \) is defined by
\[ T = \{ t(u) : u \in R^+ \}. \]

For our calculus, we will use the \( T^- \) orbits in \( \Omega \) of the elements \( \varepsilon \) of the form
\[ \varepsilon = \sum_{i=1}^{r} \varepsilon_i c_i, \quad \text{with } \varepsilon_i = \pm 1. \]

These orbits will be denoted by \( \Omega_{\varepsilon} \) and there are \( 2^r \) orbits \( \Omega_{\varepsilon} \) in \( \Omega \).

**Proposition 2.2.** The mapping
\[ \phi : R^+ \rightarrow \Omega_{\varepsilon}, \quad u \rightarrow t(u)\varepsilon, \]
is one-to-one, and if
\[ x = \sum_{j=1}^{r} x_j + \sum_{j<k} x_{jk} \]
is the decomposition of \( x \), then
\[ x_j = \varepsilon_j u_j^2 + \frac{1}{2} \sum_{k=1}^{j-1} b(u_{jk}, u_{jk}), \quad x_{jk} = \varepsilon_j u_{jk} + 2 \sum_{l=1}^{j-1} u_{lj} u_{jk}, \]
where \( b(\cdot, \cdot) \) is the bilinear form defined by (2.1).

**Proof.** By induction on the rank of \( V \), we suppose that the proposition is true in \( V(c_1, 0) \):
\[ y = P(b_2) \tau(u^{(2)}) \cdots P(b_{r-1}) \tau(u^{(r-1)}) P(b_r)(\varepsilon_2 c_2 + \cdots + \varepsilon_r c_r). \]

Then, from Lemma 2.1, there exists an element \( u^{(1)} \) of \( V(c_1, 1/2) \) such that
\[ x = P(b_1) \tau(u^{(1)})(\varepsilon_1 c_1 + y). \]

By using Eq. (2.4), we obtain that
\[ \tau(u^{(1)})(\varepsilon_1 c_1 + y) = \varepsilon_1 c_1 + u^{(1)} + L(e-c)(u^{(1)})^2 + y. \]

From [4, Lemma (2.8.2)], we obtain that
\[ a^2 = b(a, a)(c_i + c_j), \quad \text{if } a \in V_{ij}. \]

Using this last relation and [2, Proposition IV.2.4], we find that
\[ L(e-c)(u^{(1)})^2 = \frac{1}{2} \sum_{k=2}^{r} b(u_{1k}, u_{1k}) c_1 + 2 \sum_{2 \leq j < k} u_{1j} u_{1k}, \]
and finally by the recurrence hypothesis, we find that
\[ x = \varepsilon_1 u^2 c_1 + \sum_{k=2}^r u_1 u_{1k} + \frac{1}{2} \sum_{k=2}^r b(u_{1k}, u_{1k}) c_1 + \frac{1}{2} \sum_{2 \leq j < k} u_{1j} u_{1k} + y. \]

**Corollary 2.3.** If \( x = t(u) \varepsilon \) is the mapping defined in Proposition 2.2, then
\[ \int_{\Omega_{\varepsilon}} f(x) \, dx = \int_{\mathbb{R}^+} f(t(u) \varepsilon) J(u) \, du, \quad \text{with } J(u) = 2^r \prod_{i=1}^r u_i^{d(r-i)+1}, \] (2.6)
where \( d \) is the dimension of \( \mathbb{A} \), \( \mathbb{A} = \mathbb{C}, \mathbb{H}, \text{or } \mathbb{O} \).

**Proof.** The matrix of the mapping \( \phi \) defined by Proposition 2.2 is triangular and we can compute easily the jacobian by using Proposition 2.2. So, we obtain that
\[ J(u) = 2^r \prod_{i=1}^r u_i^{d(r-i)+1}. \]

**Proposition 2.4.** If
\[ \Omega' = \bigcup_{\varepsilon} \Omega_{\varepsilon}, \quad \text{then } \Omega' = \{ x \in V : \Delta_j(x) \neq 0 \}. \]

**Proof.** At first, we suppose that \( x = t(u) \varepsilon \) and we show that \( \Delta_j(x) \neq 0 \). For that, we use [2, Proposition VI.3.10]:
\[ \Delta_j(x) = (u_1 \ldots u_j)^2 \Delta_j(\varepsilon) = u_1^2 \ldots u_j^2 \varepsilon_1 \ldots \varepsilon_j, \]
and we remark that \( \forall j, \Delta_j(x) \neq 0 \) if \( u_j > 0 \). Conversely, we suppose that \( \Delta_j(x) \neq 0 \) for \( k = 1, \ldots, r \); and we show that there exists \( u \) in \( V \) such that \( x = t(u) \varepsilon \). For that, we remark that in Lemma 2.1, if we pose
\[ a_j = \varepsilon_j u_j^2, \quad u_j > 0, \quad z_{ij} = u_{ij}, \]
we obtain that \( x = t(u) \varepsilon \).

**Corollary 2.5.** The set \( \Omega' \) is dense in \( \Omega \).

### 3. Connected components of the set of invertible elements

Let \( V \) be a real simple Jordan algebra of type II. If \( \alpha \) is an involutive automorphism of \( V \) with the decomposition
\[ V_+ = \{ x \in V : \alpha(x) = x \} \quad \text{and} \quad V_- = \{ x \in V : \alpha(x) = -x \}, \]
then \( W = V_+ + i V_- \) is an euclidian Jordan algebra. If \( a \) and \( b \) are two primitive idempotents of \( V_+ \), we pose
\[ W^{ab} = W(a, 1) + W(b, 1) + W(a, 1/2) \cap W(b, 1/2) \]
and
\[ V_{+}^{ab} = W_{+}^{ab} \cap V_{+}; \quad V_{-}^{ab} = W_{-}^{ab} \cap V_{-}. \]

From [4, Paragraph 2.4.3], we obtain that
\[ W_{ab} = V_{+}^{ab} + i V_{-}^{ab}. \]

From [2, Proposition IV1.2], we can choose \( \xi \in V_{+}^{ab} \) and \( \eta \in V_{-}^{ab} \) such that \( \|\xi\|^2 = \|i\eta\|^2 = \sqrt{2} \), where \( \|\cdot\| \) is the norm on \( W \) defined by \( \|x^2\| = b(x, x). \)

**Lemma 3.1.** The mapping \( \tilde{H} \) from \( \text{Herm}(2, \mathbb{C}) \) to \( W(a+b, 1) \) defined by
\[ \begin{pmatrix} \sigma & \gamma + i\delta \\ \gamma - i\delta & \beta \end{pmatrix} \mapsto \sigma a + \beta b + \gamma \xi + i\delta \eta \]
is an homomorphism of Jordan algebras.

**Proof.** From [2, Proposition IV.1.2], we obtain that
\[ \xi^2 = \frac{\|\xi\|^2}{2}(a + b), \quad (i\eta)^2 = \frac{\|i\eta\|^2}{2}(a + b), \quad \xi\eta = b(\xi, \eta)(a + b). \]

From the orthogonality of the decomposition \( V = V_{+} + V_{-} \) we have \( \xi\eta = 0 \). Then by simple calculus, we verify that if
\[ x = \begin{pmatrix} \sigma & \gamma + i\delta \\ \gamma - i\delta & \beta \end{pmatrix}, \quad \text{then} \quad \tilde{H}(x^2) = (\tilde{H}(x))^2, \]
which means that \( \tilde{H} \) is a homomorphism of Jordan algebras. \( \Box \)

Let \( H \) be the holomorphic homomorphism from \( M(2, \mathbb{C}) \) to \( V^C = V \oplus iV \) defined by
\[ \begin{pmatrix} \sigma & \gamma + \delta \\ \gamma - \delta & \beta \end{pmatrix} \mapsto \sigma a + \beta b + \gamma \xi + \delta \eta. \]

Its restriction to \( \text{Herm}(2, \mathbb{C}) \) coincides with \( \tilde{H} \), whereas its restriction to \( M(2, \mathbb{R}) \) gives a homomorphism from \( M(2, \mathbb{R}) \) into \( V \).

If we consider the elements
\[ \varepsilon = \sum_{i=1}^{r} \varepsilon_i c_i, \quad \varepsilon' = \sum_{i=1}^{r} \varepsilon'_i c_i \quad \text{with} \quad \varepsilon_i, \varepsilon'_i = \pm 1, \]
such that
\[ \varepsilon_i = \varepsilon'_i \quad \text{if} \quad i \neq k, j \quad \text{and} \quad \varepsilon_k = -\varepsilon'_k, \quad \varepsilon_j = -\varepsilon'_j, \]
then from Lemma 3.1, we obtain the following corollary.

**Corollary 3.2.** The elements \( \varepsilon \) and \( \varepsilon' \) belong to the same connected component of \( \Omega \).
Proof. If we denote
\[ f = \sum_{i \neq k,j} c_i, \]
and pose \( a = c_k, b = c_j \) in Lemma 3.1, we obtain a continuous mapping \( \Gamma \) from \( GL(2, \mathbb{R}) \) to \( V \) defined by
\[ \Gamma(x) = f + H(x). \]
Since the group \( GL(2, \mathbb{R}) \) have two connected components:
\[
\begin{align*}
GL(2, \mathbb{R})^+ &= \{ x \in GL(2, \mathbb{R}) : \text{Det}(x) > 0 \}, \\
GL(2, \mathbb{R})^- &= \{ x \in GL(2, \mathbb{R}) : \text{Det}(x) < 0 \}.
\end{align*}
\]
From the continuity of \( \Gamma \), we deduce that the images by \( \Gamma \) of \( GL(2, \mathbb{R})^+ \) and \( GL(2, \mathbb{R})^- \) are connected and we remark that
\[
\varepsilon = \Gamma\left( \begin{pmatrix} c_k & 0 \\ 0 & c_j \end{pmatrix} \right) \quad \text{and} \quad \varepsilon' = \Gamma\left( \begin{pmatrix} c'_k & 0 \\ 0 & c'_j \end{pmatrix} \right).
\]

Proposition 3.3. The set
\[
\Omega_1 = \{ x \in V : \Delta(x) > 0 \}
\]
is connected.

Proof. Let \( \Omega_e \) be the connected component of \( \Omega \) containing the unit element \( e \). From Corollary 3.2 and from the connectedness of \( \Omega_e \), we deduce that \( \Omega_e \subset \Omega_e \) if \( \Delta(e) = 1 \). Let \( x \) be an element in \( \Omega_1 \), since \( \Omega' = \bigcup \Omega_e \) is dense in \( \Omega \), each neighborhood of \( x \) have a nonempty intersection with one \( \Omega_e \) and for this \( \varepsilon, \Delta(\varepsilon) = 1 \). So, each neighborhood of \( x \) have a nonempty intersection with \( \Omega_1 \), in consequence \( \Omega_1 = \Omega_e \). □

In the same way, we show that the set
\[
\Omega_2 = \{ x \in V : \Delta(x) < 0 \}
\]
is connected. So, we can conclude that if \( V \) is a Jordan algebra of type II, then \( \Omega \) have two connected components defined by
\[
\Omega_1 = \{ x \in V : \Delta(x) > 0 \} \quad \text{and} \quad \Omega_2 = \{ x \in V : \Delta(x) < 0 \}
\]

4. Coefficients of the functional equation

For a function \( f \) of the Schwartz space \( S(V) \) of \( V \) and for a connected component \( \Omega_j \) of \( \Omega \), the zeta distribution is defined by
\[
Z_j(f, s) = \int_{\Omega_j} f(x) |\Delta(x)|^s dx, \quad s \in \mathbb{C}, \quad j = 1, 2.
\]
(4.1)
This integral is convergent for $\text{Re}(s) > 0$, has an analytic continuation with respect to $s$ and verify the Sato–Shintani functional equation [6]. Since $\Omega$ has two components, the functional equations are:

$$Z_1(\hat{f}, s - n/r) = C_{11}(s)Z_1(f, -s) + C_{12}(s)Z_2(f, -s),$$

$$Z_2(\hat{f}, s - n/r) = C_{21}(s)Z_1(f, -s) + C_{22}(s)Z_2(f, -s).$$

To compute explicitly the coefficients $C_{ij}$, we fix a Jordan frame \{\(c_1, \ldots, c_r\)} and denote

$$E = \left\{ \sum_{i=1}^r \varepsilon_i c_i : \varepsilon_i = \pm 1 \right\}, \quad E_1 = E \cap \Omega_1, \quad E_2 = E \cap \Omega_2. \quad (4.2)$$

From Proposition 2.4 and Corollary 2.5 we deduce that if $Z_{\varepsilon}(f, s) = \int_{\Omega_{\varepsilon}} f(x) |\Delta(x)|^s dx$, then

$$Z_1(f, s) = \sum_{\varepsilon \in E_1} Z_{\varepsilon}(f, s) \quad \text{and} \quad Z_2(f, s) = \sum_{\varepsilon \in E_2} Z_{\varepsilon}(f, s). \quad (4.3)$$

Let

$$\hat{f}(\xi) = \int_{\mathbb{V}} e^{-ib(\xi, x)} f(x) dx$$

be the Fourier transform of $f$, we want to calculate

$$Z_{\varepsilon}(\hat{f}, s - n/r) = \int_{\Omega_{\varepsilon}} \hat{f}(x) |\Delta(x)|^{s-n/r} dx.$$

From Corollary 2.3, we obtain that

$$Z_{\varepsilon}(\hat{f}, s - n/r) = \int_{R^+} \left( \int_{\mathbb{V}} e^{-ib(t(u)\varepsilon, \xi)} f(\xi) d\xi \right) \alpha(u)^{2(s-n/r)} J(u) du, \quad (4.4)$$

where $J(u)$ is given by Eq. (2.6) and $\alpha(u) = u_1 \ldots u_r$. If $u = u_1 + u_{1/2} + u_0$ and $\xi_1 + \xi_{1/2} + \xi_0$ are the Peirce decompositions of $u$ and $\xi$, then

$$b(t(u)\varepsilon, \xi) = \varepsilon_1 u_1^2 \xi_1 + \varepsilon_1 u_1 b(u_{1/2}, \xi_{1/2}) + \varepsilon_1 b(u_{1/2}^2, \xi_0) + (t_0(u_0)\varepsilon_0, \xi_0). \quad (4.5)$$

where $t_0(u_0)$ is the triangular matrix associated to the element $u_0$ of the Jordan algebra $\mathbb{V}_0 = V(c_1, 0)$. Before we use the Fubini theorem for inversion of order integration, we introduce a factor of convergence in Eq. (4.4):

$$Z_{\varepsilon}(\hat{f}, s - n/r) = \lim_{\delta \to 0} \int_{R^+} \int_{\mathbb{V}} e^{-\delta(u_1^2 + |u_{1/2}^2| + |u_0^2|) - ib(t(u)\varepsilon, \xi)} f(\xi) \times \alpha(u)^{2(s-n/r)} f(u) d\xi du,$$
where \( \| \cdot \| \) is the euclidean norm on \( V(c_1, 1/2) \) and \( V(c_1, 0) \), respectively. We pose

\[
F_{\varepsilon, \delta}(\xi, s) = \int_{\mathbb{R}^+} e^{-\delta(u_1^2 + \|u_2\| + \|u_0^2\|) - ib(u)\alpha(u)\varepsilon} f(u) du
\]

(4.6)

and we compute by induction on the rank \( r \) of \( V \), the limit

\[
\lim_{\delta \to 0} F_{\varepsilon, \delta}(\xi, s).
\]

For that we will need the next results.

**Lemma 4.1.** Let \( x = x_1 + x_{1/2} + x_0 \) be the Peirce decomposition of \( x \) corresponding to an idempotent \( c \). If \( x_0 \) is invertible in \( V(c, 0) \) with the inverse \( x_0^{-1} \), then

\[
\Delta(x) = \Delta_0(x_0) \left( x_1 - b(x_0^{-1}x_{1/2}, x_{1/2}) \right),
\]

where \( \Delta_0 \) is the determinant of the Jordan algebra \( V(c, 0) \).

**Proof.** We pose \( d = d - c \), if \( z \) belongs to \( V(c, 1/2) \), then \( z \in V(d, 1/2) \). If \( \tau(z) = \exp(2z \Box d) \), then \( \Delta(\tau(z)x) = \Delta(x) \). From Eq. (2.4), we obtain

\[
\tau(z)x = x_0 + 2L(z)x_0 + x_{1/2} + 2L(c)[2L(z)x_0 + L(z)x_{1/2}] + x_1c.
\]

If \( x \) and \( y \) belong to \( V(c, 1/2) \), then \( c(xy) = \frac{1}{2}b(x_0^{-1}x_{1/2}, x_{1/2})c \), so \( 2L(c)x_1/2 = b(z, x_{1/2})c \) and \( 2L(c)(zx_0) = b(z, zx_0)c \). Let \( \Phi \) be the mapping from \( V(c, 0) \) to \( V(c, 1/2) \) defined by

\[
\Phi(u)z = 2uz.
\]

(4.7)

From [2, Proposition IV.4.2], we know that \( \Phi \) is a representation of the Jordan algebra \( V(c, 0) \) on \( V(c, 1/2) \). If we choose \( z \) such that

\[
z = -\Phi(x_0^{-1})x_{1/2} = -2(x_0^{-1})x_{1/2},
\]

then

\[
b(z, x_{1/2}) = -b(\Phi(x_0^{-1})x_{1/2}, x_{1/2}),
\]

and

\[
b(z, zx_0) = \frac{1}{2}b(z, \Phi(x_0)z) = \frac{1}{2}b(\Phi(x_0^{-1})x_{1/2}, x_{1/2}).
\]

Consequently, we find that

\[
\tau(z)x = x_0 + x_1c - \frac{1}{2}b(\Phi(x_0^{-1})x_{1/2}, x_{1/2})c
\]

\[
= x_0 + \left( x_1 - \frac{1}{2}b(\Phi(x_0^{-1})x_{1/2}, x_{1/2}) \right)c.
\]

From the last equation, we deduce that

\[
\Delta(x) = \Delta_0(x_0) \left( x_1 - b(x_0^{-1}x_{1/2}, x_{1/2}) \right).
\]

\( \square \)
Lemma 4.2. Let $A$ be a real invertible symmetric matrix of order $n$ and signature $(p, q)$. If $\alpha$ is a real number, then
\[
\lim_{\delta \to 0} \text{Det}(\delta I + iA)^\alpha = |\text{Det}(A)|^\alpha e^{i\pi\alpha(p - q)}.
\]

Proof. Since the matrix determinant is invariant under the change of basis, we can choose a basis in which $\delta I + iA = \text{diag}(\delta + ia_1, \ldots, \delta + ia_n)$, then
\[
\text{Det}(\delta I + iA)^\alpha = (\delta + ia_1)^\alpha \ldots (\delta + ia_n)^\alpha.
\]
If $\lambda$ is a real number, then
\[
\lim_{\delta \to 0} (\delta + i\lambda)^\alpha = |\lambda|^\alpha e^{i\pi\alpha \text{sgn}(\lambda)},
\]
where $\text{sgn}(\lambda)$ is the sign of $\lambda$. From the precedent equation, we deduce that
\[
\lim_{\delta \to 0} \text{Det}(\delta I + iA)^\alpha = |\text{Det}(A)|^\alpha e^{i\pi\alpha(p - q)}.
\]

Lemma 4.3. Let $B$ be a symmetric complex matrix of order $m$ with a positive-definite real part. If $v$ is a vector of $\mathbb{C}^m$, then
\[
\int_{\mathbb{R}^m} e^{-\frac{1}{2}((Bx,x)+2(v,x))} dx = (2\pi)^{\frac{m}{2}} |\text{Det}(B)|^{-\frac{1}{2}} e^{-\frac{1}{2}(B^{-1}v,v)}.
\]

Proposition 4.4. If $\text{Supp}(f) \subset \Omega_\epsilon$ and $\eta \in E$, then
\[
Z_\epsilon(\hat{f}, s - n/r) = \gamma(s) e^{-i\frac{\pi}{2} \sum_{j=1}^r \epsilon_j \eta_j (s - \frac{d}{2}(j - 1))} \int_{\Omega_\epsilon} f(\xi) |\Delta(\xi)|^{-s} d\xi,
\]
where
\[
\gamma(s) = (2\pi)^{\frac{d}{2}(r-1)} \prod_{j=1}^r \Gamma \left( s - \frac{d}{2}(j - 1) \right).
\]
and \( \Delta_0 \) and \( F_{0,\delta}^0 \) are, respectively, the determinant and the function \( F_{r,\delta} \) in \( V(c_1, o) \). Then we integrate Eq. (4.6) with respect to \( u_{1/2} \) and \( u_1 \). Let \( A(\varepsilon_1, \xi_0) \) be the symmetric endomorphism on \( V(c_1, 1/2) \) defined by
\[
A(\varepsilon_1, \xi_0) u|v = b(\Phi(\varepsilon_1 \xi_0) u, v),
\]
where \( \Phi \) is defined by Eq. (4.7). From Eqs. (4.5)–(4.7), the integration with respect to \( u_{1/2} \) become
\[
\int e^{-\frac{1}{2}[\delta I+i(A(\varepsilon_1, \xi_0) u_{1/2}|u_{1/2})+2i\varepsilon_1 u_1 b(u_1/2, \xi_1/2)]} du_{1/2}.
\]
If we pose
\[
v = 2i\varepsilon_1 u_{1/2}, \quad \text{and} \quad B = 2\delta I + i A(\varepsilon_1, \xi_0).
\]
From the Lemma 4.3 we obtain
\[
\int e^{-\frac{1}{2}[(Bx, x) + 2(v, x)]} dx = (2\pi)^\frac{1}{2} (\gamma - 1)d \text{ Det}(2\delta I + i A(\varepsilon_1, \xi_0))^{-1/2}
\times e^{-\frac{1}{2}(2\delta I+iA(\varepsilon_1, \xi_0))^{-1}v, v},
\]
By the integration of Eq. (4.6) with respect to \( u_1 \), we obtain
\[
\int_0^{+\infty} e^{-\delta + i\varepsilon_1 \xi_1 + \frac{1}{2} (2\delta I + i A(\varepsilon_1, \xi_0))^{-1} \xi_1/2} \xi_1/2 u_1^{2\gamma - 1} du_1 = \int_0^{+\infty} e^{-\lambda(\xi, \varepsilon_1, \delta)} u_1^{2\gamma - 1} du_1 = \frac{1}{2} \Gamma(\frac{\lambda}{2}, \frac{\delta}{2}, \frac{\varepsilon_1}{2}),
\]
where
\[
\lambda(\xi, \varepsilon_1, \delta) = \delta + i\varepsilon_1 \xi_1 + \frac{1}{2} (2\delta I + i A(\varepsilon_1, \xi_0))^{-1} \xi_1/2 \xi_1/2 = \delta + i\varepsilon_1 \xi_1 + b((\delta e_0 + i\varepsilon_1 \xi_0)^{-1} \xi_1/2, \xi_1/2).
\]
Considering the decomposition (2.5), each subspace \( V_{ij} \) of a Jordan algebra \( V \) of type II is isomorphic to \( \mathbb{A} \). And if \( d \) is the dimension of \( \mathbb{A} \), the dimension of \( \mathbb{A}_+ \) and \( \mathbb{A}_- \) is \( d/2 \). So, the signature of \( b(\cdot, \cdot) \) on \( V_{ij} \) is \( (d/2, d/2) \). Since we have
\[
V(c_1, 1/2) = V_{12} \oplus V_{13} \oplus \cdots \oplus V_{1r},
\]
we conclude that the signature of the bilinear form \( b(\cdot, \cdot) \) on \( V(c_1, 1/2) \) is \( (p, p) \) with \( p = \frac{d}{2}(r - 1) \). So, the matrix \( A(\varepsilon_1, \xi_0) \) is of signature \( (p, p) \) and from Lemma 4.2, we have that
\[
\lim_{\delta \to 0} \text{ Det}(2\delta I + i A(\xi_0, \varepsilon_1))^{-1/2} = |\text{ Det } A(\xi_0, \varepsilon_1)|^{-1/2}.
\]
If \( \phi \) is a representation of a Jordan algebra of rank \( r \) in a vector space of dimension \( N \), then from [2, Proposition IV.4.2] we have
\[
\text{ Det}(\phi(x)) = \Delta(x)^{N/r}.
\]
When we use this equality and the definition of $A(\xi_0, \varepsilon_1)$, we obtain that

$$|\text{Det}(A(\xi_0, \varepsilon_1))|^{-1/2} = |\Delta_0(\xi_0)|^{d/2},$$

and then

$$\lim_{\delta \to 0} \lambda(\xi, \varepsilon_1, \delta) = \lim_{\delta \to 0} \left[ \delta + i \varepsilon_1 \xi_1 + b \left( \delta \varepsilon_0 + i \varepsilon_1 \xi_0^{-1} \xi_1/2, \xi_1/2 \right) \right]$$

$$= i \varepsilon_1 \left[ \xi_1 - b \left( \xi_0^{-1} \xi_1/2, \xi_1/2 \right) \right] = i \varepsilon_1 \frac{\Delta(\xi)}{\Delta(\xi_0)}.$$  

The last equality being obtained by using Lemma 4.1. Finally, we obtain that

$$\lim_{\delta \to 0} \lambda(\xi, \varepsilon_1, \delta)^{-z} = e^{-\frac{\Delta(\xi)}{\Delta(\xi_0)}}.$$  

After the integration of Eq. (4.6) with respect to $u_{1/2}$ and $u_1$, we obtain

$$\lim_{\delta \to 0} F_{\varepsilon_1, \delta}(\xi, s) = (2\pi)^{\frac{d}{2}(r-1)} \Gamma(s) e^{-i \frac{\Delta(\xi)}{\Delta(\xi_0)}} |\Delta(\xi)|^{\frac{d}{2}}$$

$$\times \lim_{\delta \to 0} \int_{V'(c, 0)} e^{-\delta |u|/2 - i b(t_0, u_0) \varepsilon_0}$$

$$\times \left[ 2^{r-1} \prod_{j=2}^{r} u_j^{2(s-n+j)} \prod_{j=2}^{r} u_j^{d(r-j)+1} \right] du_0.$$  

However,

$$2^{r-1} \prod_{j=2}^{r} u_j^{2(s-d/2-n+j)} \prod_{j=2}^{r} u_j^{d(r-j)+1} = 2^{r-1} \prod_{i=1}^{r-1} u_{i+1}^{2(s-n+j)} \prod_{i=1}^{r-1} u_{i+1}^{d(r-i)+1}$$

$$= a_0(u_0)^{2(s-d/2-n+j)} F(u_0).$$

So

$$\lim_{\delta \to 0} F_{\varepsilon_1, \delta}(\xi, s) = (2\pi)^{\frac{d}{2}(r-1)} \Gamma(s) e^{-i \frac{\Delta(\xi)}{\Delta(\xi_0)}} |\Delta(\xi)|^{\frac{d}{2}}$$

$$\times \lim_{\delta \to 0} \int_{V'(c, 0)} e^{-\delta |u|/2 - i b(t_0, u_0) \varepsilon_0} a_0(u_0)^{2(s-d/2-n+j)} F(u_0) du_0$$

and we can write again

$$\lim_{\delta \to 0} F_{\varepsilon_1, \delta}(\xi, s) = (2\pi)^{\frac{d}{2}(r-1)} \Gamma(s) e^{-i \frac{\Delta(\xi)}{\Delta(\xi_0)}} |\Delta(\xi)|^{\frac{d}{2}}$$

$$\times \lim_{\delta \to 0} F_{\varepsilon_0, \delta}(\xi_0, s - d/2).$$

If we denote $F_\varepsilon(\xi, s) = \lim_{\delta \to 0} F_{\varepsilon_1, \delta}(\xi, s)$, we can use the recurrence hypothesis and obtain

$$F_\varepsilon(\xi, s) = \gamma(s) e^{-i \frac{(\sum_{j=1}^{r} \xi_j)}{(s-d/2}(j-1))} |\Delta(\xi)|^{-z},$$

where $\gamma(s)$ is defined by (4.8).
If we pose
\[ C_\varepsilon(\xi, s) = \gamma(s) e^{-i \frac{\pi}{2} \left( \sum_{j=1}^r \varepsilon_j \xi_j (s - \frac{d}{2} (j - 1)) \right)} \], (4.10)
and if \( \text{Supp}(f) \subset \Omega_\eta \), then we obtain
\[ Z_\varepsilon(\hat{f}, s - n/r) = C_\varepsilon(\xi, s) \int_{\Omega_\eta} f(\xi) |\Delta(\xi)|^{-s} d\xi. \]

The restriction of \( Z_\varepsilon(\hat{f}, s - n/r) \) on \( \Omega_1 \) and \( \Omega_2 \) are, respectively, equal to \( C_{\varepsilon_1} |\Delta(\xi)|^{-s} \) and \( C_{\varepsilon_2} |\Delta(\xi)|^{-s} \). So, from the invariance of the zeta distributions, we can choose, respectively, \( \xi = e \) and \( \xi = \bar{e} = -e_1 + e_2 + \cdots + e_r \) for computing \( C_{\varepsilon_1} |\Delta(\xi)|^{-s} \) and \( C_{\varepsilon_2} |\Delta(\xi)|^{-s} \), we obtain
\[ C_{11} = \sum_{\varepsilon \in E_1} C_\varepsilon(\xi, s), \quad C_{12} = \sum_{\varepsilon \in E_2} C_\varepsilon(\xi, s), \]
\[ C_{21} = \sum_{\varepsilon \in E_2} C_\varepsilon(\xi, s), \quad C_{22} = \sum_{\varepsilon \in E_2} C_\varepsilon(\xi, s), \]
(4.11)
where \( E_1 \) and \( E_2 \) are defined by Eq. (4.2). From Eqs. (4.3), (4.10), and (4.11), we deduce that \( C_{11} = C_{22} \) and \( C_{12} = C_{21} \).

If \( \varepsilon = \varepsilon_1 c_1 + \cdots + \varepsilon_r c_r \) is an element of \( E \) and if \( j \) is the number of negative \( \varepsilon_i, i = 1, \ldots, r \), then \( \sum_{i=1}^r \varepsilon_i = r - 2j \). Moreover,
\[ \varepsilon \in E_1 \iff j = 2k, \quad \varepsilon \in E_2 \iff j = 2k + 1, \quad \text{where } k \in \mathbb{Z}. \] (4.12)

If \( \alpha \in \mathbb{N} \), we pose
\[ S_\alpha = \sum_{\varepsilon \in E} e^{i \frac{\pi}{2} \alpha (\sum_{j=1}^r \varepsilon_j - r)} C_\varepsilon(\xi, s). \]
This sum depends only of the parity of \( \alpha : S_{\alpha+2} = S_\alpha \), and from the previous equations, we deduce that
\[ S_\alpha = \sum_{\varepsilon \in E_1} C_\varepsilon(\xi, s) + e^{i \pi \alpha} \sum_{\varepsilon \in E_2} C_\varepsilon(\xi, s). \] (4.13)

From Eqs. (4.11) and (4.13), we deduce that
\[ S_0 = C_{11}(s) + C_{21}(s) \quad \text{and} \quad S_1 = C_{11}(s) - C_{21}(s). \] (4.14)

\textbf{Lemma 4.5.} If \( \alpha \in \mathbb{N} \), then
\[ S_\alpha = \gamma(s) \prod_{j=1}^r \left[ e^{i \frac{\pi}{2} (s - \frac{d}{4} (j - 1))} + e^{-i \pi \alpha} e^{i \frac{\pi}{2} (s - \frac{d}{4} (j - 1))} \right], \]
where \( \gamma(s) \) is given by Eq. (4.8).

\textbf{Proof.} In fact
\[ S_\alpha = \gamma(s) e^{-i \frac{\pi}{2} \alpha} \left[ \sum_{\varepsilon \in E} e^{i \frac{\pi}{2} (\sum_{j=1}^r \varepsilon_j (\alpha + s - \frac{d}{4} (j - 1)))} \right] = \gamma(s) e^{-i \frac{\pi}{2} \alpha} \left[ \sum_{\varepsilon = \pm 1} \prod_{j=1}^r a_{\varepsilon j}^{\varepsilon j} \right]. \]
If we use the next equality which can be easily shown by induction:

\[
\sum_{\varepsilon=\pm 1} \prod_{j=1}^{r} a_j^{\varepsilon} = \prod_{j=1}^{r} (a_j + 1/a_j),
\]

we find the result. \(\square\)

By using Lemma 4.5, we remark that

\[
S_0 = 2^r \gamma(s) \prod_{j=1}^{r} \cos \frac{\pi}{2} \left( s - \frac{d}{2}(j-1) \right),
\]

\[
S_1 = (2i)^r \gamma(s) \prod_{j=1}^{r} \sin \frac{\pi}{2} \left( s - \frac{d}{2}(j-1) \right),
\]

so, we can write the main result.

**Theorem 4.6.** If \(V\) is a real Jordan algebra of type II, then the coefficients of the zeta functional equation are given by

\[
C_{11} = C_{22} = 2^{r-1} \gamma(s) \left[ \prod_{j=1}^{r} \cos \frac{\pi}{2} \left( s - \frac{d}{2}(j-1) \right) \right. \\
+ i^r \prod_{j=1}^{r} \sin \frac{\pi}{2} \left( s - \frac{d}{2}(j-1) \right) \left. \right],
\]

\[
C_{12} = C_{21} = 2^{r-1} \gamma(s) \left[ \prod_{j=1}^{r} \cos \frac{\pi}{2} \left( s - \frac{d}{2}(j-1) \right) \right. \\
- i^r \prod_{j=1}^{r} \sin \frac{\pi}{2} \left( s - \frac{d}{2}(j-1) \right) \left. \right].
\]

**Corollary 4.7.** We pose

\[
Z^\chi(f, s) = \frac{1}{i^2} \int f(x) |\Delta(x)|^{\frac{1}{2}} \left[ \text{sign} (\Delta(x)) \right]^{\chi} dx, \quad \chi \in \{0, 1\},
\]

and denote \(Z^0(f, s)\) and \(Z^1(f, s)\), respectively, if \(\chi = 0\) and \(\chi = 1\). Then we obtain

\[
Z^\chi(\hat{f}, s - n/r) = C^\chi(s) Z^\chi(f, -, s),
\]

where

\[
C^0(s) = 2^r \gamma(s) \prod_{j=1}^{r} \cos \frac{\pi}{2} \left( s - \frac{d}{2}(j-1) \right),
\]

\[
C^1(s) = (2i)^r \gamma(s) \prod_{j=1}^{r} \sin \frac{\pi}{2} \left( s - \frac{d}{2}(j-1) \right).
\]
Proof. We have only to remark that

\[ Z^T(f, s) = Z_1(f, s) + (-1)^T Z_2(f, s). \]

\[ \square \]

References