Population models with diffusion, strong Allee effect and constant yield harvesting

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Abstract

We study the steady state distribution of reaction diffusion equations with strong Allee effect type growth and constant yield harvesting (semipositone) in heterogeneous bounded habitats. Assuming the exterior of the habitat is completely hostile, we establish existence results for positive solutions. We also establish a multiplicity result for the non-harvested case. We obtain our results via the method of sub–super solutions.

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1. Introduction

A typical model of reaction diffusion equations that describes the spatiotemporal distribution and abundance of organisms is

\[ u_t = d \Delta u + uf(x,u) \]

where \( u(x,t) \) is the population density, \( d > 0 \) is the diffusion coefficient, \( \Delta u \) is the Laplacian of \( u \) with respect to the \( x \) variable, and \( f(x,u) \) is the per capita growth rate, which is affected by the heterogeneous environment. Such ecology models were first studied by Skellam in [33]. A classic example is Fisher’s equation (see [10]) with \( f(x,u) = (1 - u) \). Similar reaction diffusion biological models have been studied by Kolmogoroff, Petrovsky, and Piscounoff [18] earlier. Since then reaction diffusion models have been used to describe various spatiotemporal phenomena in biology, physics, chemistry and ecology, see Fife [11], Okuba and Levin [28], Smoller [34], Murray [26], and Cantrell and Cosner [7]. Since the pioneer work by Skellam [33], the logistic growth rate \( f(x,u) = m(x) - b(x)u \) has been used in population dynamics to model the crowding effect (see Oruganti, Shi and Shivaji [29]). A more general logistic type model can be characterized by a declining growth rate per capita function, i.e., \( f(x,u) \) is decreasing with respect to \( u \).

However observational data (see [2,12,13,15,19–21,23,37]) witnesses an increase in the per capita growth rate at low densities, which Odum [27] first recognized as the Allee principle and is called the Allee effect (see Allee [1], Dennis [8], Lewis and Kareiva [24] and Shi and Shivaji [36]). Allee effects can be caused by many reasons: less efficient feeding at low densities (Way and Banks [39], and Way and Cammell [40]), reduced effectiveness of vigilance and anti-predator defenses (Kruuk [17], and Kenward [16]), inbreeding depression (Ralls et al., 1986), shortage of mates (Hopf and Hopf [14], Veit and Lewis [38]), lack of effective pollination (Groom [13]), predator saturation (de Roos et al. [9]), and cooperative behaviors (Wilson and Nisbet [42]) and several other factors (Folt, 1987; Foster and Trehern, 1981; Turchin and Kareiva, 1989; Pulliam and Caraco, 1984).

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This Allee effect is strong if the per capita growth rate is negative at low population densities, that is, \( f(x, u) \) is negative for \( u \) small and is weak if per capita growth rate is positive at low population densities, that is, \( f(x, u) \) is positive for \( u \) small. A strong Allee effect introduces a threshold in the population. The population must overcome this threshold to grow, where as weak Allee does not have a threshold (see [41]). In Clark [6] a strong Allee effect is called critical depensation and a weak Allee effect is called a noncritical depensation. A population with strong Allee effect is also called asocial by Philip [30]. Most people regard the strong Allee effect as the Allee effect (see [8,13,18,41]).

In this paper, we consider the dispersal and evolution of species on a bounded domain \( \Omega \) (in \( \mathbb{R}^N \)) when the per capita growth rate is

\[
 f(x, u) = a(x)u + b(x)u^2 - h(x)u^3
\]

where \( a, b, h \) are \( C^1 \) (Hölder continuous) functions such that \( b(x), h(x) \) are strictly positive functions on \( \overline{\Omega} \) and \( a(x) \) is negative at least for some \( x \in \Omega \) (strong Allee effect). Let \( a_0, a_1, b_0, b_1, h_0 \) and \( h_1 \) be defined as \( a_0 := -\inf_{x \in \Omega} a(x) \), \( a_1 := \sup_{x \in \Omega} a(x) \), \( b_0 := \inf_{x \in \Omega} b(x) \), \( b_1 := \sup_{x \in \Omega} b(x) \), \( h_0 := \inf_{x \in \Omega} h(x) \) and \( h_1 := \sup_{x \in \Omega} h(x) \).

We also consider the constant yield harvesting of the population and assume the exterior of the habitat is completely hostile. Hence we study the model:

\[
\begin{align*}
 u_t &= d \Delta u + a(x)u + b(x)u^2 - h(x)u^3 - ca(\alpha); & x & \in \Omega, \quad t > 0, \\
 u(x, t) &= 0; & x & \in \partial \Omega, \quad t > 0, \\
 u(0, x) &= u_1(x) \geq 0; & x & \in \Omega.
\end{align*}
\]

Here \( ca(\alpha) \), with \( \alpha : \Omega \to [0, 1] \) and \( c > 0 \) a parameter, represents the constant yield harvesting. We assume \( \alpha \) is a \( C^1 \) function. In applications, a typical \( \alpha(x) \) is not only zero on the boundary, but it is also zero close to the boundary.

In the literature there have been many studies that consider density dependent harvesting. However, constant yield harvesting is favored in fishery management problems since harvesting is regulated by respective authorities (see e.g., [5,25] for the case of Atlantic bluefin tuna). This was also confirmed by Selgrade and Roberds in [35].

The main goal here is to determine the long time dynamical behavior of the population, i.e., to study the (steady state) solutions to:

\[
\begin{align*}
 -\Delta u &= a(x)u + b(x)u^2 - h(x)u^3 - ca(\alpha); & x & \in \Omega, \\
 \mu &\in \partial \Omega, & x & \in \partial \Omega.
\end{align*}
\]

Here we have assumed \( d = 1 \). We consider also the case without harvesting i.e.,

\[
\begin{align*}
 -\Delta u &= a(x)u + b(x)u^2 - h(x)u^3; & x & \in \Omega, \\
 u &= 0; & x & \in \partial \Omega.
\end{align*}
\]

Let \( \lambda_1 \) be the principal eigenvalue of \( -\Delta \) with the Dirichlet boundary conditions. We establish the following results:

**Theorem 1.1.**

(a) Let \( \frac{b_1^2 + 4a_1h_0}{4a_0} \leq \lambda_1 \) and \( c > 0 \). Then \( (1.3) \) has no positive solution.

(b) For \( c \) large \( (1.3) \) has no positive solution.

**Theorem 1.2.** There exists positive constants \( \tilde{b}_0 := \tilde{b}_0(a_0, h_1, \Omega) \) and \( c^* := c^*(a_0, h_1, b_0, \Omega) \) such that for \( b_0 \geq \tilde{b}_0 \) and \( c \leq c^* \), \( (1.3) \) has a positive solution. Further, \( c^* \) is an increasing function of \( b_0 \) and \( \lim_{b_0 \to \infty} c^* = \infty \).

**Theorem 1.3.** Suppose \( (1.3) \) has a positive solution for some \( c > 0 \). Then there exists a positive constant \( c^{**} := c^{**}(a_0, h_1, b_0, \Omega) \) such that

(a) for \( 0 < c < c^{**} \), \( (1.3) \) has a maximal positive solution \( \bar{u}(x, c) \);

(b) for \( c > c^{**} \), \( (1.3) \) has no solution;

(c) \( \bar{u}(x, c) \) decreases with respect to the parameter \( c \) on \( (0, c^{**}) \), \( \forall x \in \Omega \).

**Theorem 1.4.** Let \( a_1 < \lambda_1 \), then for \( b_0 \geq \tilde{b}_0(a_0, h_1, \Omega) \), \( (1.4) \) has at least two positive solutions.

**Theorem 1.5.** Let \( \sup \alpha = K \subset \Omega \). Then given \( c > 0 \), there exists a positive constant \( b^* := b^*(c, \alpha, a_0, h_1, \Omega) (> \tilde{b}_0) \) such that \( (1.3) \) has a positive solution \( u \) with \( u(x) > c \alpha(x) \) on \( \Omega \) for \( b_0 \geq b^* \).

Note that \( (1.3) \) is a semipositone problem due to the presence of the constant yield harvesting term. It is well known in the literature that the study of positive solutions to semipositone problems is mathematically challenging (see [4,22,29]). See [29] where such a model was discussed for the logistic growth case with constant coefficients. Here we deal with the more difficult strong Allee effect growth. We also do not restrict our analysis to models with just constant coefficients.
We establish our result via the method of sub–super solutions. By a sub-solution we mean a function \( w \in C^2(\overline{\Omega}) \) such that
\[
\begin{aligned}
\begin{align*}
-\Delta w &\leq a(x)w + b(x)w^2 - h(x)w^3 - c\alpha(x), \\
& w \leq 0,
\end{align*}
\end{aligned}
\]
and by a super-solution a function \( v \in C^2(\overline{\Omega}) \) such that
\[
\begin{aligned}
\begin{align*}
-\Delta v &\geq a(x)v + b(x)v^2 - h(x)v^3 - c\alpha(x), \\
v &\geq 0,
\end{align*}
\end{aligned}
\]
Then it is well known (see [3,31]) that if there exists a sub-solution \( w \) and by a super-solution a function \( v \) such that \( w \leq v \) in \( \Omega \), then there exists a solution \( u \in C^2(\overline{\Omega}) \) such that \( w \leq u \leq v \) in \( \Omega \). We will prove Theorem 1.1 in Section 2, Theorem 1.2 in Section 3, Theorem 1.3 in Section 4, Theorem 1.4 in Section 5, and Theorem 1.5 in Section 6.

2. Proof of Theorem 1.1

We will first prove Theorem 1.1(a) by a contradiction.

Let \( \lambda_1 \) be the principal eigenvalue of the operator \( -\Delta \) with Dirichlet boundary conditions and \( \phi_1 > 0 \) in \( \Omega \) be a corresponding eigenvector. Let \( u \) be a positive solution of (1.3). Then by the Green’s identity, we have
\[
0 = \int_{\Omega} (u \Delta \phi_1 - \phi_1 \Delta u) \, dx = \int_{\Omega} \{ \phi_1 [a(x)u + b(x)u^2 - h(x)u^3 - c\alpha(x)] - \lambda_1 \phi_1 u \} \, dx. \tag{2.1}
\]

Let \( \tilde{f}(s) = a_1 + b_1 s - h_2 s^2 \), and \( f^*(u) = u \tilde{f}(u) \). Then \( \tilde{f}_0 = \sup_{[0, \infty)} \tilde{f}(s) = \frac{b_1^2 + 4a_1 h_2}{4h_2} \) and \( a(x)u + b(x)u^2 - h(x)u^3 - c\alpha(x) \leq f^*(u) - c\alpha(x) \) for \( u \geq 0 \). Rewriting (2.1), we have
\[
0 = \int_{\Omega} (u \Delta \phi_1 - \phi_1 \Delta u) \, dx \leq \int_{\Omega} \{ \phi_1 f^*(u) - \lambda_1 \phi_1 u - c\alpha(x) \phi_1 \} \, dx. \tag{2.2}
\]

But for \( \lambda_1 \geq \frac{b_1^2 + 4a_1 h_2}{4h_2} \),
\[
\int_{\Omega} (\phi_1 f^*(u) - \lambda_1 \phi_1 u - c\alpha(x) \phi_1) \, dx = \int_{\Omega} (\tilde{f}(u) - \lambda_1) \phi_1 u \, dx + c \int_{\Omega} \alpha(x) \phi_1 \, dx
\]
\[
< \int_{\Omega} (f_0 - \lambda_1) \phi_1 u \, dx
\]
\[
\leq 0 \quad \text{(since } f_0 \leq \lambda_1)\]

a contradiction to (2.2). Hence (1.3) has no positive solution and Theorem 1.1(a) is proven. Note that conclusion of Theorem 1.1(a) holds for \( \frac{b_1^2 + 4a_1 h_2}{4h_2} < \lambda_1 \) when \( c = 0 \).

We now prove Theorem 1.1(b). We observe that
\[
c \int_{\Omega} \alpha(x) \phi_1(x) \, dx = \int_{\Omega} \Delta u \phi_1 + \int_{\Omega} [a(x)u + b(x)u^2 - h(x)u^3] \phi_1 \, dx
\]
\[
\leq -\lambda_1 \int_{\Omega} u \phi_1 + \int_{\Omega} [a(x)u + b(x)u^2 - h(x)u^3] \phi_1 \, dx
\]
\[
\leq \int_{\Omega} \phi_1 \, dx. \tag{2.3}\]

Clearly (2.3) is not satisfied for \( c \) large and hence Theorem 1.1(b) holds.

3. Proof of Theorem 1.2

In this section, we first prove an existence result for
\[
\begin{aligned}
\begin{cases}
-\Delta u = -a_0 u + b_0 u^2 - h_1 u^3 - c\alpha(x), & x \in \Omega, \\
u = 0, & x \in \partial \Omega.
\end{cases}
\end{aligned} \tag{3.1}
\]
Let \( \lambda_1 \) be the principal eigenvalue of the operator \(-\Delta\) with Dirichlet boundary conditions and \( \phi_1 > 0 \) in \( \Omega \) be the corresponding eigenvector such that \( \| \phi_1 \|_\infty = 1 \). It is well known that \( \frac{\partial \phi_1}{\partial n} < 0 \) on \( \partial \Omega \) where \( n \) is the unit outward normal. Hence there exists \( \delta > 0 \), \( \mu \in (0, 1) \) and \( m > 0 \) such that

\[
|\nabla \phi_1|^2 - \lambda_1 \phi_1^2 \geq m \quad \text{on } \Omega_{\delta},
\]

\[
\phi_1 \geq \mu \quad \text{on } \Omega - \bar{\Omega}_\delta
\]

where \( \Omega_\delta := \{ x \in \Omega \mid d(x, \partial \Omega) < \delta \} \).

Let \( b_0 > 2 \sqrt{a_0 h_1} \) and \( g(s) = -a_0 s + b_0 s^2 - h_1 s^3 \). The zeros of \( g \) are \( 0, r := \frac{b_0 - \sqrt{b_0^2 - 4a_0 h_1}}{2a_0} \) and \( R := \frac{b_0 + \sqrt{b_0^2 - 4a_0 h_1}}{2a_0} \) and hence

\[
g(s) = -h_1(s - r)(s - R). \]

Let \( r^* \) be the first positive zero of \( g' \). In fact, \( r^* = \frac{b_0 - \sqrt{b_0^2 - 3a_0 h_1}}{3a_0 h_1} < b_0 \). But \( g \) is convex on \( (0, b_0 \frac{r_0}{a_1}) \).

Hence \( \sigma := \inf_{s \in [0, R]} g(s) < a_0 \left[ \frac{b_0 - \sqrt{b_0^2 - 3a_0 h_1}}{2a_0} \right] = a_0 r^* \). (See Fig. 1.)

We first note that

\[
\frac{\sigma}{R} < \frac{a_0 [b_0 - \sqrt{b_0^2 - 3a_0 h_1}]/3h_1}{[b_0 + \sqrt{b_0^2 - 4a_0 h_1}]/2h_1} = \frac{2a_0 [b_0 - \sqrt{b_0^2 - 4a_0 h_1}]/3h_1}{[b_0 + \sqrt{b_0^2 - 4a_0 h_1}]/2h_1} = \frac{2a_0^2 h_1}{[b_0 + \sqrt{b_0^2 - 4a_0 h_1}]/2h_1},
\]

and thus RHS tends to zero as \( b_0 \) tends to infinity. Hence there exists \( b_0^{(1)} := b_0^{(1)}(a_0, h_1, \Omega) \) such that for every \( b_0 > b_0^{(1)} \), we have

\[
m > \frac{\sigma}{R} \quad (3.4)
\]

Next we also note that

\[
\frac{R}{r} = \left( \frac{b_0 + \sqrt{b_0^2 - 4a_0 h_1}}{b_0 - \sqrt{b_0^2 - 4a_0 h_1}} \right) = \frac{[b_0 + \sqrt{b_0^2 - 4a_0 h_1}]^2}{4a_0 h_1} \to \infty \quad \text{as } b_0 \to \infty.
\]

Hence there exists \( b_0^{(2)} := b_0^{(2)}(a_0, h_1, \Omega) \) such that for every \( b_0 > b_0^{(2)} \), we have

\[
\left[ \frac{R}{2} \mu^2, \frac{R}{2} \right] \subset (r, R).
\]

and \( K_\mu := \inf_{s \in [\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]} g(s) > 0 \). Finally

\[
K_\mu = \frac{\min(g(\frac{R}{2} \mu^2), g(\frac{R}{2}))}{R} = \min \left\{ \frac{R}{2} \mu^2 \left( \frac{R}{2} \mu^2 - r \right) \left( 1 - \frac{\mu^2}{2} \right), h_1 \frac{R}{4} \left( \frac{R}{2} - r \right) \right\}
\]

(3.6)

tends to infinity as \( b_0 \) tends to infinity. Thus there exists \( b_0^{(3)} := b_0^{(3)}(a_0, h_1, \Omega) > b_0^{(2)} \) such that for every \( b_0 > b_0^{(3)} \) we have

\[
\lambda_1 < \frac{K_\mu}{R} \quad (3.7)
\]

For a given \( a_0 > 0 \), \( h_1 > 0 \), define \( \tilde{b}_0 := \max(b_0^{(1)}, b_0^{(3)}) := \min(Rm - \sigma, K_\mu - R\lambda_1) \). For \( c \leq c^* \), \( (3.1) \) has a positive solution.

Lemma 3.1. Let \( b_0 > \tilde{b}_0 \) and \( c^* := c^*(a_0, h_1, b_0, \Omega) := \min \{ Rm - \sigma, K_\mu - R\lambda_1 \} \). For \( c \leq c^* \), \( (3.1) \) has a positive solution.
Hence (1.3) has a maximal positive solution \( \bar{c} \).

It is easy to see that \( q = R \) is a super-solution. Hence (3.1) has a positive solution. \( \square \)

We now prove Theorem 1.2.

**Proof of Theorem 1.2.** We observe that \( \psi = \frac{\phi_1^2}{2} \) is a positive sub-solution for (1.3), since \( \psi \) satisfies

\[
\begin{align*}
-\Delta \psi &\leq g(\psi) - c\sigma(x) \leq a(x)\psi + b(x)\psi^2 - h(x)\psi^3 - c\sigma(x); \quad x \in \Omega, \\
\psi &> 0; \quad x \in \partial\Omega.
\end{align*}
\]

It is easy to see that \( \varphi = M \) where \( M > 0 \) sufficiently large, is a super-solution of (1.3). Hence (1.3) has a positive solution.

We recall here that \( \sigma = -g(r^*, b_0) \). Differentiating \( \sigma \) with respect to \( b_0 \), we have

\[
\frac{d\sigma}{db_0} = -\frac{\partial g(r^*, b_0)}{\partial r^*} \frac{dr^*}{db_0} - \frac{\partial g}{\partial b_0},
\]

\[
= -\frac{\partial g}{\partial b_0} \quad \text{(since \( \frac{\partial g(r^*, b_0)}{\partial r^*} = 0 \))}
\]

\[
= -\left(\frac{r^*}{b_0}\right)^2 < 0.
\]

Also \( R \) is an increasing function of \( b_0 \). Thus \( Rm - \sigma \) is an increasing function of \( b_0 \). Next since \( \frac{r^*}{b_0} \) decreases as \( b_0 \) increases, we deduce from (3.6) that \( K_\mu - R\lambda_1 \) increases as \( b_0 \) increases. Therefore \( K_\mu - R\lambda_1 = R[\frac{K_\mu}{R} - \lambda_1] \) also increases as \( b_0 \) increases. Hence by the definition of \( c^* \), it is clear that \( c^* \) is an increasing function of \( b_0 \). Finally, since \( R \to \infty \), \( \frac{r^*}{b_0} \to 0 \) and \( \frac{K_\mu}{R} \to \infty \) as \( b_0 \to \infty \), it is easy to see that \( \lim_{b_0 \to \infty} c^* = \infty \). Hence the proof of Theorem 1.2. \( \square \)

### 4. Proof of Theorem 1.3

Suppose (1.3) has a positive solution for \( c = \hat{c} \). Let \( M = M(a_1, b_1, h_0) > 0 \) be the largest zero of \( f^*(s) = a_1s + b_1s^2 - h_0s^3 \). Since \( a(x)s + b(x)s^2 - h(x)s^3 - c\sigma(x) \leq f^*(s) \) for \( s \geq 0 \), by the maximum principle every solution \( u \) of (1.3) (for any \( c \)) must be such that \( u \leq M \) (note that \( M \) is independent of \( c \)). Also it is easy to see that \( \phi = \frac{\phi_1^2}{2} \) is a super-solution of (1.3). Hence (1.3) has a maximal positive solution \( \bar{u}(c, \hat{c}) \) at \( c = \hat{c} \). Clearly \( \bar{u}(c, \hat{c}) \) is a strict sub-solution for \( c < \hat{c} \) and again using the super-solution \( \phi = \frac{\phi_1^2}{2} \), (1.3) must have a positive solution for \( c < \hat{c} \). Using the argument as before, we can now conclude that (1.3) has a maximal positive solution \( \bar{u}(c, c) \) for \( c \leq \hat{c} \). Thus Theorem 1.3(a) and (b) hold by combining the above discussion with Theorems 1.1(b) and 1.2. Finally given \( c_1 < c_2 \) it is easy to see that \( \bar{u}(c_2, c_2) \) is a strict sub-solution of (1.3) with \( c = c_1 \). Again noting the fact that \( \phi = \frac{\phi_1^2}{2} \) is a super-solution, if \( \bar{u}(x, c_2) \geq \bar{u}(x, c_1) \) for some \( x_0 \in \Omega \) then \( \bar{u}(c_1, c_1) \) cannot be the maximal positive solution of (1.3) at \( c = c_1 \). Hence \( \bar{u}(x, c_2) < \bar{u}(x, c_1) \) \( \forall x \in \Omega \) and Theorem 1.3(c) is proven.

### 5. Proof of Theorem 1.4

In this section we prove the multiplicity of solutions to (1.4). To prove our result we first recall a multiplicity result discussed in [3,32] for the problem:

\[
\begin{align*}
-\Delta u(x) + qu(x) &= f(x, u(x)); \quad x \in \Omega, \\
u = 0; \quad x \in \partial\Omega,
\end{align*}
\]

where \( q \) is a non-negative constant and \( f \in C^\mu(\overline{\Omega} \times I) \), with \( \mu = 1 \) if \( N = 1 \) and \( 0 < \mu < 1 \) if \( N \geq 2 \), where \( I \) is a closed interval in \( \mathbb{R} \) with non-empty interior.
Lemma 5.1. (See Theorem 1.4 in [32].) Suppose that there exists a strict sub-solution \( \psi_1 \), a strict super-solution \( \phi_1 \), a strict sub-solution \( \psi_2 \) and a super-solution \( \phi_2 \) for (5.1) such that \( \psi_1 < \phi_1 < \psi_2 < \phi_2 \), \( \psi_1 < \psi_2 < \phi_2 \) and \( \psi_1 < \phi_1 \). Then (5.1) has at least three distinct solutions \( u_i \) (\( i = 1, 2, 3 \)) such that \( \psi_1 < u_1 < u_2 < u_3 < \phi_2 \).

We now prove our Theorem 1.4.

Proof of Theorem 1.4. For the boundary value problem (1.4) we will construct a sub-solution \( \psi_1 \), a strict sub-solution \( \psi_2 \), a strict super-solution \( \phi_1 \), and a super-solution \( \phi_2 \) such that \( \psi_1 < \phi_1 < \psi_2 < \phi_2 \). Then by Lemma 5.1 there exists solutions \( u_1 \in \{ \psi_1, \phi_1 \} \), \( u_2 \in \{ \psi_2, \phi_2 \} \) and \( u_3 \in \{ \psi_1, \phi_1 \} \). Hence there exists a positive solution \( u_1 \). Then by Lemma 5.1 (see Theorem 1.2), and consider

\[
\begin{align*}
-\Delta u &= a(x)u + b(x)u^2 - h(x)u^3 - c_1(x); & x \in \Omega, \\
 u &= 0; & x \in \partial \Omega. \tag{5.2}
\end{align*}
\]

By Theorem 1.2, (5.2) has a solution say \( w \) such that \( w \geq \frac{\phi_1}{2} \phi_2^2 \). We observe that \( w \) satisfies

\[
\begin{align*}
-\Delta w &= a(x)w + b(x)w^2 - h(x)w^3 - c_1(x) \leq a(x)w + b(x)w^2 - h(x)w^3; & x \in \Omega, \\
 u &= 0; & x \in \partial \Omega. \tag{5.3}
\end{align*}
\]

Let \( \psi_2 = w \). Then \( \psi_2 \) is a strict sub-solution of (1.4). Note that for \( \epsilon > 0 \) sufficiently small, \( \psi_2 \) and \( \psi_1 \) satisfy \( \psi_2 < \psi_1 \). Hence there exists solutions \( u_1 \in \{ \psi_1, \psi_2 \} \) and \( u_2 \in \{ \psi_2, \phi_2 \} \). Since \( \psi_1 \equiv 0 \) is a solution it may turn out that \( u_1 \equiv \psi_1 \). In any case we have two positive solutions \( u_2 \) and \( u_3 \). Hence Theorem 1.4 holds. \( \Box \)

6. Proof of Theorem 1.5

For a given \( c > 0 \), by Theorem 1.2, if \( b_0 \) (\( > \hat{b}_0 \)) is large enough then \( c < c^* \), and hence by Theorem 1.2, (1.3) has a positive solution \( u \geq \psi = \frac{b}{2} \phi_1^2(x) \). But \( R \) is an increasing function of \( b_0 \) and \( R \uparrow \infty \) as \( b_0 \uparrow \infty \). Hence there exists a positive constant \( \hat{b}_0 := \hat{b}_0(c, \alpha, a_0, h_1, \Omega) \) such that if \( b_0 \geq \hat{b}_0 \) then \( \psi \geq \frac{\phi_1}{2} \phi_1^2(x) \geq c_1(x) \). Note that \( |\nabla \psi| = 0; \partial \Omega \). However since \( \phi_1 > 0 \); \( \Omega \) and we are assuming \( \text{sup} \alpha \subset \Omega \) such a \( \hat{b}_0 \) must exists. Let \( b^* = \max(\hat{b}_0, \hat{b}_0) \). Then Theorem 1.5 clearly holds for \( b_0 \geq b^* \).

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