JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 9, 303-314 (1964)

Notes on Linear Inequalities, I: The Intersection of the Nonnegative Orthant with Complementary Orthogonal Subspaces*

ADI BEN-ISRAEL

Technion-Israel Institute of Technology, Haifa, Israel
and
Northwestern University, Evanston, Illinois

The intersections of the nonnegative orthant in E^n with pairs of complementary orthogonal subspaces are investigated. Applications to linear inequalities and linear programming are then made by using Fredholm's alternative theorem.

INTRODUCTION

The theory of linear inequalities, the classic reference on which is [1],[1] is closely related to the theory of linear equations.[2] This relation is the subject of the present paper. We show that the main results in the theory of linear inequalities (in finite dimensional vector spaces over arbitrary ordered fields) follow from two basic facts: (a) Theorem 4 below which is an elementary property of the intersections of the nonnegative orthant with pairs of complementary orthogonal subspaces,[3] (b) Fredholm's "alternative theorem."[4] Thus new proofs, valid in arbitrary ordered fields, are given for some well-known theorems in [10-12] and [13] (corollaries 7, 8, and 9 below) and for the duality theorem of linear programming[5] (remark 10 below).

0. NOTATIONS. The notations used in this paper are:

$\{x : f(x)\}$ the *set of elements x for which f(x) is true*
$\{x\}$ the *set consisting of x*

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* This research was partly supported by the Office of Naval Research, contract Nonr-1228(10), project NR 047-021, and by the National Science Foundation, project G-14102. Reproduction of this paper in whole or in part is permitted for any purpose of the United States Government.

[1] See also [2, 3] and the bibliography in [4, pp. 305-322].
[2] E.g., [5] and [3, § 16].
[3] Studied earlier in [6, 7, 8].
[4] E.g., [9, p. 340].
[5] A relation between duality and orthogonality was recently given in [14].

ϕ	the *empty set*
ϵ, $\subset$	respectively *element* and *set containment*
$\cup$, $\cap$, $-$	respectively *set union, intersection* and *difference*
$\mathscr{F}$	an *arbitrary ordered field*
E^n	the n-dimensional vector space over $\mathscr{F}$
θ	the zero vector in E^n

$$C\{f_i : i = 1, \cdots, k\} = \left\{x : x \in E^n, x = \sum_{i=1}^{k} \alpha_i f_i, \alpha_i \in \mathscr{F}, \alpha_i \geqq 0, i = 1, \cdots, k\right\}$$

the *cone spanned by the vectors* $\{f_i : i = 1, \cdots, k\}$ in E^n

$$\text{int}\, C\{f_i : i = 1, \cdots, k\} = \left\{x : x \in E^n, x = \sum_{i=1}^{k} \alpha_i f_i, \alpha_i \in \mathscr{F}, \alpha_i > 0, i = 1, \cdots, k\right\}$$

the *interior* of $C\{f_i : i = 1, \cdots, k\}$

$$\text{bdry}\, C\{f_i : i = 1, \cdots, k\} = C\{f_i : i = 1, \cdots, k\} - \text{int}\, C\{f_i : i = 1, \cdots, k\}$$

the *boundary* of $C\{f_i : i = 1, \cdots, k\}$

$\{e_i : i = 1, \cdots, n\}$ *a* fixed *orthonormal base in* E^n

$E^n_+ = C\{e_i : i = 1, \cdots, n\}$

the *nonnegative orthant in* E^n.

For $x, y \in E^n$ let:

$$x \geqq y \quad \text{denote} \quad x - y \in E^n_+$$

$$x \geq y \quad \text{denote} \quad x - y \in E^n_+ - \{\theta\}$$

$$x > y \quad \text{denote} \quad x - y \in \text{int}\, E^n_+$$

For $x, y \in E^n$ let:

$(x, y) = \sum_{i=1}^{n} x_i y_i$ denote the *inner product of* x, y

$\| x \| = \sqrt{(x, x)}$ the *norm* of x

$x \perp y \quad$ denote $\quad (x, y) = 0$

For a subspace L in E^n let:

$x \perp L$ denote $x \perp y$ for all $y \in L$

$L^\perp = \{x : x \in E^n, x \perp L\}$, *the orthogonal complement of* L

$\dim L$: *the dimension of* L

$x + L = \{y : y \in E^n, y = x + l, l \in L\}$ *a translate of* L

P_L: *the perpendicular projection on* L, i.e.

$$P_L = P_L^2 = P_L^T, \qquad L = \{x : x \in E^n, P_L x = x\}$$

For an $m \times n$ matrix A over $\mathscr{F}$ let:

A^T denote the *transpose of* A

$N(A) = \{x : x \in E^n, Ax = \theta\}$ the *null space of* A

$R(A^T) = \{y : y \in E^n, y = A^T v$ for some $v \in E^m\}$ the *range space of* A^T.

1. Lemma. *Let C be a cone in E^n, generated by a finite number of positively linear independent vectors. Then C is a halfspace of E^n if, and only if, $n = 1$.*

Proof. *If:* Obvious.

Only if: Recall that the vectors $\{f_i : i = 1, \cdots, k\}$ are *positively linearly independent* if $\sum_{i=1}^k \alpha_i f_i = \theta$ implies that all $\alpha_i = 0$ or that there is a pair of indices $1 \leqslant i,\ j \leqslant k$ for which: $\alpha_i \alpha_j < 0$. Let $C = C\{f_i : i = 1, \cdots, k\}$ where the $\{f_i\}$ are positively linearly independent. Then $C \cap \{-C\} = \{\theta\}$. For if $x \in C \cap \{-C\}$, $x \neq \theta$, then

$$x = \sum_{i=1}^k \alpha_i f_i, \qquad \alpha_i \geqslant 0$$

and

$$x = \sum_{i=1}^k \beta_i f_i, \qquad \beta_i \leqslant 0$$

hence

$$\theta = \sum_{i=1} (\alpha_i - \beta_i) f_i,$$

with $(\alpha_i - \beta_i) \geqslant 0$ and $(\alpha_j - \beta_j) > 0$ for at least one $1 \leqslant j \leqslant k$. This contradicts the positively linear independence of the $\{f_i : i = 1, \cdots, k\}$. If $n > 1$ then $C \cup \{-C\}$ is not identical with the whole space E^n, as $C \cup \{-C\}$ does not contain vectors of the form: $\sum_{i=1}^k \alpha_i f_i$ with some α_i's of opposite signs. Thus C is not a halfspace of E^n.

Remarks. (a) This lemma can be proved as a corollary to Weyl's fundamental theorem [21].

(b) If $\mathscr{F}$ is an arbitrary ordered field and C is not finitely generated, then the conclusion of this lemma generally does not hold. Let for example Q be the rational field, and

$$C = \left\{ \binom{x}{y} : x, y \in Q, x \leqslant \sqrt{2}\, y \right\}.$$

Then C is a closed convex pointed cone in Q^2. It is also a closed halfspace of Q^2.

2. THEOREM. *Let L be a subspace in E^n, of dimension:* $\dim L \leqq n - 2$, *such that $L \cap E^n_+ = \{\theta\}$. Then L is contained in a subspace M, with*

$$\dim M = \dim L + 1 \qquad \text{and} \qquad M \cap E^n_+ = \{\theta\}.$$

PROOF. Let L satisfy the above assumptions. We define

$$S_L = \{x \in L^\perp : x + y \geqq \theta \text{ for some } y \in L\}.$$

S_L is a cone in $L^\perp$, generated by the positively linearly independent vectors $\{P_{L^\perp} e_i : i = 1, \cdots, n\}$, the perpendicular projections on $L^\perp$ of the $\{e_i : i = 1, \cdots, n\}$.

We show first that $S_L = P_L \perp E^n_+$:

$P_L \perp E^n_+ \subset S_L$: For $x \in P_L \perp E^n_+$, i.e. $x = P_L \perp u$ where $u \in E^n_+$ we take $y = P_L u$. Thus $x + y \geqq \theta$ and $x \in S_L$.

$S_L \subset P_L \perp E^n_+$: Suppose there is an $x \in L^\perp$ which is not in $P_L \perp E^n_+$. If there exists a $y \in L$ such that $x + y \geqslant \theta$, then on one hand

$$P_L \perp (x + y) \in P_L \perp E^n_+$$

and on the other hand

$$P_L \perp (x + y) = P_L \perp x = x \notin P_L \perp E^n_+,$$

a contradiction. Thus $x \notin S_L$. Next we notice that the positively linear independence of the $\{P_L \perp e_i : i = 1, \cdots, n\}$ is implied by $L \cap E^n_+ = \{\theta\}$. Thus

$$S_L = C\{P_L \perp e_i : i = 1, \cdots, n\},$$

as a cone in $L^\perp$, satisfies the assumptions of Lemma 1.

Suppose that there is no subspace M in E^n such that $L \subset M$,

$$\dim M = \dim L + 1 \qquad \text{and} \qquad M \cap E^n_+ = \{\theta\}.$$

In other words, for every $x \in L^\perp$ there is an $l \in L$ such that $x + l \geqq \theta$ or $-x + l \geqq \theta$. But this means that S_L is a halfspace of $L^\perp$, and by using

Lemma 1 we conclude that $\dim L^{\perp} = 1$. Hence $\dim L = n - 1$, a contradiction.

REMARK. We have shown that any subspace L of E^n, with dimension $\leqq n - 2$, satisfying $L \cap E^n_+ = \{\theta\}$ can be extended to a subspace M, $\dim M = \dim L + 1$ and the same property. The maximal subspace $H \supset L$ with $H \cap E^n_+ = \{\theta\}$ is a hyperplane.

3. COROLLARY. *Let P be a perpendicular projection in E^n. Then the following are equivalent:*

(i) *$Py = \theta$ has no solution $y \geqslant \theta$.*
(ii) *$Px = x$ has some solution $x > \theta$.*

PROOF. (i) $\Rightarrow$ (ii). Let rank $P = k$, $k = 0, 1, \cdots, n$. Since both (i) and (ii) are false for $k = 0$, we prove (i) $\Rightarrow$ (ii), first for $k = 1$ and then for $k = 2, \cdots, n$.

Let P be a perpendicular projection of rank 1. $R(P)$, the subspace of all solutions of[6]

$$Px = x \qquad x \in E^n \tag{1}$$

is of dimension 1 and therefore representable as

$$R(P) = \{x : x = \alpha u, \alpha \in \mathscr{F}, u \text{ a nonzero solution of (1)}\}. \tag{2}$$

Suppose now that (ii) is false. This is possible only in two (not mutually exclusive) cases, where $\{u_j\}_{j=1}^n$ are the coordinates of the vector u in (2):

Case A: $u_i = 0$ for some $1 \leqslant i \leqslant n$.

Case B: $u_k u_l < 0$ for some $1 \leqslant k, l \leqslant n$, i.e., some two components of u are of opposite sign.

In each case consider the vector v, given by its coordinates $\{v_j\}_{j=1}^n$ as follows:

Case A: $v_i = 1, v_j = 0$ for $j \neq i$.

Case B: $v_k = 1, v_l = - u_k/u_l,\ v_j = 0$ for $j \neq k, l$.

The vector v satisfies

$$v \geqslant \theta, \tag{3}$$
$$v \perp u. \tag{4}$$

Combining (4) and (2) we conclude that $v \perp R(P)$ and therefore satisfies[7]

$$Pv = \theta. \tag{5}$$

[6] E.g. [5, § 41, theorem 2].
[7] E.g. [5, p. 146].

But (3) and (5) imply (i) to be false, and thus (i) $\Rightarrow$ (ii) is proved for $k = 1$. Now let P be a perpendicular projection of rank $k \geqslant 2$, satisfying (i). In other words $N(P)$ is a subspace of dimension $n - k \leqslant n - 2$, and

$$N(P) \cap E_+^n = \{\theta\}.$$

By Theorem 2 $N(P)$ is contained in some hyperplane H with $H \cap E_+^n = \{\theta\}$. This fact is expressed as

$$P = Q_1 + Q_2 \tag{6}$$

where Q_1, Q_2 are perpendicular projections of ranks 1, $k - 1$ respectively, satisfying

$$H = N(Q_1) \tag{7}$$

and

$$Q_1 Q_2 = O = Q_2 Q_1 . \tag{8}$$

Since rank $Q_1 = 1$, $N(Q_1) \cap E_+^n = \{\theta\}$ it follows that Q_1 satisfies (ii). Thus there exists a vector $x > \theta$ such that $Q_1 x = x$. By (6) and (8) we conclude that $x \in R(P)$, i.e., $Px = x$ and the proof of (i) $\Rightarrow$ (ii) is completed.

(ii) $\Rightarrow$ (i). If $Px = x > \theta$ then for any $y \geqslant \theta$,

$$0 < (x, y) = (Px, y) = (x, Py).$$

Therefore $Py \neq \theta$.[8]

REMARK. Corollary 3 can be rewritten as

3′. COROLLARY. *Let L, $L^\perp$ be complementary orthogonal subspaces in E^n. Then the following are equivalent:*

(i) $$L \cap E_+^n = \{\theta\}$$

(ii) $$L^\perp \cap \operatorname{int} E_+^n \neq \phi.$$

PROOF. Let $L = N(P)$, P as in Corollary 3. A somewhat stronger result is given below.

4. THEOREM. *Let L be a subspace of E^n. Then the following are equivalent:*

(i) $L \cap E_+^n = \{\theta\}$.

(ii) *$L^\perp$ has a basis in* $\operatorname{int} E_+^n$.

(iii) *For every $x \in E^n$, $\{x + L\} \cap E_+^n$ is bounded, may be empty.*

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[8] Only the symmetry of P was used here.

PROOF.[9] The part (ii) $\Rightarrow$ (i) follows from Corollary 3'. Also (iii) $\Rightarrow$ (i) is obvious, for if (i) is false then (iii) is false with $x = \theta$. It remains to prove that *(i)* $\Rightarrow$ *(ii)* and *(i)* $\Rightarrow$ *(iii)*.

(i) $\Rightarrow$ *(ii)*. Let $\dim L = k$. From (i) and Corollary 3' it follows that $L^\perp$ has a basis: $x_1, x_2, \cdots, x_{n-k}$ with $x_1 \in \operatorname{int} E_+^n$. Let m be the minimum of the coordinates of x_1 (relative to the basis: $e_1, \cdots, e_n$ of E^n) and let M be the maximum of the absolute values of the coordinates of all the vectors $x_2, \cdots, x_{n-k}$.

The vectors: $x_1, x_1 + (m/2M)\, x_2, x_1 + (m/2M)\, x_3, \cdots, x_1 + (m/2M)\, x_{n-k}$ form a basis of $L^\perp$ and are all in int E_+^n. (If $k = n - 1$ then x_1 is the basis of $L^\perp$ in int E_+^n, for $k = n$ both (i) and (ii) are trivially false.)

(i) $\Rightarrow$ *(iii)*. Clearly it is sufficient to consider vectors $x \in L^\perp$. By Corollary 3', (i) implies the existence of a vector $z \in L^\perp \cap \operatorname{int} E_+^n$. Let m be the minimum of the coordinates of z (relative to $e_1, \cdots, e_n$). If $y \in \{x + L\} \cap E_+^n$, then

$$y_k z_k \leqslant \sum_{i=1}^{n} y_i z_i = (y, z) = (x, z) \qquad \text{for} \qquad 1 \leqslant k \leqslant n.$$

Thus

$$0 \leqslant y_k \leqslant \frac{(x, z)}{z_k} \leqslant \frac{(x, z)}{m}$$

which proves the boundedness of $\{x + L\} \cap E_+^n$. We conclude by noting that for any $l \in L$, $\alpha \in \mathscr{F}$ the vector $x + \alpha l$ is $\geqslant \theta$ only if the scalar α is bounded by

$$\max_{l_i > 0} -\frac{x_i}{l_i} \leqslant \alpha \leqslant \min_{l_i < 0} \frac{x_i}{l_i}. \tag{9}$$

Thus $\{x + L\} \cap E_+^n$ is empty if the left hand side in (9) exceeds the right hand side for all $l \in L$.

5. COROLLARY. *Let L be a subspace of E^n of dimension k, $k = 1, \cdots n$. Then the following are equivalent:*

(i) $L \cap \operatorname{bdry} E_+^n = C\{e_1, \cdots, e_p\}, 1 \leqslant p \leqslant k$.

(ii) $L^\perp$ has a basis in int $C\{e_{p+1}, \cdots, e_n\}$.

PROOF. *(i)* $\Rightarrow$ *(ii)*. Consider E^p, the space spanned by $\{e_1, \cdots, e_p\}$ as a subspace of E^n, and the quotient space of E^n modulo $E^p : E^n/E^p$. From

[9] This proof is due to Dr. Micha Perles of the Hebrew University of Jerusalem.

(i) it follows that $E^p \subset L$. Hence in E^n/E^p the subspace L/E^p satisfies (i) of theorem 4. Therefore $L^\perp/E^p$ has a basis in

$$\text{int}\,(E^n_+/E^p) = \text{int}\,C\{e_{p+1}, \cdots, e_n\}.$$

But $L^\perp/E^p = L^\perp$ and (ii) is established.

(ii) ⇒ (i). From (ii) it follows that $C\{e_1, \cdots, e_p\} \subset L$ and consequently (i).

REMARKS. (a) If $p = k$ then (ii) can be rewritten as

(ii′) $L^\perp \cap E^n_+ = C\{e_{p+1}, \cdots, e_n\}$.

(b) If $\dim L \leqslant n - 2$ it can be shown as in Theorem 2 that L is contained in a subspace M with

$$\dim M = \dim L + 1 \qquad \text{and} \qquad M \cap \text{bdry}\, E^n_+ = C\{e_1, \cdots, e_p\}.$$

The maximal subspace H with $H \supset L$ and

$$H \cap \text{bdry}\, E^n_+ = C\{e_1, \cdots, e_p\}$$

is a hyperplane, e.g., [16, p. 316, Theorem 33 (2)].

6. COROLLARY. *Let L, $L^\perp$ be any pair of complementary orthogonal subspaces in E^n. Then there is a vector x in* int E^n_+ *such that $x = y + z$, $y \in L \cap E^n_+$, $z \in L^\perp \cap E^n_+$.*

PROOF. Since the case: $L = E^n$, $L^\perp = \{\theta\}$ is trivial, let $\dim L = 1, \cdots, n - 1$. Now there are three mutually exclusive cases:

(i) $L \cap E^n_+ = \{\theta\}$.

(ii) $L \cap \text{bdry}\, E^n_+ = C\{e_1, \cdots, e_p\}$, $1 \leqslant p \leqslant \dim L$ and $L \cap \text{int}\, E^n_+ = \phi$.

(iii) $L \cap \text{int}\, E^n_+ \neq \phi$.

In case (i) we use Corollary 3′ and choose x as a vector in $L^\perp \cap \text{int}\, E^n_+$. Thus $x = z$ and $y = \theta$.

In case (ii), Corollary 5 is used to construct x as $x = y + z$ where

$$z \in L^\perp \cap \text{int}\,\{e_{p+1}, \cdots, e_n\}_+$$

and y is any vector in int $C\{e_1, \cdots, e_p\}$. By remark (a) following Corollary 5, if $p = \dim L$ then any vector x in int E^n_+ can be so represented.

Case (iii) is, by Corollary 3′, case (i) with L, $L^\perp$ permuted.

7. COROLLARY (Tucker [10]). *Let A be any $m \times n$ matrix over $\mathscr{F}$. Then the following system of equations and inequalities*

$$Ax = \theta \qquad A^T u \geqslant \theta \qquad x \geqq \theta$$

has solutions x^0, u^0 satisfying

$$A^T u^0 + x^0 > \theta.$$

PROOF. Follows immediately from Corollary 6, by letting $L = R(A^T)$ and using Fredholm's alternative theorem [9, p. 340] which for a linear operator $A : E^n \to E^m$ can be stated as: *$R(A^T)$ and $N(A)$ are complementary orthogonal subspaces in E^n*, e.g., [15, § 49].

REMARK. This corollary is fundamental in the theories of linear inequalities and linear programming e.g., [10]. We proved it here as a consequence of two facts:

(a) Corollary 6 which states a simple property of the intersections of E_n^+ with arbitrary pairs of complementary orthogonal subspaces.

(b) Fredholm's alternative theorem

$$E^n = R(A^T) \oplus N(A)$$

which is basic to the theory of linear equations.

The abundance of theorems (e.g., the Minkowski-Farkas-Weyl theorems [12] and their consequences) which follow Corollary 7, e.g., [10] emphasizes the merits of a unified treatment of linear inequalities and equations, e.g. [5].

8. COROLLARY (Jackson [11]. Charnes-Cooper [12]). *Let A be any $m \times n$ matrix over $\mathscr{F}$. Then the following are equivalent:*

(i) *$Ax = \theta$ has no solution $x \geqslant \theta$.*

(ii) *$A^T u > \theta$ has solutions.*

(iii) *For any $b \in E^m$ the set $\{x : x \in E^n, Ax = b, x \geqq \theta\}$ is bounded, maybe empty.*

PROOF. Setting, by Fredholm's alternative theorem, $L = N(A)$ and $L^\perp = R(A^T)$ it follows that statements (i), (ii), and (iii) are equivalent to the corresponding statements in Theorem 4.

REMARK. In the real case this theorem was proved by Jackson [11]. The part (i) $\Rightarrow$ (iii) is close to the "opposite sign theorem" of Charnes-Cooper [12, 13], which states, more precisely, that (i) above is equivalent to:

(iv) *The set*

$$\{x : x \in E^n, Ax = b, x \geqq \theta\}$$

is spanned by its extreme points.

Property (iv), see [17], is not restricted to bounded sets when x is in an infinite dimensional vector space.

9. Corollary (Tucker [10]). *Let K be a skew symmetric matrix over $\mathscr{F}$.*[10] *Then the system of inequalities*

$$Kw \geqq \theta \qquad w \geqq \theta$$

has a solution w^0 such that

$$Kw^0 + w^0 > \theta. \tag{10}$$

Proof. Consider the system of equations and inequalities

$$(I, K^T)\begin{pmatrix}x\\ y\end{pmatrix} = \theta \qquad \begin{pmatrix}I\\ K\end{pmatrix} u \geqq \theta.$$

$$\begin{pmatrix}x\\ y\end{pmatrix} \geqq \theta.$$

By Corollary 7 this system has solutions $\binom{x^0}{y^0}$ and u^0 such that

$$\begin{pmatrix}x^0\\ y^0\end{pmatrix} + \begin{pmatrix}I\\ K\end{pmatrix} u^0 > \theta. \tag{11}$$

Combining (11) and the fact that $x^0 = -K^Ty^0 = Ky^0$, it follows that $w^0 = y^0 + u^0$ satisfies (10).

10. Remark. Corollary 9 was used to prove the duality theorem of linear programming, e.g., [18] and [19]. We conclude this paper by outlining an alternative proof which, like our other results above, rests upon the "Fredholm alternative" theorem.

Consider the pair of dual problems

$$\begin{array}{llll} \text{maximize} & c^Tx & \text{minimize} & w^Tb \\ & Ax \leqq b & & w^TA \geqq c^T \\ & x \geqq \theta & & w \geqq \theta \end{array}$$

where A is an $m \times n$ matrix over $\mathscr{F}$, $b \in E^m$, and $c \in E^n$.

[10] I.e., $K = -K^T$.

To these problems there corresponds the $(m + 1) \times (m + n + 1)$ matrix:

$$B_t = \left(\begin{array}{c|c|c} t & -c^T & \theta^T \\ \hline -b & A & I \end{array}\right)$$

where t is in $\mathscr{F}$. For any given value of t we consider the subspaces $N(B_t)$ and $R(B_t^T)$—which are complementary orthogonal by Fredholm's theorem—and their intersection with E_+^{m+n+1}:

$$N(B_t) \cap E_+^{m+n+1} = \left\{ \begin{pmatrix} \alpha \\ x \\ y \end{pmatrix} : \begin{array}{r} \alpha t - c^T x = 0 \\ -\alpha b + Ax + y = \theta \\ \alpha \geqq 0, x, y \geqq \theta \end{array} \right\}$$

$$R(B_t^T) \cap E_+^{m+n+1} = \left\{ \begin{pmatrix} \beta t - b^T w \\ -\beta c + A^T w \\ w \end{pmatrix} : \begin{array}{r} \beta t - w^T b \geqq 0 \\ -\beta c^T + w^T A \geqq \theta^T \\ w \geqq \theta \end{array} \right\}.$$

The duality theorem of linear programming (as well as the "complementary slackness" property [19], which is the statement that $R(B_t^T) \cap E_+^{m+n+1}$ and $N(B_t) \cap E_+^{m+n+1}$ are orthogonal sets)[11] follows now by considering the above intersections; the keys to the whole situation being the vanishing of the scalars α, β and the value of t. The details are left to the reader.

Acknowledgment

The author is greatly indebted to Professor A. Charnes, to Dr. A. J. Hoffman, and to Dr. M. Perles for their helpful comments on a draft of this paper.

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