Optimal Algorithms for a Problem of Optimal Control

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This paper deals with optimal algorithms for the approximate solution of a problem of optimal control. The control problem in question is the minimization of a quadratic energy functional, which is equivalent to the solution of a mildly nonlinear two-point second-order elliptic boundary-value problem. The only restriction on the algorithms considered is that they can use only a finite amount of information about the problem element $f$ appearing in the definition of the energy functional. An algorithm having error $\varepsilon$ is said to be optimal if its cost is minimal among all algorithms that solve the problem to within $\varepsilon$. We first suppose that the information available about $f$ consists of a finite set of linear functionals of $f$; that is, we allow arbitrary linear information. We then show that there is a finite element method (whose degree depends on the smoothness of $f$) which is an optimal algorithm for the optimal control problem. Note that this finite element method requires the evaluation of the inner products of $f$ with finite element basis functions. These inner products are not usually available in practice; often, only "standard information" is available (meaning that we can evaluate $f$ at a finite set of points). So, we next consider the case where the only information that is available is standard information. We then find that there is a "finite element method with quadrature" which is an optimal algorithm among all algorithms using this standard information. Moreover, we find that standard information is weaker than inner-product information. The asymptotic penalty for using standard information instead of inner-product information is unbounded as $\varepsilon$ tends to 0.


1. Introduction

We analyze a simple problem of optimal control that was presented to us by Dr. Jo Bollen, formerly of the Twente University of Technology, Enschede, The Netherlands. Since these problems do not have a closed-form solution, we can find only an approximate solution. Hence, we are
interested in finding algorithms with minimal cost for computing an approximation with given accuracy. We investigate two cases, the first being when any finite set of inner products of the function \( f \) defining the optimal control problem are permissible, and the second being when we can only evaluate the standard information consisting of the values of \( f \) at a finite set of points.

The optimal control problem studied in this paper is equivalent to a mildly nonlinear boundary-value problem. Since finite element methods (FEMs) have been successful in the solution of boundary-value problems, it is only natural to investigate whether FEMs can be optimal algorithms for our problem. Indeed, we show that if the parameters of the FEM are properly chosen (depending on the regularity of \( f \)), then the FEM is optimal.

Moreover, we compare the strengths of inner-product information and standard information. We find that there is a loss in going from inner-product to standard information. If \( r \) represents the regularity of \( f \), then the complexity (i.e., minimal cost) of finding an \( \varepsilon \)-accurate approximation is proportional to \( \varepsilon^{-1/(r+1)} \) if inner products can be used but is proportional to \( \varepsilon^{-1/r} \) if only standard information is available.

This problem has also been studied recently in a different setting by B. Z. Kacewicz, who discusses optimal convergence properties of a multiple shooting method based on a spline approximation of \( f \). For further details, see Kacewicz (1989).

2. **Statement of the Problem**

Let \( M \) and \( T \) be given positive real numbers. Let \( f \) be a nonnegative function of smoothness \( r \) over the interval \( I = [0, T] \), bounded by \( M \) in an appropriate norm. That is, we assume that \( f \) belongs to the class

\[
F_r = \begin{cases} 
\{ f \in H^r(I) : \|f\|_{H^r(I)} \leq M \text{ and } f \geq 0 \} & \text{for } r \geq 1, \\
\{ f \in L^\infty(I) : \|f\|_{L^\infty(I)} \leq M \text{ and } f \geq 0 \} & \text{for } r = 0,
\end{cases}
\]

where \( H^r(I) \) is the usual Sobolev space of order \( r \) (see, e.g., Babuška and Aziz, 1972; Ciarlet, 1978; Oden and Reddy, 1976). Define an energy functional \( J(\cdot ; f) \) on \( H^1(I) \) by

\[
J(v; f) = \int_0^T \left[ (v'(t))^2 + f(t)(v(t))^2 \right] dt \quad \forall v \in H^1(I).
\]

For a given real number \( c \), we wish to find a function \( u^* \in H^1(I) \) such that \( u^*(0) = c \) and
Alternatively, we seek a solution \( u^* : I \to \mathbb{R} \) to the nonlinear two-point boundary-value problem

\[-(u^*)''(x) + f(x)u^*(x) = 0 \quad \text{for } 0 < x < T,\]

\[u^*(0) = c \quad \text{and} \quad (u^*)'(T) = 0.\]

In what follows, we shall use a slightly different version of our problem. Namely, for \( f \in F_r \), we seek \( u \in \mathcal{H}_E \) satisfying

\[B(u, v; f) = (f, v) \quad \forall v \in \mathcal{H}_E. \tag{2.1}\]

Here,

\[\mathcal{H}_E = \{ v \in H^1(I) : v(0) = 0 \}\]

denotes what Strang and Fix (1973) call the energy space for our problem, the bilinear form \( B \) on \( \mathcal{H}_E \) is given by

\[B(v, w; f) = \int_0^T (v'(x)w'(x) + f(x)v(x)w(x)) \, dx \quad \forall v, w \in \mathcal{H}_E,\]

and \((\cdot, \cdot)\) denotes the inner product in \( L_2(I) \). Equivalently, we seek \( u : I \to \mathbb{R} \) such that

\[-u''(x) + f(x)u(x) = f(x) \quad \text{for } 0 < x < T,\]

\[u(0) \ u'(T) = 0. \tag{2.2}\]

Of course, if \( u \) is the solution to the new problem, then \( u^* = c - cu \) is the solution to the original problem. It is straightforward to check that for any \( f \in F_r \), there exists a unique solution \( u \) to the problem. We show this by writing \( u = Sf \). Since \( u \) depends nonlinearly on \( f \), the operator \( S : F_r \to \mathcal{H}_E \) is nonlinear.

3. INFORMATION AND ALGORITHMS

Since a closed-form solution to the problem is not available, we are interested in finding algorithms for approximating \( u \). We suppose that the only knowledge an algorithm has about any \( f \) is the values of a finite set of linear functionals of \( f \). Let \( \Lambda \) denote a class of permissible linear functionals. Then information \( N \) of cardinality \( n \) has the form
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\[
Nf = \begin{bmatrix}
\lambda_1(f) \\
\lambda_2(f) \\
\vdots \\
\lambda_n(f)
\end{bmatrix} \quad \forall f \in F_r,
\]

where \(\lambda_1, \ldots, \lambda_n \in \Lambda\) are permissible linear functionals.

For \(f \in F_r\), we wish to approximate \(Sf\), knowing only the information \(Nf\). Hence, the approximation must be of the form \(\varphi(Nf)\). Here, we say that \(\varphi\) is an algorithm using \(N\), i.e., a mapping \(\varphi: N(F_r) \to \mathcal{H}_E\). Note that we allow any mapping to be an algorithm.

**Remark 3.1.** Note that we are restricting ourselves to algorithms using nonadaptive information. That is, the number \(n\) of evaluations, as well as the functionals \(\lambda_1, \ldots, \lambda_n\) to be evaluated, does not depend on the problem element \(f\). This has been done only for the sake of exposition. We could also consider adaptive information, in which these restrictions are lifted. Since adaptive information is more general than nonadaptive information, one might think that it is more powerful. This is not the case. Using techniques of Wasilkowski (1985) and Wasilkowski (1986), one can slightly modify the statements and proofs of the main results of this paper to see that nonadaptive information is roughly as strong as adaptive information.

The quality of an algorithm \(\varphi\) using \(N\) is measured by its error \(e(\varphi, N)\), which is given by

\[
e(\varphi, N) = \sup_{f \in F_r} \|Sf - \varphi(Nf)\|_{H(f)}.
\]

Note that this is a "worst-case" setting for defining the error of an algorithm. Hence, if we know that \(e(\varphi, N) \leq \varepsilon\), this means that we are guaranteed that \(\|Sf - \varphi(Nf)\|_{H(f)} \leq \varepsilon\) for any problem element \(f \in F_r\).

We seek algorithms using \(N\) whose error is as small as possible. Let

\[
r(N) = \inf_{\varphi} e(\varphi, N)
\]

be the minimal error among all algorithms using information \(N\). We say that \(r(N)\) is the radius of information \(N\). (The terminology is that of Traub and Woźniakowski, 1980.) An algorithm \(\varphi\) using \(N\) for which \(e(\varphi, N) = r(N)\) is said to be an optimal error algorithm using \(N\).

We also seek to determine the best permissible information of given cardinality \(n\). We say that the \(n\)th minimal radius of information \(r(n)\) is given by
the infimum being over all permissible information $N$ whose cardinality is at most $n$. Permissible information $N^*_n$ of cardinality at most $n$ is said to be $n$th optimal information if

$$r(N^*_n) = r(n).$$

An algorithm $\varphi$ using information $N$ of cardinality at most $n$ for which $e(\varphi, N) = r(n)$ is said to be an $n$th optimal error algorithm. Where necessary, we will explicitly show the dependence of the $n$th minimal radius on the class $\Lambda$ by writing $r(n, \Lambda)$ instead of $r(n)$.

4. Finite Element Methods

We briefly illustrate the ideas of information and algorithms by the finite element method of degree $k$, as well as the finite element method with quadrature. A more detailed description may be found in Babuška and Aziz (1972), Ciarlet (1978), and Oden and Reddy (1976).

Let $Y$ be an $m$-dimensional space of continuous piecewise polynomials of degree $k$ which vanish at the origin, based on a uniform partition $\mathcal{T}_I$ of $I$. We then seek $U \in Y$ satisfying

$$mh^{-1} s, f) = \sum_i \lambda_i(s_i) \varphi_i \in Y.$$

(4.1)

Let $\{s_1, \ldots, s_m\}$ be a basis for $Y$, and let $\lambda_1, \ldots, \lambda_n$ denote the nonzero linear functionals among

$$\{(\cdot, s_i)\}_{i=1}^m \cup \{(\cdot, s_j s_i)\}_{i,j=1}^m.$$

For each $f \in F_r$, let

$$N_n f = \begin{bmatrix} \lambda_1(f) \\
\vdots \\
\lambda_n(f) \end{bmatrix}.$$

Then $N_n$ is finite element information (FEI) of cardinality $n$. Since $u_m$ depends on $f$ only through $N_n f$, we may write

$$u_m = \varphi_{n,k}(N_n f).$$
We refer to $\varphi_{n,k}$ as the finite element method (FEM) of degree $k$ using finite element information $N_n$ of cardinality $n$. Note that the finite element method depends nonlinearly on its information.

Of course, the FEM is well-defined if and only if the linear functionals $\lambda_1, \ldots, \lambda_n \in \Lambda$. This holds, for instance, if the class $\Lambda$ of permissible linear functionals coincides with the class $\Lambda_{\text{ip}}$ of all bounded linear functionals on $H'(I)$, i.e., if inner products of arbitrary elements of $F$, with finite element basis functions are permissible. More often, we can only compute the value of a function at a point. Thus, information $N$ of cardinality $n$ has the form

$$Nf = \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix},$$

where $x_1, \ldots, x_n \in I$. Such information is said to be standard information, and we write $\Lambda = \Lambda_{\text{std}}$ when this holds.

So, we now suppose that only standard information is available. The obvious solution is to approximate the integrals appearing in the FEM via quadratures. This leads to the finite element method with quadrature.

More precisely, let $n$ be a positive integer. Let

$$\mathcal{I}_l = \{0 = \xi_0 < \xi_1 < \cdots < \xi_l = T\} \quad \text{with} \quad \xi_i = \frac{i}{l} T \quad \text{for} \quad 0 \leq i \leq l$$

be a uniform partition of $I$. Choose points $x_1, \ldots, x_n$ so that $x_{ak+1}, \ldots, x_{ak+k}$ are the Gauss quadrature points for the interval $[\xi_a, \xi_{a-1}]$, where $0 \leq a \leq l - 1$. Choose $\mathcal{H}_n^q$ to be a subspace of $\mathcal{H}_E$ which is an $n$-dimensional finite element space of degree $k$, based on the partition $\mathcal{I}_l$; clearly, we have $n = kl$. (The superscript $q$ reminds us that the sampling points for the degrees of freedom for the space are Gauss quadrature points.) The information $N_n^q$ defined by

$$N_n^q f = \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}, \quad \forall f \in F_r$$

is called finite element quadrature information (FEQI). Clearly, $N_n^q$ is standard information of cardinality $n$.

For each index $a$, let $\omega_{ak+1}, \ldots, \omega_{ak+k}$ denote the Gauss quadrature weights corresponding to the nodes $x_{ak+1}, \ldots, x_{ak+k}$. For continuous
v: I → ℝ, we let

\[ Q_n(v) = \sum_{j=1}^{n} \omega_j v(x_j) \]

denote the resulting piecewise Gaussian quadrature for approximating the integral \( \int_I v(x) \, dx \).

Now for \( f \in F_r \), we seek \( u_n^q \in \mathcal{F}_n^q \) satisfying

\[ B_n(u_n^q, s; f) = f_n(s) \quad \forall s \in \mathcal{F}_n^q. \quad (4.2) \]

Here, the bilinear form \( B_n \) on \( \mathcal{F}_n^q \) is defined by

\[ B_n(v, w; f) = Q_n(v'w' + fvw) \quad \forall v, w \in \mathcal{F}_n^q, \]

and the linear functional \( f_n \) on \( \mathcal{F}_n^q \) is given by

\[ f_n(w) = Q_n(fw) \quad \forall w \in \mathcal{F}_n^q. \]

Since \( u_n^q \) depends on \( f \) only through the information \( N_n^q f \), we write

\[ u_n^q = \varphi_{n,k}^q(N_n^q f). \]

We refer to \( \varphi_{n,k}^q \) as the finite element method with quadrature (FEMQ) of degree \( k \), using finite element quadrature information \( N_n^q \) of cardinality \( n \).

5. Statement of Optimal Error Properties

Our results are stated first for the case \( \Lambda = \Lambda^{ip} \); i.e., we assume that any continuous linear functional is permissible. We then state results for the case \( \Lambda = \Lambda^{std} \). These results are stated using the \( \Theta \)-notation of Knuth (1976), which may be thought of as being a "two-sided" \( O \)-notation. The proofs of the results in this section may be found in the Appendix.

Recall that \( r \) measures the regularity of the problem, and that \( k \) is the degree of the FEM and FEMQ.

**Theorem 5.1.** Suppose that any continuous linear functional is permissible. Then the following hold:

1. The \( n \)th minimal radius is given by

\[ r(n, \Lambda^{std}) = \Theta(n^{-(r+1)}) \quad \text{as } n \to \infty. \]
(2) The error of the FEM is given by
\[ e(\varphi_{n,k}, N_n) = \Theta(n^{-\mu}) \quad \text{as } n \to \infty, \]
where
\[ \mu = \min\{k, r + 1\}. \]

(3) The radius of FEI is given by
\[ r(N_n) = \Theta(n^{-(r+1)}) \quad \text{as } n \to \infty. \]

From (1) and (2) of Theorem 5.1, we see that the FEM has quasi-minimal error iff \( k \geq r + 1 \). Suppose this inequality is violated. There are two reasons explaining why the FEM no longer has quasi-minimal error. On the one hand, the finite element information used by the FEM may not be strong enough; that is, there may be no method using FEI of degree \( k < r + 1 \) whose error is quasi-minimal. On the other hand, it may be the case that the FEM does not make sufficiently good use of its information; that is, there may be another method using FEI of degree \( k < r + 1 \) whose error is quasi-minimal. From (3) of Theorem 5.1, we see that the latter is the case. That is, no matter what the values of \( k \) and \( r \), there always exists an algorithm using FEI of degree \( k \) having quasi-minimal error. Thus FEI is always quasi-optimal information, while the FEM has quasi-minimal error iff \( k \geq r + 1 \).

**Theorem 5.2.** Suppose that only standard information is permissible. Then the following hold:

(1) The nth minimal radius is given by
\[ r(n, \Lambda_{\text{std}}) = \Theta(n^{-r}) \quad \text{as } n \to \infty. \]

(2) If \( k \geq r \), then the error of the FEM is given by
\[ e(\varphi_{n,k}^s, N_n^s) = \Theta(n^{-r}) \quad \text{as } n \to \infty. \]

Hence there is a loss in going from arbitrary continuous linear information to standard information. The nth minimal error changes from \( \Theta(n^{-(r+1)}) \) to \( \Theta(n^{-r}) \). This loss is slight if \( r \) is reasonably large. However, if \( r \) is small, this loss of one in the exponent is felt more strongly. For instance, let \( r = 0 \). Then the problem is not convergent if we are restricted to standard information; that is, there is not a sequence of algorithms whose error goes to zero. However, if FEI is permissible, then the problem is convergent even when \( r = 0 \).
Remark 5.1. It is not too difficult to see that the FEMQ has almost minimal error iff $k \geq r$ (see the remarks at the end of Appendix C). What happens when this inequality is violated? Is the reason for the nonoptimality of the FEMQ that its information (FEQI) is nonoptimal, or that the FEMQ uses FEQI nonoptimally? A simple adaption of the results in Section 3.2 of Kacewicz (1989) shows that the latter is the case; there always exists an algorithm using FEQI with error $\Theta(n^{-r})$, even if $k < r$.

Remark 5.2. We have considered only standard information using the values of $f$ itself. One could also look at standard information containing derivatives of $f$. It turns out that there is no advantage in doing this; see Section 4 of Kacewicz (1989) for details.

6. Implementation of the FEM and FEMQ

We show briefly how the FEM and FEMQ may be reduced to the solution of linear systems of equations.

6.1. The FEM

Recall that $\mathcal{G}_m$ is a standard finite element space of degree $k$ and dimension $m$, based on a uniform partition

$$
\mathcal{G}_I = \{0 = \xi_0 < \xi_1 < \cdots < \xi_l = T\} \quad \text{with } \xi_i = \frac{i}{l} T \text{ for } 0 \leq i \leq l
$$

of $I$. That is, each $s \in \mathcal{G}_m$ is continuous on $I$, satisfies $s(0) = 0$, and is a polynomial of degree at most $k$ on each subinterval $[\xi_i, \xi_{i+1}]$. (Of course, $m = kl$.) Let the degrees of freedom of $\mathcal{G}_m$ be given by evaluations at the points $x_1, \ldots, x_m$ (with $x_j = j/m$), and let $s_1, \ldots, s_m$ be the resulting dual basis. In other words, any $s \in \mathcal{G}_m$ may be represented in the form

$$
s(x) = \sum_{j=1}^{m} s_j(x) s_j(x),
$$

where the basis functions satisfy

$$
s_j(x_i) = \delta_{ij}, \quad 1 \leq i, j \leq m.
$$

From this, we see that the basis functions $s_1, \ldots, s_m$ have "small" or "local" supports, i.e., the number of subintervals in the support of any basis function is independent of $m$. 

From Section 4, we let $\lambda_1, \ldots, \lambda_n$ denote the nonzero linear functionals among

$$\{\langle \cdot, s_i \rangle \}_{i=1}^m \cup \{\langle \cdot, s_j s_i \rangle \}_{i,j=1}^m.$$  

Since the basis functions have local support, $n = \Theta(m)$.  

Given a positive integer $n$, we first compute the finite element information

$$w^*_{n \mathcal{F}} = \sum_{j=1}^m f \mathcal{G} \left[ 1 \sum_{i=1}^m f \mathcal{G} \right].$$

We then seek $u_m \in \mathcal{F}_m$ satisfying (4.1).  

We claim that the mapping $N_n f \mapsto u_m$ is well-defined. To see this, write

$$u_m(x) = \sum_{j=1}^m y_j s_j(x).$$

Then

$$A y = b,$$

where

$$a_{ij} = B(s_j, s_i; f) \quad \text{and} \quad b_i = (f, s_i).$$

Since $f \geq 0$, it is a standard exercise to see that $A$ is a symmetric positive definite matrix. Thus the linear system has a unique solution, and so there exists a unique $u_m$ satisfying (4.1). Clearly, $u_m$ depends only on $f$ through $A$ and through $b$, i.e., through $N_n f$. Hence the mapping $N_n f \mapsto u_m$ is well-defined, as claimed.

Finally, note that the linear system above is a banded system, whose bandwidth is independent of $n$. Hence, we can find $u_m$, once we are given $N_n f$, in $\Theta(n)$ arithmetic operations.

6.2. The FEMQ

Recall the definitions and notation of Section 4 leading to the definition of the FEMQ $\varphi^k_n$ of degree $k$ using FEI $N^*_n$ of cardinality $n$.

We claim that the mapping $N^*_n f \mapsto u^*_n$ is well-defined. Indeed, write

$$u^*_n(x) = \sum_{j=1}^n y_j s_j(x).$$
Then

\[ Ay = b, \]

where

\[ a_{ij} = B_n(s_j, s_i; f) \quad \text{and} \quad b_i = f_n(s_i). \]

Since \( f \geq 0 \) and the weights are positive, the exactness properties of Gauss quadrature lead us to see that

\[ B_n(s, s; f) = Q_n((s')^2) = \|s'\|^2_{L^2(I)} \quad \forall s \in \mathcal{S}_n^q \]

(see, e.g., Ciarlet, 1978, Exercise 4.1.5(a)). From this and the Poincaré inequality

\[ \|v'\|_{L^2(I)} \leq \frac{2T}{\pi} \|v\|_{L^2(I)} \quad \forall v \in \mathcal{H}_E \]  \hspace{1cm} (6.1)

(see, e.g., Schultz, 1973, Exercise 1.3), we find

\[ B_n(s, s; f) \leq \frac{\pi^2}{\pi^2 + 4T^2} \|s'\|^2_{H^2(I)} \quad \forall s \in \mathcal{S}_n^q. \]  \hspace{1cm} (6.2)

The positive definiteness of \( A \) follows immediately. Hence, for any \( f \) and \( n \), there exists a unique \( u_n \in \mathcal{S}_n^q \) satisfying (4.2). Clearly, \( u_n \) depends only on \( f \) through \( A \) and \( b \), i.e., through \( N_n^q f \). Hence the mapping \( N_n^q f \mapsto u_n \) is well-defined, as claimed.

Finally, note that the linear system above is a banded system, whose bandwidth is independent of \( n \). Hence, we can find \( u_n \), once we are given \( N_n f \), in \( \Theta(n) \) arithmetic operations.

### 7. Complexity Analysis

In this section, we discuss the complexity (minimal cost) of computing \( \varepsilon \)-approximations to the solution of the variational boundary-value problem. Recall that \( F_r \) denotes the class of problem elements of smoothness \( r \). We show that if inner products are permissible information, then the FEM of degree \( k \) is optimal (meaning that it produces approximations with minimal cost) iff \( k \geq r + 1 \), and that the penalty for using an FEM of too-low degree is unbounded. If only standard information is available, then we show that the FEMQ of degree \( k \geq r \) is optimal. Finally, we show that
standard information is weaker than inner-product information; the asymptotic penalty for using standard information instead of inner-product information is unbounded.

Let $N$ be information and let $\varphi$ be an algorithm using $N$. The cost of $\varphi$ is defined via the model of computation discussed in Traub and Woźniakowski (1980, Chap. 5). That is, we assume that any permissible linear functional can be evaluated with finite cost $c$, and that the cost of an arithmetic operation is unity. We denote the cost of an algorithm $\varphi$ using information $N$ by $\text{cost}(\varphi, N)$.

Let $\varepsilon > 0$. An algorithm $\varphi$ using information $N$ produces an $\varepsilon$-approximation to the problem if

$$ e(\varphi, N) \leq \varepsilon. $$

We then define, for $\varepsilon > 0$, the $\varepsilon$-complexity $\text{COMP}(\varepsilon)$ of the problem to be

$$ \text{COMP}(\varepsilon) = \inf \{ \text{cost}(\varphi, N) : \varphi \text{ and } N \text{ such that } e(\varphi, N) < \varepsilon \}. $$

For $\varepsilon > 0$, an algorithm $\varphi_\varepsilon$ using information $N_\varepsilon$ for which

$$ e(\varphi_\varepsilon, N_\varepsilon) \leq \varepsilon \quad \text{and} \quad \text{cost}(\varphi_\varepsilon, N_\varepsilon) = \text{COMP}(\varepsilon) $$

is said to be an optimal complexity algorithm for computing $\varepsilon$-approximations.

**Remark 7.1.** Note that we distinguish between the cost of an algorithm and the complexity of the problem. An optimal complexity algorithm for computing $\varepsilon$-approximations is an algorithm producing an $\varepsilon$-approximation with minimal cost. In addition, the problem complexity tacitly depends on the class $\Lambda$ of permissible linear functionals. When necessary, we shall show this dependence by writing $\text{COMP}(\varepsilon, \Lambda)$.

We first suppose that $\Lambda = \Lambda^{ip}$; i.e., information is allowed to consist of inner products. Let

$$ \text{FEM}(\varepsilon) = \inf \{ \text{cost}(\varphi_{n,k}, N_n) : n \text{ is an index such that } e(\varphi_{n,k}, N_n) \leq \varepsilon \} $$

denote the cost of using the FEM of degree $k$ to compute an $\varepsilon$-approximation. The following theorem gives necessary and sufficient conditions for the FEM to be almost optimal. Its proof is immediate from Theorem 5.1 and the results in Section 6.1.

**Theorem 7.1.** Suppose that inner products are permissible information.

1. $\text{COMP}(\varepsilon) = \Theta(\varepsilon^{-1/(r+1)})$ as $\varepsilon \to 0$. 
(2) If $k \geq r + 1$, then
\[ FEM(\varepsilon) = \Theta(\varepsilon^{-1/(r+1)}) \quad \text{as } \varepsilon \to 0. \]

Hence, the FEM is an almost optimal complexity algorithm.

(3) If $k < r + 1$, then
\[ FEM(\varepsilon) = \Theta(\varepsilon^{-1/k}) \quad \text{as } \varepsilon \to 0. \]

Hence,
\[ \frac{FEM(\varepsilon)}{COMP(\varepsilon)} = \Theta(\varepsilon^{-\lambda_1}) \quad \text{as } \varepsilon \to 0, \]

where
\[ \lambda_1 = \frac{1}{k} - \frac{1}{r + 1} > 0, \]
so that
\[ \lim_{\varepsilon \to 0} \frac{FEM(\varepsilon)}{COMP(\varepsilon, \Lambda)} = \infty. \]

Hence when $k$ is too small for a given value of $r$, there is an infinite asymptotic penalty for using the FEM.

We now suppose that $\Lambda = \Lambda^{std}$; i.e., we restrict our attention to standard information. That is, we suppose that the only information available about problem elements is their values at a finite set of points in $I$. Let
\[ FEMQ(\varepsilon) = \inf\{\text{cost}(\varphi_{n,k}^q, N_n^q) : n \text{ is an index such that } e(\varphi_{n,k}^q, N_n^q) \leq \varepsilon\} \]
denote the cost of solving the problem using the FEMQ. We then have the following result, whose proof is immediate from Theorem 5.2 and the results of Section 6.2.

**Theorem 7.2.** Suppose that only standard information is permissible.

(1) \[ COMP(\varepsilon) = \Theta(\varepsilon^{-1/r}) \text{ as } \varepsilon \to 0. \]

(2) If $k \geq r$, then
\[ FEMQ(\varepsilon) = \Theta(\varepsilon^{-1/(r+1)}) \quad \text{as } \varepsilon \to 0. \]

Hence, the FEM is an almost optimal complexity algorithm.

(3) If $k < r + 1$, then
\[ FEMQ(\varepsilon) = \Theta(\varepsilon^{-1/r}) \quad \text{as } \varepsilon \to 0. \]
Hence,

\[
\frac{\text{FEM}(\epsilon)}{\text{COMP}(\epsilon)} = \Theta(\epsilon^{-\lambda_2}) \quad \text{as } \epsilon \to 0,
\]

where

\[
\lambda_2 = \frac{1}{k} - \frac{1}{r} > 0,
\]

so that

\[
\lim_{\epsilon \to 0} \frac{\text{FEM}(\epsilon)}{\text{COMP}(\epsilon)} = \infty.
\]

Again, we see that when \( k \) is too small for a given value of \( r \), there is an infinite asymptotic penalty for using the FEM.

Finally, using Theorems 7.1 and 7.2, we compare the strength of standard information and inner-product information:

**Theorem 7.3.** Let

\[
\lambda_3 = \frac{1}{r} - \frac{1}{r+1} > 0.
\]

Then

\[
\frac{\text{COMP}(\epsilon, \Lambda^{\text{std}})}{\text{COMP}(\epsilon, \Lambda^{\text{ip}})} = \Theta(\epsilon^{-\lambda_3}) \quad \text{as } \epsilon \to 0,
\]

and so

\[
\lim_{\epsilon \to 0} \frac{\text{COMP}(\epsilon, \Lambda^{\text{std}})}{\text{COMP}(\epsilon, \Lambda^{\text{ip}})} = \infty.
\]

Hence the penalty for using standard information instead of inner-product information is unbounded as \( \epsilon \to 0 \).

8. **Numerical Results for the FEMQ**

In this section, we report the results of numerical experiments that confirm theoretical properties of the finite element method with quadrature. The problems considered were of the form
\[-u''(x) + f(x)u(x) = f(x) \quad \text{for } 0 < x < 1,\]
\[u(0) = u'(1) = 0.\]

We considered three sample problems, with \(r = 1\) for simplicity. For Problem 1, we chose

\[f(x) = \frac{2}{2 - 2x + x^2},\]

with exact solution

\[u(x) = x - \frac{1}{2}x^2.\]

For Problem 2, we chose

\[f(x) = \frac{2}{(x + 2)(x + 1)^2},\]

with exact solution

\[u(x) = 1 - \frac{1}{3 + 4 \ln 3 - 4 \ln 2} \left( 2 \frac{x + 2}{x + 1} \ln \frac{3}{x + 2} + x + 3 \right).\]

Finally, for Problem 3, we chose

\[f(x) = \frac{1}{1 - xe^{1-x} + e^{-x}},\]

with exact solution

\[u(x) = \frac{1}{2}(1 - e^x + ex).\]

Note that Problems 2 and 3 are taken from Kacewicz (1989).

Since \(r = 1\), we chose the FEMQ using piecewise linear elements. This FEMQ uses information \(N^q_n\) of cardinality \(n\) given by

\[N^q_n f = \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix},\]

with

\[x_i = (i - \frac{1}{2})h \quad (1 \leq i \leq n)\]
A PROBLEM OF OPTIMAL CONTROL

and $h = 1/n$. The matrix $A$ is given as follows. For the first row of $A$, we have

$$
a_{1j} = \begin{cases} 
2 + \frac{1}{2}h^2(f(x_1) + f(x_2)) & \text{for } j = 1, \\
-1 + \frac{1}{2}h^2f(x_2) & \text{for } j = 2, \\
0 & \text{otherwise}.
\end{cases}
$$

For the intermediate rows 2 through $n - 1$ of $A$, we have

$$
a_{ij} = \begin{cases} 
-1 + \frac{1}{2}h^2f(x_i) & \text{for } j = i - 1, \\
2 + \frac{1}{2}h^2(f(x_i) + f(x_{i+1})) & \text{for } j = i, \\
-1 + \frac{1}{2}h^2f(x_{i+1}) & \text{for } j = i + 1, \\
0 & \text{otherwise}.
\end{cases}
$$

For the $n$th row of $A$, we have

$$
a_{nj} = \begin{cases} 
-1 + \frac{1}{2}h^2f(x_n) & \text{for } j = n - 1, \\
1 + \frac{1}{2}h^2f(x_n) & \text{for } j = n, \\
0 & \text{otherwise}.
\end{cases}
$$

The vector $b$ is given by

$$
b_i = \begin{cases} 
\frac{1}{2}h^2(f(x_i) + f(x_{i+1})) & \text{for } i \neq n, \\
\frac{1}{2}h^2f(x_n) & \text{for } i = n.
\end{cases}
$$

The program that tested these problems was written in C and run on a VAX 11/750 operating under 4.3BSD UNIX. Double-precision arithmetic was used, with a precision of roughly 17 decimal digits. The test problems were run with the number $n$ of mesh points varying from 10 to 10,000. Table I gives, for each $n$, the value of $n$ multiplied by the $H^1$-error, as well as the value of $n^2$ multiplied by the $L^2$-error.

As predicted by the results of this paper, the $H^1(I)$-error is $\Theta(n^{-1})$. In addition, we see that the $L^2(I)$-error appears to be $\Theta(n^{-2})$. Although we do not establish the latter result in this paper, it is reasonable, in light of well-known results for the finite element method for linear problems; see Ciarlet (1978, Chap. 4) for further discussion. Also, note that these results compare favorably with those in Kacewicz (1989).

1 UNIX is a trademark of AT&T Bell Laboratories.
TABLE I

<table>
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<tr>
<th>n</th>
<th>Problem 1</th>
<th>Problem 2</th>
<th>Problem 3</th>
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<td>$n^2 \cdot \text{err}_{L_n}$</td>
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APPENDIX: PROOF OF THEOREMS 5.1 AND 5.2

We first use ideas of Wasilkowski (1985) to show how our (nonlinear) optimal control problem may be reduced to a simpler linear problem. Using this reduction, we will find that the $n$th minimal errors for the two problems are roughly the same. Then, we prove Theorem 5.1 by applying this result to inner-product information and prove Theorem 5.2 by considering standard information.

In this appendix, we shall assume that $M = 1$ and $T = 1$. This is no serious loss of generality, and it will simplify some of the formulas in what follows.

A. Reduction to a Simpler Problem

The description of the new problem will require a slightly nonstandard Hilbert space, namely the dual space $\mathcal{H}_E^*$ of the energy space $\mathcal{H}_E$ mentioned previously. This space $\mathcal{H}_E^*$ is a Hilbert space under the norm $\|\cdot\|_{H^{-1}(I)}$ which is defined by

$$
\|f\|_{H^{-1}(I)} = \sup_{g \in \mathcal{H}_E} \frac{|(f, g)|}{\|g\|_{H^{1}(I)}} \quad \forall f \in \mathcal{H}_E^*.
$$

(See, e.g., Babuška and Aziz (1972) and Oden and Reddy (1976) for further discussion of dual norms.)

We now describe our new problem. First, we give our new class $\tilde{F}_r$ of problem elements (which is analogous to $F_r$ in Section 2). Suppose first
that $r \geq 1$. Then by the Sobolev embedding theorem, there is a positive constant $\zeta$ such that

$$\| \cdot \|_{C(I)} \leq \zeta \| \cdot \|_{H^r(I)}.$$ 

Let

$$\eta = \min\{\frac{1}{2}, 1/2\zeta\}.$$ 

Then we choose

$$\tilde{F}_r = \{f \in H^r(I) : \|f\|_{H^r(I)} \leq \eta\}$$

as our new set of problem elements. In the case $r = 0$, we choose

$$\tilde{F}_0 = \{f \in L^2(I) : \|f\|_{H^0(I)} \leq \frac{1}{2}\}$$

as our new set of problem elements.

Our new problem will then be to approximate $f \in \tilde{F}_r$ in the norm of $\mathcal{H}^r_{\omega}$. This means that the solution operator $S : F_r \rightarrow \mathcal{H}_{\omega}$ described in Section 2 is now replaced by the embedding $E : H^r(I) \rightarrow \mathcal{H}^r_{\omega}$. Note that this embedding is a compact linear operator (see Ciarlet, 1978, p. 114). We shall refer to this new problem as the embedding problem.

To do this, we must know something about each $f$. Let $\Lambda$ denote a class of permissible linear functionals. As in Section 3, information $N$ of cardinality $n$ is defined by

$$Nf = \begin{bmatrix} \lambda_1(f) \\ \vdots \\ \lambda_n(f) \end{bmatrix},$$

where $\lambda_1, \ldots, \lambda_n \in \Lambda$ are permissible. Following the ideas of Section 3, an algorithm $\hat{\phi}$ using information $N$ is a mapping $\hat{\phi} : N(\tilde{F}_r) \rightarrow \mathcal{H}^r_{\omega}$. (The tilde will serve to distinguish between quantities defined for the control problem and those defined for the embedding problem.) The error of an algorithm $\hat{\phi}$ using $N$ is now given by

$$\hat{e}(\hat{\phi}, N) = \sup_{f \in \tilde{F}_r} \|f - \hat{\phi}(Nf)\|_{H^{-1}(I)}.$$ 

Once again, the minimal error among all algorithms using information $N$ is given by the radius of information.
Optimal error algorithms are defined as in Section 3.

Of course, we are interested in finding the best possible information for the embedding problem. The \( n \)th minimal radius for the embedding problem is given by

\[
\tilde{r}(n, \Lambda) = \inf_{N} \tilde{r}(N),
\]

the infimum being over all information \( N \) consisting of at most \( n \) linear functionals from \( \Lambda \). Optimal information of given cardinality remains as defined in Section 3, as do \( n \)th minimal error algorithms.

We now wish to relate optimal informations and minimal radii for the control and embedding problems. Let

\[
\gamma_1 = \frac{1}{2} \pi^2 \sqrt{2}
\]

and

\[
\gamma_2 = \beta_1^2 [1 + \sqrt{3} \max\{\beta_1, \beta_2\}],
\]

where

\[
\alpha = \frac{2e^{1/\sqrt{2}}}{1 + e^{1/\sqrt{2}}},
\]

\[
\beta_1 = \sqrt{1 + 4/\pi^2},
\]

and

\[
\beta_2 = 1 + \beta_1.
\]

The main result of this subsection (whose proof we momentarily postpone) is

**Theorem A.1.** Let \( n \) be a positive integer, and let \( \Lambda \) be a class of permissible linear functionals.

1. For any \( \Lambda \)-information \( N \) of cardinality \( n \),

\[
\gamma_1 \tilde{r}(N) \leq r(N) \leq \gamma_2 \tilde{r}(N).
\]

2. The \( n \)th minimal radii for the control problem and the embedding problem are roughly the same, i.e.,
Before we can give the proof of Theorem A.1, we will need the following

**Lemma A.1.** For any \( f \in \mathcal{F}, \)

1. \( \| (Sf)' \|_{L_2(I)} \leq \beta_1 \| f \|_{H^{-1}(I)} \leq \beta_1, \)
2. \( \| Sf \|_{L_2(I)} \leq \beta_1, \)
3. \( \| (Sf)' \|_{L_2(I)} \leq \beta_2 \| f \|_{L_2(I)} \leq \beta_2, \) and
4. \( \| (Sf)' \|_{L_2(I)} \leq \beta_2. \)

**Proof.** Let \( f \in \mathcal{F}, \) and \( u = Sf. \) To prove part (1), replace \( v \) in (2.1) by \( u. \) Since \( f \geq 0, \) we then find

\[
\| u' \|_{L_2(I)} \leq B(u, u) = (f, u) \leq \| f \|_{H^{-1}(I)} \| u \|_{H(I)} \leq \sqrt{1 + 4T^2/n^2} \| f \|_{H^{-1}(I)} \| u' \|_{L_2(I)},
\]

the last line following from Poincaré's inequality (6.1). The second inequality in (1) now holds because \( f \in \mathcal{F}, \) implies that \( \| f \|_{H^{-1}(I)} \leq 1. \)

Part (2) follows immediately by using part (1) and the Sobolev inequality

\[
\| u \|_{L_2(I)} \leq \| u' \|_{L_2(I)},
\]

which holds because \( u(0) = 0 \) (see Strang and Fix, 1973, Chap. 1).

We now turn to part (3). From the differential equation for \( u, \) we have

\[
u'' = f(u - 1),
\]

so that

\[
\| u'' \|_{L_2(I)} \leq (1 + \| u \|_{L_2(I)}) \| f \|_{L_2(I)}.
\]

Now use part (2) to prove the first inequality in (1). The second inequality follows because \( f \in \mathcal{F}, \) implies that \( \| f \|_{L_2(I)} \leq 1. \)

Finally, part (4) follows by using part (3) and the Sobolev inequality

\[
\| u' \|_{L_2(I)} \leq \| u'' \|_{L_2(I)},
\]

which holds because \( u'(1) = 0. \)

We are now ready for the

**Proof of Theorem A.1.** It suffices to prove (1), since (2) follows immediately from (1) and Traub and Woźniakowski (1980, Theorem 2.7.1). Let
Let $f^*$ be the constant function

$$f^*(x) = \frac{1}{2} \quad \forall x \in I.$$  

From the definition of $\tilde{F}_r$ and $f^*$, it follows that

$$f^* + h \in F \quad \forall h \in \tilde{F}_r. \quad (A.1)$$

Suppose we can establish

$$\|S(f^* + h) - Sf^*\|_{H^1(I)} \geq \frac{1}{2} \alpha^2 \sqrt{2}\|h\|_{H^{-1}(I)} \quad \forall h \in \tilde{F}_r. \quad (A.2)$$

Using property (A.1) and inequality (A.2), the desired lower bound on $r(N)$ will follow immediately from Wasilkowski (1985, Lemma 3.1).

In what follows, it will be useful to let $u^* = Sf^*$, i.e.,

$$u^*(x) = 1 - \frac{e^{\sqrt{2}e^{-x\sqrt{2}}} + e^{x\sqrt{2}}}{1 + e^{\sqrt{2}}} \quad \forall x \in I.$$  

Then we have

$$0 < \alpha \leq 1 - u^*(x) \leq 1 \quad \forall x \in I. \quad (A.3)$$

Now let $h \in \tilde{F}_r$; we must prove (A.2). Let

$$e = S(f^* + h) - Sf^*.$$  

Subtracting (2.2) with $u$ replaced by $u^* = Sf^*$ and $f$ replaced by $f^*$ from (2.2) with $u$ replaced by $S(f^* + h)$ and $f$ replaced by $f^* + h$, we find that

$$\int_0^1 (e'u' + (f^* + h)e) v = \int_0^1 (1 - u^*)hv \quad \forall v \in \mathcal{H}_E. \quad (A.4)$$

Since $h \in \mathcal{H}_E^k$, the Lax–Milgram theorem implies that there exists a unique $w \in \mathcal{H}_E$ such that

$$\int_0^1 (w'g' + wg) = \int_0^1 hg \quad \forall g \in \mathcal{H}_E.$$
Moreover, it is straightforward to check that
\[
\|w\|_{H^1(I)} = \|h\|_{H^{-1}(I)},
\]
and so
\[
\|h\|_{H^{-1}(I)} \|w\|_{H^1(I)} = \|w\|_{H^1(I)}^2 = \int_0^1 ((w')^2 + w^2) = \int_0^1 hw \quad (A.5)
\]
\[
= \int_0^1 \left( (1 - u^*)h \left( \frac{w}{1 - u^*} \right) \right).
\]

Now let \( v = w/(1 - u^*) \) in (A.4). Note that (A.3) implies that this replacement is legitimate; i.e., the function \( w/(1 - u^*) \) is in \( \mathcal{H}_E \). Since \( h \in \tilde{F}_r \) implies that \( 0 \leq f^* + h \leq 1 \), we may use the Cauchy–Schwarz inequality to find that
\[
\int_0^1 \left[ (1 - u^*)h \left( \frac{w}{1 - u^*} \right) \right] = \int_0^1 \left[ e' \left( \frac{w}{1 - u^*} \right) + (f^* + h)e \left( \frac{w}{1 - u^*} \right) \right] \leq \|e\|_{H^1(I)} \left\| \frac{w}{1 - u^*} \right\|_{H^1(I)}. \quad (A.6)
\]

We now estimate the term \( \|w/(1 - u^*)\|_{H^1(I)} \) appearing in (A.6). Clearly, (A.3) yields that
\[
\int_0^1 \left[ \frac{w}{1 - u^*} \right]^2 \leq \frac{1}{\alpha^2} \int_0^1 w^2. \quad (A.7)
\]

Using the quotient rule for differentiation and the triangle inequality, as well as the explicit formula (A.2) for \( u^* \) and the bound (A.3), we find that
\[
\left| \left( \frac{w}{1 - u^*} \right)' \right| \leq \frac{1}{1 - u^*} |w'| + \frac{1}{(1 - u^*)^2} |w| |(1 - u^*)'| \leq \frac{1}{\alpha} |w'| + \frac{1}{\sqrt{2\alpha^2}} e^{\sqrt{2}} - 1 |w| \leq \frac{1}{\sqrt{2\alpha^2}} (|w| + |w'|).
\]

Since \( (a + b)^2 \leq 2(a^2 + b^2) \), this implies that
which (when integrated) implies that

$$\int_0^1 \left| \left( \frac{w}{1 - u^*} \right)' \right|^2 \leq \frac{1}{\alpha^4} \left| w \right|_{H^1(\Omega)}^2. \tag{A.8}$$

Since \( \alpha \leq 1 \), we may use (A.7) and (A.8) to find that

$$\left| \frac{w}{1 - u^*} \right|_{H^1(\Omega)}^2 \leq \frac{2}{\alpha^4} \left| w \right|_{H^1(\Omega)}^2. \tag{A.9}$$

Finally, combining (A.5) and (A.9), we have

$$\| h \|_{H^{-1}(\Omega)} \left| w \right|_{H^1(\Omega)} \leq \| e \|_{H^1(\Omega)} \left| \frac{w}{1 - u^*} \right|_{H^1(\Omega)} \leq \frac{\sqrt{2}}{\alpha^2} \left| e \right|_{H^1(\Omega)} \left| w \right|_{H^1(\Omega)},$$

so that

$$\| e \|_{H^1(\Omega)} \geq \frac{1}{2} \alpha^2 \sqrt{2} \| h \|_{H^{-1}(\Omega)},$$

which establishes (A.2), completing the proof of the lower bound (A.1).

(ii) We now turn to the proof of the upper bound. Let \( f_1, f_2 \in F \). From Wasilkowski (1985, Lemma 3.2), it suffices to show that

$$\| h \|_{H^{-1}(\Omega)} \leq \gamma_2 \| f_1 - f_2 \|_{H^1(\Omega)}. \tag{A.10}$$

Set \( u_1 = S f_1 \) and \( u_2 = S f_2 \). Let \( h = f_1 - f_2 \) and \( e = u_1 - u_2 \). From (2.1), we find that

$$\int_0^1 [e'v' + f_1 ev] = \int_0^1 [(1 - u_2)hv] \quad \forall v \in \mathcal{H}_E.$$

Set \( v = e \) in this equation. Since \( f_1 \succeq 0 \), we find that

$$\| e' \|_{L^2(\Omega)}^2 \leq \int_0^1 [(e')^2 + f_1 e^2] = \int_0^1 [(1 - u_2)he].$$

Using Poincaré's inequality and the duality of the norms \( \| \cdot \|_{H^1(\Omega)} \) and \( \| \cdot \|_{H^{-1}(\Omega)} \), we find
\[ \|e\|_{H(U)}^2 \leq \left(1 + \frac{4}{\pi^2}\right) \|e'\|_{L^2(U)}^2 \leq \left(1 + \frac{4}{\pi^2}\right) \int_0^1 [(1 - u_2)h] \]
\[ \leq \left(1 + \frac{4}{\pi^2}\right) \|h\|_{H^{-1}(U)} \|(1 - u_2)e\|_{H(U)} \]
\[ \leq \left(1 + \frac{4}{\pi^2}\right) \|h\|_{H^{-1}(U)} \|[e\|_{H(U)} + \|u_2e\|_{H(U)}\]. \quad (A.11) \]

Setting
\[ M = \max\{\|u_2\|_{L^2}, \|u_2'\|_{L^2}\}, \quad (A.12) \]
we find that
\[ \|u_2e\|_{H(U)} = \int_0^1 [u_2^2e^2 + (u_2e' + u_2'^2)^2] \]
\[ \leq \int_0^1 [(u_2^2 + 2(u_2^2)^2)e^2 + 2u_2^2(e')^2] \]
\[ \leq 3M^2\|e\|_{H(U)}^2. \quad (A.13) \]

The desired bound (A.10) follows immediately from (A.11)–(A.13), Lemma A.1, and the definition of \( \gamma_2 \). \[ \blacksquare \]

B. Proof of Theorem 5.1

We divide the proof of Theorem 5.1 into three sections. First, we determine the \( n \)th minimal radius \( r(n, \Lambda^p) \). Next, we determine the error of the FEM. Finally, we compute the radius of FEI.

**THEOREM B.1.** \( r(n, \Lambda^p) = \Theta(n^{-(r+1)}) \) as \( n \to \infty \).

**Proof.** Using Theorem A.1, we have
\[ r(n, \Lambda^p) = \Theta(\hat{r}(n, \Lambda^p)) \quad \text{as} \quad n \to \infty. \]

But Theorem 6.1 of Traub and Woźniakowski (1980, Chap. 2) yields that
\[ \hat{r}(n, \Lambda^p) = d^n(\hat{F}, \mathcal{H}^+_E), \]
where \( d^n \) denotes the Gelfand \( n \)-width. Suppose first that \( r \geq 1 \). Using results of Babuška and Aziz (1972, Theorem 2.5.1) and Pinkus (1985, Theorems IV.2.2, VII.1.1), it is easy to see that
\[ d^n(\hat{F}, \mathcal{H}^+_E) = \Theta(n^{-(r+1)}) \quad \text{as} \quad n \to \infty. \]
Now suppose that \( r = 0 \). Using differentiation and duality, one may check that

\[
d^n(\mathcal{F}_0, \mathcal{H}_E^*) = \Theta(d^n(\mathcal{B} W^{1,\infty}(I), L_2(I))) \quad \text{as } n \to \infty,
\]

where \( \mathcal{B}X \) denotes the unit ball of the normed linear space \( X \). From Pinkus (1985, Theorem VII.2.2), we have the lower bound

\[
d^n(\mathcal{B} W^{1,\infty}(I), L_2(I)) = \Omega(n^{-1}) \quad \text{as } n \to \infty,
\]

whereas the continuous embedding of \( W^{1,\infty}(I) \) in \( H^1(I) \), Pinkus (1985), Theorem IV.2.2, and Jerome (1968) yield

\[
d^n(\mathcal{B} W^{1,\infty}(I), L_2(I)) = O(d^n(\mathcal{B} H^1(I), L_2(I))) = \Theta(n^{-1}) \quad \text{as } n \to \infty.
\]

So,

\[
d^n(\mathcal{F}_0, \mathcal{H}_E^*) = \Theta(d^n(\mathcal{B} W^{1,\infty}(I), L_2(I))) = \Theta(n^{-1}) \quad \text{as } n \to \infty,
\]

completing the proof of the theorem. \( \blacksquare \)

Recall that \( N_n \) is finite element information of cardinality \( n \), based on the finite element space \( \mathcal{S}_m \) of degree \( k \), and that \( \varphi_{n,k} \) is the finite element method using \( N_n \). We next determine the error \( e(\varphi_{n,k}, N_n) \) of the finite element method of degree \( k \).

**Theorem B.2.** Let

\[
\mu = \min\{k, r + 1\}.
\]

Then

\[
e(\varphi_{n,k}, N_n) = \Theta(n^{-\mu}) \quad \text{as } n \to \infty.
\]

**Proof.** We first establish the upper bound

\[
e(\varphi_{n,k}, N_n) = O(n^{-\mu}) \quad \text{as } n \to \infty. \quad \text{(B.1)}
\]

Let \( f \in F_r \). From Schultz (1973, Theorem 7.20), we have

\[
\|Sf - \varphi_{n,k}(N_n f)\|_{H^1(I)} \leq \sqrt{1 + \frac{4}{\alpha^2}} \inf_{s \in \mathcal{S}_m} \|Sf - s\|_{H^1(I)}.
\]
The results of Babuška and Aziz (1972, Chap. 4) show that there exists a positive constant $C_1$, independent of $f$, such that

$$\inf_{s \in \mathcal{S}_m} \|Sf - s\|_{H^1(I)} \leq C_1 m^{-\mu} \|Sf\|_{H^{1+\mu}(I)}.$$ 

Since $m = \Theta(n)$ as $n \to \infty$, this implies that there exists a positive constant $C_2$, independent of $f$, such that

$$\inf_{s \in \mathcal{S}_m} \|Sf - s\|_{H^1(I)} \leq C_2 n^{-\mu} \|Sf\|_{H^{1+\mu}(I)}.$$ 

From the definition of $F$, as a bounded subset of $H^s(I)$, it is easy to check that there is a positive constant $C_3$, independent of $f$, for which

$$\|Sf\|_{H^{1+\mu}(I)} \leq C_3 \quad \forall f \in F.$$ 

(The proof is similar to that of (i) and (ii) in Lemma A.1.) Hence, we find that

$$e(\varphi_{n,k}, N_n) = \sup_{f \in F_r} \|Sf - \varphi_{n,k}(N_n f)\|_{H^1(I)} \leq \sqrt{1 + \frac{4}{\pi^2}} C_2 C_3 n^{-\mu},$$

establishing the upper bound (B.1).

To prove the lower bound

$$e(\varphi_{n,k}, N_n) = \Omega(n^{-\mu}) \quad \text{as } n \to \infty,$$ 

we consider two cases.

(i) Suppose first that $k \geq r + 1$. Then $\mu = r + 1$ and so we have

$$e(\varphi_{n,k}, N_n) \geq r(N_n) \geq r(n, \Lambda^p) = \Theta(n^{-(r+1)}) \quad \text{as } n \to \infty,$$

establishing the lower bound (B.3) for the case $k \geq r + 1$.

(ii) We now consider the remaining case $k < r + 1$. Since $k \geq 1$ must hold for the finite element method to be conforming (see, e.g., Werschulz, 1987), this implies that $r \geq 1$ in what follows. To establish the lower bound (B.3), it now suffices to show that

$$e(\varphi_{n,k}, N_n) = \Omega(n^{-k}) \quad \text{as } n \to \infty.$$ 

For $\alpha \in (0, 1/k)$, let

$$u_\alpha(x) = \alpha((k + 1)x - x^{k+1})$$
and

\[ f_\alpha(x) = \frac{\alpha k(k + 1)x^{k-1}}{1 - \alpha((k + 1)x - x^{k+1})}. \]

Then

\[ -u''_\alpha + f_\alpha u_\alpha = f_\alpha \quad \text{in } (0, T), \]

with

\[ u_\alpha(0) = u_\alpha'(1) = 0. \]

Using the Leibniz rule, it is easy to see that

\[ \lim_{\alpha \to 0} f^{(j)}_\alpha(x) = 0 \quad (0 \leq j \leq r) \]

uniformly for \( x \in [0, 1] \). Choose \( \alpha > 0 \) such that

\[ |f^{(j)}_\alpha(x)| \leq \sqrt{1/r} \quad (0 \leq x \leq 1, 0 \leq j \leq r). \]

Then

\[ \|f_\alpha\|_{H^r(\Omega)} \leq 1. \]

Since

\[ f_\alpha(x) \geq 0 \quad (0 \leq x \leq 1), \]

we see that \( f_\alpha \in F \), and that \( u_\alpha = Sf_\alpha \).

Now let

\[ u_1(x) = (k + 1)x \quad \text{and} \quad u_2(x) = x^{k+1}, \]

so that

\[ u_\alpha = \alpha(u_1 - u_2). \]

Define \( P: \mathcal{H}_E \to \mathcal{S}_m \) by

\[ \|v - Pv\|_{H^r(\Omega)} = \inf_{s \in \mathcal{S}_m} \|v - s\|_{H^r(\Omega)}. \]
Since $P$ is a linear projection,

$$Pu_\alpha = \alpha Pu_1 + \alpha Pu_2.$$ 

But $u_1 \in S_m$ implies that $u_1 = Pu_1$. Hence

$$u_\alpha - Pu_\alpha = -\alpha(u_2 - Pu_2).$$

Since $\varphi_{n,k}(N_n f_\alpha) \in S_m$, the minimum properties of the projector $P$ yield that

$$\|Sf_\alpha - \varphi_{n,k}(N_n f_\alpha)\|_{H^1(I)} \geq \|Sf_\alpha - PSf_\alpha\|_{H^1(I)} = |\alpha| \|u_2 - Pu_2\|_{H^1(I)}.$$ 

From the proof of Werschulz (1986, Theorem 4.2(i)), there exists a positive constant $C_1$ (depending only on $k$ and $T$), such that

$$\|u_2 - Pu_2\|_{H^1(I)} \geq C_1 m^{-k}.$$ 

Since $m = \Theta(n)$ as $n \to \infty$, this implies that there exists a positive constant $C_2$ such that

$$\|u_2 - Pu_2\|_{H^1(I)} \geq C_2 n^{-k}.$$ 

Combining the previous inequalities and using the definition of the error of an algorithm, we find that

$$e(\varphi_{n,k}, N_n) \geq \|Sf_\alpha - \varphi_{n,k}(N_n f_\alpha)\|_{H^1(I)} \geq C_2|\alpha|n^{-k},$$

establishing the required lower bound (B.4), and completing the proof of the theorem. □

Finally, we compute the radius of $\text{FEI}$.

THEOREM B.3. \quad $r(N_n) = \Theta(n^{-(r+1)})$ as $n \to \infty$.

Proof. From Theorem B.1, we already have the lower bound

$$r(N_n) \geq r(n, \Lambda^p) = \Theta(n^{-(r+1)}) \quad \text{as } n \to \infty.$$ 

It remains only to establish the upper bound

$$r(N_n) = O(n^{-(r+1)}) \quad \text{as } n \to \infty.$$ 

We do this by exhibiting an algorithm $\varphi^*$ using $N_n$ such that

$$e(\varphi^*, N_n) \leq C\gamma_2 n^{-(r+1)},$$
where $\gamma_2$ is defined above and $C$ is a positive constant that is independent of $n$.

Let

$$\hat{F}_r = \{ f \in H^r(I) : \| f \|_{H^r(I)} \leq 1 \}. $$

Let $\varphi^*$ be the spline algorithm for approximating the embedding $E : H^r(I) \rightarrow \mathcal{H}^*_E$ for problem elements in $\hat{F}_r$, using the finite element information $N_n$. Recalling the notational conventions of Section 2 and using the results of Traub and Woźniakowski (1980, Chaps. 2, 4), we find

$$\bar{e}(\varphi^*, N_n) = \sup_{f \in \hat{F}_r} \| f - \varphi^*(N_n f) \|_{H^{-r}(I)} = \bar{r}(N_n)$$

We claim that there exists a positive constant $C$, independent of $n$, such that

$$\bar{e}(\varphi^*, N_n) \leq C n^{-r+1}. \quad (B.6)$$

Indeed, let $h \in \hat{F}_r \cap \text{ker } N_n$. By the Lax–Milgram theorem, there exists a unique $w \in \mathcal{H}_E$ satisfying

$$\int_0^1 (w' g' + wg) = \int_0^1 hg \quad \forall g \in \mathcal{H}_E.$$ 

Then

$$\| h \|_{H^{-r}(I)} = \| w \|_{H^r(I)},$$

and so

$$\| h \|_{H^{-r}(I)} \| w \|_{H^r(I)} = \| w \|_{\mathcal{H}_E} = \int_0^1 hw.$$ 

Since $n = \Theta(m)$, we may use Theorem 4.11 of Babuška and Aziz (1972) to see that there exists a positive constant $C$, independent of $n$ and $h$, as well as $s \in S_m$, such that

$$\| w - s \|_{H^{-r}(I)} \leq C n^{-r+1} \| w \|_{H^r(I)}.$$
Since \( s \in S_m \) and \( h \in \ker N_n \cap \hat{F}_r \), we have

\[
\int_0^1 h w = \int_0^1 h(w - s) \leq \|h\|_{H^1(I)} \|w\|_{H^{-1}(I)} \leq \|w - s\|_{H^{-1}(I)}.
\]

Combining these last three results, we find

\[
\|h\|_{H^{-1}(I)} \leq C n^{-(r+1)}.
\]

Since \( h \in \hat{F}_r \cap \ker N_n \) is arbitrary, the desired result (B.6) follows immediately from this inequality and (B.5).

We now consider the spline algorithm \( \varphi^* \) given by

\[
\varphi^*(N_n f) = S \varphi^*(N_n f) \quad \forall f \in F_r.
\]

Since \( F_r \subseteq \hat{F}_r \), we may use (A.9) and (B.6) to find that

\[
e(\varphi^*, N_n) = \sup_{f \in F_r} \|S f - S \varphi^*(N_n f)\|_{H^1(I)} \leq \gamma_2 \sup_{f \in F_r} \|f - \varphi^*(N_n f)\|_{H^{-1}(I)}
\]

\[
= \gamma_2 \theta(\varphi^*, N_n) \leq C \gamma_2 n^{-(r+1)},
\]

completing the proof of the theorem. ■

Thus FEI is always quasi-optimal information, while the FEM has quasi-minimal error iff \( k \geq r + 1 \). Note, however, that (unlike the FEM) the spline algorithm \( \varphi^* \) appears to be difficult to implement. It requires the exact solution of the problem (2.1), with right-hand-side \( f \) replaced by the spline element \( \varphi^*(N_n f) \) interpolating \( f \). So, we suspect that it may be difficult to find a useful quasi-minimal error algorithm using FEI of degree \( k \), when \( k < r + 1 \).

C. Proof of Theorem 5.2

We divide the proof of Theorem 5.2 into two sections. First, we show a lower bound on the \( n \)th minimal radius of standard information. Then, we show that this lower bound is sharp. To do this, we consider the cases \( r = 0 \) and \( r \geq 1 \) separately. The proof for the case \( r = 0 \) consists of showing optimality of the zero algorithm; the proof for the case \( r \geq 1 \) follows immediately once we know that the error of the FEMQ is \( O(n^{-r}) \) when \( k \geq r \).

**Theorem C.1.** \( r(n, \Lambda^{\text{std}}) = \Omega(n^{-r}) \) as \( n \to \infty \).

**Proof.** By Theorem A.1, it suffices to show that

\[
\tilde{r}(n, \Lambda^{\text{std}}) = \Omega(n^{-r}) \quad \text{as} \quad n \to \infty.
\]
Let standard information \( N \) of cardinality \( n \) be given by

\[
Nf = \begin{bmatrix}
  f(x_1) \\
  \vdots \\
  f(x_n)
\end{bmatrix} \quad \forall f \in \tilde{F}_r.
\]

Let \( I' = [\frac{1}{10}T, \frac{4}{10}T] \). Let \( l \) denote the number of sample points \( x_1, \ldots, x_n \) which actually lie in the interval \( I' \). Without loss of generality, we may assume that \( x_1, \ldots, x_l \in I' \).

We show that there exists a nonzero function \( h \in \tilde{F}_r \cap \ker N \) such that

\[
\int_{I'} h(x) \, dx \geq Cl^{-r}.
\]  

(C.1)

First suppose that \( r = 0 \). Define \( h \in L_\infty(I') \) by

\[
h(x) = \begin{cases}
0 & \text{for } x \in \{x_1, \ldots, x_l\} \\
\frac{1}{l} & \text{otherwise}
\end{cases}
\]

Extend \( h \) from \( I' \) to all of \( I \) by letting \( h \) be zero outside \( I' \). Then \( h \in \tilde{F}_0 \cap \ker N \), with

\[
\int_{I'} h(x) \, dx = \frac{1}{l}.
\]

Since \( r = 0 \), this yields (C.1) with \( C = \frac{1}{l} \). We now consider the case \( r \geq 1 \).

By Poincaré's inequality, there exists a positive constant \( \kappa \) such that

\[
\kappa \|v(r)\|_{L_2(I')} \leq \|v\|_{H^r(I')} \quad \forall v \in H^r_0(I').
\]

Recall the definition of the constant \( \eta \) above. From Bakhvalov (1977, pp. 301–304), there exists a positive constant \( C \) (independent of \( n \) and \( l \)) and a nonnegative function \( h \in H^r_0(I') \) such that

\[
\|h^{(r)}\|_{L_2(I')} \leq \kappa \eta,
\]

\[
h(x_1) = \cdots = h(x_l) = 0,
\]

and

\[
\int_{I'} h(x) \, dx \geq Cl^{-r}.
\]
Extending \( h \) from \( I' \) to \( I \) by letting \( h \) be zero outside \( I' \), we find that \( h \in \ker N \). Moreover,

\[
\|h\|_{H(I')} = \|h\|_{H(I')} \leq \frac{1}{\kappa} \|h^{(r)}\|_{L^2(I)} \leq \eta;
\]

i.e., \( h \in \tilde{F}_r \cap \ker N \) is the desired function satisfying (C.1).

Choose a function \( v \in H_E \) such that

\[ v = 1 \quad \text{on } I'. \]

Since \( v = 1 \) on the support \( I' \) of \( h \) and \( l \leq n \), (C.1) implies that

\[
\|h\|_{H^{-l}(I)} = \int_I h(x) v(x) \, dx = \int_I h(x) \, dx \geq C l^{-r} \geq C n^{-r},
\]

and so

\[
\|h\|_{H^{-l}(I)} \geq \frac{C}{\|v\|_{H^l(I)}} n^{-r}.
\]

Since \( h \in \tilde{F}_r \cap \ker N \), the results of Traub and Woźniakowski (1980, Chap. 2) yield that

\[
\bar{r}(N) \geq \frac{C}{\|v\|_{H^l(I)}} n^{-r}.
\]

Since \( N \) is arbitrary standard information of cardinality \( n \), the desired result follows immediately. ■

In the remainder of this section, we show that the lower bound of Theorem C.1 is sharp, i.e., that there exists an algorithm using standard information of cardinality \( n \) whose error is proportional to \( n^{-r} \).

We first deal with the case \( r = 0 \). Recalling the definition of \( \gamma_2 \) from Section 3, we have

**Theorem C.2.** Let \( r = 0 \). For any standard information \( N \) of cardinality \( n \), let \( \varphi_0 \) be the zero algorithm

\[
\varphi_0(Nf) = 0 \quad \forall f \in F_0.
\]

Then

\[
e(\varphi_0, Nf) \leq \gamma_2.
\]
Thus, the zero algorithm is a quasi-minimal error algorithm using standard information when \( r = 0 \).

**Proof.** Let \( f \in F_0 \). Using (A.10), we have

\[
\| Sf - \varphi_0(N_n f) \|_{H(I)} = \| Sf \|_{H(I)} \leq \gamma_2 \| f \|_{H^{-1}(I)} \leq \gamma_2,
\]

from which

\[
e(\varphi_0, Nf) \leq \gamma_2
\]

follows immediately. Thus the zero algorithm is an algorithm using standard information whose error is bounded from above by the constant \( \gamma_2 \). Since Theorem C.1 implies that the error of any algorithm using standard information is bounded from below by a constant that is independent of the algorithm and the cardinality of the information, the zero algorithm is a quasi-minimal error algorithm using standard information. \( \blacksquare \)

Since we have disposed of the case \( r = 0 \), we need only consider the case \( r \geq 1 \). Let \( k \) be a positive integer. Recall that \( \varphi_{a,k}^q \) is the finite element method with quadrature (FEMQ) of degree \( k \) using finite element quadrature information \( N_{n}^q \) of cardinality \( n \).

The remainder of this section will be devoted to establishing

**Theorem C.3.** Let \( r \geq 1 \). If \( k \geq r \), then

\[
e(\varphi_{a,k}^q, N_{n}^q) = \Theta(n^{-r}) \quad \text{as } n \to \infty.
\]

Before proving Theorem C.3, we need to establish an auxiliary lemma. Recall the notation of Section 4. For \( 0 < a < l - 1 \), let \( I_a = [\xi_a, \xi_{a+1}] \) denote the \( a \)-th subinterval of the partition \( \mathcal{F}_I \), and let \( Q_{la} \) denote Gauss quadrature (using \( k \) nodes) on \( I_a \), so that

\[
Q_{a}(g) = \sum_{a=0}^{l-1} Q_{la}(g) \quad \forall g : I \to \mathbb{R}.
\]

Let \( E_l \) denote the error in the piecewise Gauss quadrature rule \( Q_l \) over \( I \), i.e.,

\[
E_l(g) = \int_I g(x) \, dx - Q_l(g) \quad \forall g : I \to \mathbb{R}.
\]

and (for \( 1 \leq a \leq l \)) let \( E_{la} \) denote the error in the Gauss quadrature rule \( Q_{la} \) over \( I_a \), i.e.,
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\[ E_{lu}(g) = \int_{I_a} g(x) \, dx - Q_{lu}(g) \quad \forall g : I_a \to \mathbb{R}. \]

**Lemma C.1.** Let \( k \geq r \). There exists a positive constant \( C \), independent of \( n \), such that

\[ \| f \|_{H^r(I_a)} \| \nabla \|_{L^2(I_a)} \| w \|_{L^2(I_a)} \]

for \( 0 \leq a \leq k - 1 \).

**Proof.** We use the notation of Ciarlet (1978, Sect. 4.1). Let \( \tilde{I} \) denote the reference interval \([-1, 1]\). Define (for \( 0 \leq a \leq l - 1 \)) an affine bijection \( B_a : \tilde{I} \to I_a \) by

\[ B_a(\tilde{\xi}) = \frac{\xi_{a+1} - \xi_a}{2} (\tilde{\xi} + 1) + \xi_a \quad \forall \tilde{\xi} \in \tilde{I}. \]

Then

\[ E_{lu}(fsw) = (\det B_a) \hat{E}(\tilde{f}\tilde{w}). \]  

(C.2)

Here, functions \( g \) on \( I_a \) and \( \hat{g} \) on \( \tilde{I} \) are related by

\[ \hat{g}(\tilde{\xi}) = g(B_a(\tilde{\xi})), \]  

(C.3)

and we write

\[ E(g) = (\det B_a) \hat{E}(\hat{g}) \]

for functions \( g \) and \( \hat{g} \) related by (C.3). For \( w \in P_k(\tilde{I}) \), define a linear functional \( \hat{\lambda}_\psi \) on \( H^r(\tilde{I}) \) by

\[ \hat{\lambda}_\psi(\hat{g}) = \hat{E}(\hat{g}\hat{w}) \quad \forall \hat{g} \in H^r(\tilde{I}). \]

Since \( r \geq 1 \), the space \( H^r(\tilde{I}) \) is continuously embedded in the space \( C^0(\tilde{I}) \). Since the norms \( \| \cdot \|_{C^0(\tilde{I})} \) and \( \| \cdot \|_{L^2(\tilde{I})} \) are equivalent on the finite-dimensional space \( P_k(\tilde{I}) \), this implies that there exists a constant \( \hat{C} \) such that

\[ |\hat{\lambda}_\psi(\hat{g})| \leq \hat{C} \| \hat{g} \|_{H^r(\tilde{I})} \| \hat{w} \|_{L^2(\tilde{I})}, \]

and so \( \hat{\lambda}_\psi \) is a bounded linear functional on \( H^r(\tilde{I}) \), with

\[ \| \hat{\lambda}_\psi \| \leq \hat{C} \| \psi \|_{L^2(\tilde{I})}. \]
Since \( r \leq k \), we see that if \( g \in P_{r-1}(I) \), then \( \hat{g} \hat{w} \in P_{2k-1}(I) \). Thus
\[
\hat{\lambda}_0(\hat{g}) = 0 \quad \forall \hat{g} \in P_{r-1}(I).
\]

By the Bramble–Hilbert lemma (Ciarlet, 1978, Lemma 4.1.3), we thus have
\[
|\hat{\mathcal{E}}(\hat{g} \hat{w})| = |\hat{\lambda}_0(\hat{g})| \leq \hat{C} \|\hat{g}\|_{H'(I)} \|\hat{w}\|_{L_2(I)}.
\]

Setting
\[
\hat{g} = \hat{f} \hat{s}, \quad \text{where } \hat{f} \in H'(I), \hat{s} \in P_k(I),
\]
we have
\[
|\hat{\mathcal{E}}(\hat{f} \hat{s} \hat{w})| = \hat{C} \|\hat{f}\|_{H'(I)} \|\hat{w}\|_{L_2(I)}.
\]

The Sobolev space version of Leibniz's rule (Ciarlet, 1978, Sect. 4.1) yields
\[
\|\hat{f} \hat{s}\|_{H'(I)} \leq \hat{C} \sum_{j=0}^{r} |\hat{f}|_{H^{r-j}(I)} |\hat{s}|_{W^j(I)}.
\]

Recall the following inequalities of Ciarlet (1978, Theorems 3.1.2, 3.1.3) (in which \( C \) is a constant, independent of \( f, s, w, \) and \( n \)):
\[
|\hat{f}|_{H^{r-j}(I)} \leq C(B_a)^{-1/2} n^{-j} |\hat{f}|_{H^{r-j}(I)},
\]
\[
|\hat{s}|_{W^j(I)} \leq C n^{-j} |\hat{s}|_{W^j(I)},
\]
\[
\|\hat{w}\|_{L_2(I)} \leq C(B_a)^{-1/2} \|w\|_{L^2(I)}.
\]

The desired inequality now follows immediately from (C.2), (C.4), (C.5), and (C.6).

We now complete the

**Proof of Theorem C.3.** From the first Strang lemma (Ciarlet, 1978, Lemma 4.1.1) and (6.2), we have
\[
e(\varphi_{n,k}^q, N_n^q) \leq \left(1 + \frac{4}{\pi^2}\right) \inf_{s \in \mathcal{G}^q_n} \left\{ \sup_{f \in \mathcal{F}, s} \|Sf - s\|_{H'(I)} + \sup_{w \in \mathcal{G}^q_n} \frac{|B(s, w) - B_n(s, w)|}{\|w\|_{H'(I)}} + \sup_{w \in \mathcal{G}^q_n} \frac{|f(w) - f_n(w)|}{\|w\|_{H'(I)}} \right\}.
\]
In what follows, we will let $\Pi_n : \mathcal{H}_E \to \mathcal{F}_n^q$ denote the $\mathcal{F}_n^q$-interpolation operator given by

$$(\Pi_n v)(x) = \sum_{j=1}^n v(x_j) s_j(x) \quad \forall v \in \mathcal{H}_E.$$ 

We will estimate the right-hand side of (C.7) by setting $s = \Pi_n Sf$ for each $f \in F_r$. Let $f \in F_r$. Using Theorem 3.2.1 of Ciarlet (1978), we find that there exists a positive constant $C_1$, independent of $f$ and $n$, such that

$$\|Sf - \Pi_n Sf\|_{H^q(I)} \leq C_1 n^{-r} \|Sf\|_{H^q(I)} \leq C_1 n^{-r} \|Sf\|_{H^{q-r}(I)} \leq C_1 C_3 n^{-r},$$

(C.8)

where the positive constant $C_3$ is the same as that in the inequality (B.2). Now for any $w \in \mathcal{F}_n^q$, the exactness properties of Gauss quadrature and Lemma C.1 yield

$$|B(\Pi_n Sf, w) - B_n(\Pi_n Sf, w)| \leq \sum_{a=0}^{l-1} E_{la}(f \cdot \Pi_n Sf \cdot w)$$

$$\leq C n^{-r} \sum_{a=0}^{l-1} \|f\|_{H^q(I)} \|\Pi_n Sf\|_{W^{q-r}(I)} \|w\|_{L^q(I)}.$$

(C.9)

Since $k \geq r$, we may use the results of Ciarlet (1978, Chap. 3.1) to find that there exists a positive constant $C_4$, independent of $n$, $f$, and $w$, such that for any $a$ with $0 \leq a \leq l - 1$,

$$\|\Pi_n Sf\|_{W^r(I)} \leq \|Sf\|_{W^r(I)} + \|Sf - \Pi_n Sf\|_{W^r(I)} \leq C_4 \|Sf\|_{W^r(I)}. \quad (C.10)$$

Substituting (C.10) into (C.9), we may use the Cauchy–Schwarz inequality, the Sobolev embedding theorem, and (B.2) to find that there exists a positive constant $C_5$, independent of $n$ and $f$, such that

$$|B(\Pi_n Sf, w) - B_n(\Pi_n Sf, w)| \leq C_5 n^{-r} \|f\|_{H^q(I)} \|w\|_{L^q(I)} \quad \forall w \in \mathcal{F}_n^q.$$ 

(C.11)

Since $k \geq r$, one can show that there is a positive constant $C_6$, independent of $n$ and $f$, such that for $0 \leq a \leq l - 1$,
The proof follows that of Ciarlet (1978, Theorem 4.1.5), while using the relation $k \geq r.$ Summing this inequality over all $a$ and using the Cauchy–Schwarz inequality, we find

$$|f(w) - f_n(w)| = \sum_{a=0}^{i-1} E_{la}(fw) \leq C_6 n^{-r} \|f\|_{H^1(U)} \|w\|_{H^1(U)} \quad \forall w \in \mathbb{S}_n^q.$$  

(C.12)

Since $f \in F,$ implies that $\|f\|_{H^1(U)} \leq 1,$ we may combine (C.7), (C.8), (C.11), and (C.12) to find that

$$e(q_{n,k}, N^q_n) = \Omega(n^{-r}) \quad \text{as } n \to \infty.$$

This upper bound, when combined with the lower bound of Theorem C.1, completes the proof of the theorem. □

Remark C.1. Note that we have a somewhat stronger result, namely, that the FEMQ $\varphi_{n,k}^q$ is almost optimal iff $k \geq r.$ Indeed, we need only show that $e(q_{n,k}^q, N^q_n) = \Omega(n^{-k})$ when $k < r.$ Since $u_n^q \in \mathbb{S}_n^q,$ this follows immediately by using the same technique as that used in the proof of (ii) in the lower bound of Theorem B.1.

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