Zeros of Brauer characters and linear actions of finite groups

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Abstract

Let $G$ be a finite group, and $p$ a prime number greater than 3. It is known that, if every irreducible $p$-Brauer character of $G$ does not vanish on any $p'$-element of $G$, then $G$ is solvable. The primary aim of this work is to describe the structure of groups satisfying the above condition; among other more specific properties, we show that the $p'$-length of $G$ is at most 2 (the bound being the best possible). The structural results are obtained as an application of the main theorem in this paper, that deals with particular linear actions of solvable groups on finite vector spaces.

1. Introduction

In a recent paper, G. Malle studied the class of finite groups whose Brauer character table for a given prime $p$ does not contain any zero. He focuses on nonabelian simple groups, proving that no group $G$ of this kind satisfies the above condition, unless $p = 2$ and every $\phi \in \text{IBr}_2(G)$ has 2-power degree (see [4, Theorem 1.1]). As a consequence, in [4, Theorem 1.3] it is established that a finite group whose $p$-Brauer character table contains no zeros is necessarily solvable if $p \neq 2$ (and it is not difficult to find examples of such groups: for instance, consider the 3-Brauer character table of the alternating group Alt(4)).

In this spirit, the aim of the present work is to describe the structure of finite groups $G$ such that, for a given prime $p$, every irreducible $p$-Brauer character of $G$ does not take the value 0. Since the maximal normal $p$-subgroup $O_p(G)$ of $G$ lies in the kernel of every $p$-Brauer character of $G$, we can...
assume $O_p(G) = 1$. In fact, we describe the structure of $G$ modulo its Fitting subgroup $F(G)$ when $p$ is at least 5.

**Theorem A.** Let $G$ be a finite group and $p \geq 5$ a prime number. Assume that $O_p(G) = 1$. If the $p$-Brauer character table of $G$ does not contain any zero, then the Hall $p'$-subgroups of $G/F(G)$ are abelian of squarefree exponent and the $p'$-length of $G/F(G)$ is at most 1.

The following bounds concerning the $p$-length and the $p'$-length can be immediately derived from Theorem A.

**Corollary B.** Let $G$ be a finite group and $p \geq 5$ a prime number. If the $p$-Brauer character table of $G$ does not contain any zero, then we have

$$l_p'(G) \leq 2 \quad \text{and} \quad l_p(G/O_p(G)) \leq 2.$$

We remark that, as shown by Example 4.1, the bounds in Corollary B cannot be improved.

As regards our proof of Theorem A, the main idea is the following. By Gaschütz' theorem, denoting by $F$ the Fitting subgroup of the solvable group $G$, the factor group $F/\Phi(G)$ can be viewed as a faithful completely reducible $G/F$-module (possibly in “mixed characteristic”), and the same holds for the dual group $V := \text{Irr}(F/\Phi(G))$. Now, the assumptions of Theorem A yield that every $p'$-element of $G/F$ fixes at least one element in each $G/F$-orbit on $V$. Therefore the following result on linear actions, that may be of independent interest, turns out to be a key tool in our analysis. In the statement, $\Gamma(q^n)$ denotes the semilinear group on the field with $q^n$ elements.

**Theorem C.** Let $G$ be a finite solvable group, $p \geq 5$ and $q$ prime numbers, and $V$ a faithful irreducible $G$-module over $GF(q)$. Assume that every $p'$-element of $G$ fixes an element in each $G$-orbit on $V$. Then the following conclusions hold:

(a) Either $G$ is a $p$-group, or there exist $H \leq \Gamma(q^n)$ (for a suitable $n \in \mathbb{N}$) and a (possibly trivial) $p$-group $K$ such that $G$ is isomorphic to a subgroup of $H : K$. Moreover, $H$ is a Frobenius group with cyclic kernel of $p$-power order and Frobenius complement of prime order $r$.

(b) The Hall $p'$-subgroups of $G$ are elementary abelian $r$-groups, and the $p'$-length of $G$ is at most 1.

Note that the order of a group satisfying the assumptions of Theorem C has at most one prime divisor different from $p$.

We also point out that part (a) of Theorem C does not extend to the cases $p = 2$ and $p = 3$ (see Examples 3.3 and 3.5). On the other hand, it remains an open question whether part (b) holds also for $p \in \{2, 3\}$. As a consequence, we do not know whether the assumption $p \geq 5$ is really needed in Theorem A (at any rate, in order to extend Theorem A to the prime 2, the solvability of $G$ must be assumed).

Theorem C is proved after a result (Theorem 3.1) which analyzes the case when the $G$-module $V$ is primitive. Interestingly, the groups that satisfy the assumptions of Theorem 3.1 turn out to be the same that satisfy the following (in principle) stronger hypothesis: every element of $V$ is centralized by a Hall $p'$-subgroup of $G$ (see [6, Corollary 10.6]).

Finally, a remark concerning character degrees.

**Proposition D.** Let $G$ be a finite group and $p$ a prime number. Assume that either $p \neq 2$ or that $G$ is $p$-solvable. If the $p$-Brauer character table of $G$ does not contain any zero, then the degree of every irreducible Brauer character of $G$ is a multiple of $p$.

Proposition D is an easy consequence of the Fong–Swan Theorem (which applies here in view of Malle’s result), and of a theorem by G. Malle, G. Navarro and J. Olsson (see [5, Theorem A]). It is
worth noticing that, in contrast to what happens for nonabelian simple groups, the degrees of the irreducible $p$-Brauer characters of $G$ are not necessarily $p$-powers: an instance of this fact can be observed looking at the 7-Brauer character table of the normalizer of a Sylow 2-subgroup in the Suzuki group $Suz(8)$.

To close with, all the groups considered in the following discussion are assumed to be finite groups, and all the vector spaces will be finite-dimensional.

2. Preliminaries

We start by recalling some well-known facts, also establishing some notation.

**Remark 2.1.** Let $G$ be a group, $T$ a subgroup of $G$, and $L$ a normal subgroup of $T$ such that $core_T(L) = 1$. Denote by $H$ the factor group $T/L$. We adopt the “bar convention” for the natural homomorphism of $T$ onto $H$. Also, $\{g_1 = 1, g_2, \ldots, g_s\}$ will be a right transversal for $T$ in $G$, and $\Sigma$ will denote the set $\{1, \ldots, s\}$.

Now, it is possible to define a monomorphism of $G$ into $H \wr K$ where $K \simeq G/core_T(T)$ is a transitive subgroup of $\text{Sym}(\Sigma)$. This monomorphism is defined as follows.

Given an element $g \in G$, there exist $t_{g,i} \in T$ and $\sigma \in \text{Sym}(\Sigma)$ such that, for every $i \in \{1, \ldots, s\}$, we get $g_i g = t_{g,i}g_i \sigma$. The $s$-tuple $(t_{g,1}, \ldots, t_{g,s})$ and $\sigma$ are uniquely determined by $g$ (although $(t_{g,1}, \ldots, t_{g,s})$ depends of course on the choice of the right transversal of $T$ in $G$), and we can define a function $\phi : G \rightarrow H \wr \text{Sym}(\Sigma)$ mapping $g$ to $(t_{g,1}, \ldots, t_{g,s})\sigma$. This $\phi$ is in fact the composition map of the monomorphism $g \mapsto (t_{g,1}, \ldots, t_{g,s})\sigma$ (see [1, 13.3]) with the natural homomorphism from $T \wr \text{Sym}(\Sigma)$ onto $H \wr \text{Sym}(\Sigma)$. As we are assuming $core_T(L) = 1$, it can be checked that $\phi$ is injective.

Now we define $K$ to be the transitive subgroup of $\text{Sym}(\Sigma)$ obtained as the image of $\phi(G)$ under the top projection $H \wr \text{Sym}(\Sigma) \rightarrow \text{Sym}(\Sigma)$. (Of course, $K = 1$ if $T = G$.)

We shall apply the above setting to the following two contexts:

(a) Let $\Omega$ be a finite nonempty set, $G$ a transitive subgroup of $\text{Sym}(\Omega)$, and $\Delta$ a block for the action of $G$ on $\Omega$. In this situation, the role of $T$ is played by the setwise stabilizer of $\Delta$ in $G$, and $L$ is defined as $\bigcap_{\delta \in \Delta} G_\delta$. Note that $T := T/L$ is a transitive subgroup of $\text{Sym}(\Delta)$.

(b) Let $G$ be a group, and $V$ a faithful irreducible $G$-module over a suitable field. In this case, set $T$ to be a subgroup of $G$ such that $V_T$ has a submodule $W$ with $V \simeq W^G$, and take $L := C_T(W)$. Note that $W$ is a faithful irreducible $H$-module.

**Remark 2.2.** In the setting (a) of Remark 2.1, consider the cartesian product $\Delta \times \Sigma$, and define an action of $H \wr K$ on it as follows:

$$(\delta, i) \cdot (h_1, \ldots, h_s)\sigma = (\delta \cdot h_1, i\sigma).$$

Restricting this action to $\phi(G)$ and identifying $G$ with $\phi(G)$, it turns out that $\Omega$ and $\Delta \times \Sigma$ are equivalent $G$-sets (independently on the choice of the right transversal for $T$ in $G$): see [3, I.15.3].

**Remark 2.3.** In the setting (b) of Remark 2.1, consider the direct sum $W^\oplus s$ of $s$ copies of $W$, and define an action of $H \wr K$ on it as follows:

$$(W_1 + \cdots + W_s)^{(h_1, \ldots, h_s)}\sigma = W_{1\sigma^{-1}}^{h_{1\sigma^{-1}}} + \cdots + W_{s\sigma^{-1}}^{h_{s\sigma^{-1}}}.$$

Restricting this action to $\phi(G)$ and identifying $G$ with $\phi(G)$, it turns out that $V$ and $W^\oplus s$ are equivalent $G$-modules (independently on the choice of the right transversal for $T$ in $G$): see for instance [6, Lemma 2.8].

As mentioned in the Introduction, one particular orbit property will play a central role in our discussion. In view of that, it will be convenient to introduce some terminology.
Definition 2.4. Let $\Omega$ be a finite nonempty set, and let $G$ be a subgroup of $\text{Sym}(\Omega)$. Also, let $\mathcal{O}$ be an orbit of the action of $G$ on $\Omega$, and $\pi$ a set of prime numbers. We say that the orbit $\mathcal{O}$ is $\pi$-deranged if there exists a $\pi$-element of $G$ which does not fix any element in $\mathcal{O}$.

We shall also make use of the following notation.

Definition 2.5. Let $\Omega$ be a finite nonempty set. Given a positive integer $k$, we define $\mathcal{P}_k(\Omega)$ to be the set of ordered $(k+1)$-tuples $(\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_{k+1})$, where the $\mathcal{E}_j$ are (possibly empty) subsets of $\Omega$ such that $\mathcal{E}_j \cap \mathcal{E}_i = \emptyset$ whenever $j \neq i$, and $\bigcup_{j=1}^{k+1} \mathcal{E}_j = \Omega$. We shall write $\mathcal{P}(\Omega)$ rather than $\mathcal{P}_1(\Omega)$.

Observe that, if $G$ is a subgroup of $\text{Sym}(\Omega)$, then $G$ also embeds into $\text{Sym}(\mathcal{P}_k(\Omega))$ in a natural way (under the convention that the empty set is fixed by every element of $G$). Moreover, there is an obvious bijection between $\mathcal{P}_k(\Omega)$ and the subset of $\mathcal{P}_{k+1}(\Omega)$ consisting of the elements $(\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_{k+2})$ such that $\mathcal{E}_{k+2} = \emptyset$; this subset is clearly $G$-invariant, and the action of $G$ on it is equivalent to that on $\mathcal{P}_k(\Omega)$.

We are ready to prove the following lemma. After that, we gather the counterpart of it in the context of modules as well.

Lemma 2.6. Assume the setting (a) of Remark 2.1, let $k$ be a positive integer, and let $p$ be a prime number. If there exists a $p'$-deranged orbit for the action of $H$ on $\mathcal{P}_k(\Delta)$, then there exists a $p'$-deranged orbit for the action of $G$ on $\mathcal{P}_k(\Omega)$.

Proof. Let $(\Gamma_1, \Gamma_2, \ldots, \Gamma_{k+1})$ be an element lying in a $p'$-deranged orbit for the action of $H$ on $\mathcal{P}_k(\Delta)$. We define an ordered $(k+1)$-tuple of subsets of $\Delta \times \Sigma$, setting

$$\mathcal{E}_j = \{(\gamma, i) \mid \gamma \in \Gamma_j, \ i \in \{1, \ldots, s\}\}$$

for every $j \in \{1, \ldots, k+1\}$. By assumption, there exists a $p'$-element $h$ of $H$ which does not fix any element in the $H$-orbit of $(\Gamma_1, \Gamma_2, \ldots, \Gamma_{k+1})$; we take an element $t \in T$ such that $t = h$, and we can certainly assume that $t$ is a $p'$-element as well. Considering the monomorphism $\phi$ defined in Remark 2.1, we claim that $\phi(t)$ does not fix any element in the $\phi(G)$-orbit of $(\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_{k+1})$.

In view of Remark 2.2, this will yield the desired conclusion.

In fact, for a proof by contradiction, assume that there exists $x \in G$ such that $\phi(t)$ fixes $(\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_{k+1})$. Writing $\phi(t) = (h_1, \ldots, h_k)\tau$ for suitable $h_i \in H$ and $\tau \in K$, we get that $h_1 = h$ (recall that the element $g_1$ of the transversal for $T$ in $G$ was set to be 1), and $\tau$ fixes the symbol 1. Also, write $\phi(x) = (l_1, \ldots, l_j)\sigma$ for suitable $l_i \in H$ and $\sigma \in K$. For $j \in \{1, \ldots, k+1\}$ and $\gamma \in \Gamma_j$, take $i \in \{1, \ldots, s\}$ and $\epsilon \in \Delta$ such that

$$\left(\gamma, 1\sigma^{-1}\right) \cdot \phi(x) \phi(t) = (\epsilon, i) \cdot \phi(x)$$

holds. Note that $i = 1\sigma^{-1}$, as $\tau$ fixes 1. Observe also that $\epsilon \in \Gamma_j$, because $\phi(t^{-1})$ fixes $(\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_{k+1})$. Let $u$ be an element of $T$ such that $\bar{u} = 1\sigma^{-1}$.

We get $\gamma \cdot u = \epsilon \cdot u$, thus $\gamma \cdot t u^{-1}$ lies in $\Gamma_j$. Since this holds for every $\gamma \in \Gamma_j$, we conclude that $t u^{-1}$ fixes $\Gamma_j$ and, as this happens for every $j \in \{1, \ldots, k+1\}$, we get that $h = t$ fixes $(\Gamma_1, \Gamma_2, \ldots, \Gamma_{k+1}) \cdot \bar{u}$, contradicting our choice of $h$ and $(\Gamma_1, \Gamma_2, \ldots, \Gamma_{k+1})$. \ \Box

Lemma 2.7. Assume the setting (b) of Remark 2.1, and let $p$ be a prime number. If there exists a $p'$-deranged orbit for the action of $H$ on $W$, then there exists a $p'$-deranged orbit for the action of $G$ on $V \simeq W^G$.

Proof. Let $w$ be an element lying in a $p'$-deranged orbit of $H$ on $W$, thus there exists a $p'$-element $h$ of $H$ which does not fix any element in the $H$-orbit of $w$. We take an element $t \in T$ such that $t = h$,
and we can certainly assume that $t$ is a $p'$-element as well. Considering the monomorphism $\phi$ defined in Remark 2.1, we claim that $\phi(t)$ does not fix any element in the $\phi(G)$-orbit of $w + \cdots + w \in W^{\otimes n}$. In view of Remark 2.3, this will yield the desired conclusion.

In fact, for a proof by contradiction, assume that there exists $x \in G$ such that $\phi(t)$ fixes $(w + \cdots + w)^{\phi(x)}$. Writing $\phi(t) = (h_1, \ldots, h_3)\tau$ for suitable $h_i \in H$ and $\tau \in K$, we get that $h_1 = h$, and $\tau$ fixes the symbol 1. Also, write $\phi(x) = (l_1, \ldots, l_3)\sigma$ for suitable $l_i \in H$ and $\sigma \in K$. If $u$ is an element of $T$ such that $\alpha = l_1\sigma^{-1}$, we get $w^u = w^{\alpha}$ (here we are focusing on the first component of the vector $w + \cdots + w$), thus $t^{\alpha^{-1}}$ fixes $w$. But this is a contradiction, as we get that $h = \bar{t}$ fixes $w^{\alpha}$, against our choice of $h$ and $w$. □

The next result will be a key ingredient, together with Theorem 3.1, in our proof of Theorem C.

**Lemma 2.8.** Let $p$ be a prime number. In the setting (a) of Remark 2.1, assume that $G$ is a transitive solvable subgroup of $\text{Sym}(\Omega)$ such that $|G|$ is not a $p$-power, and take $\Delta$ to be a minimal nontrivial block (i.e. $|\Delta| > 1$, but allowing $\Delta = \Omega$). Then the following conclusions hold:

(a) There exists a $p'$-deranged orbit for the action of $G$ on $P_2(\Omega)$.

(b) Either there exists a $p'$-deranged orbit for the action of $G$ on $P(\Omega)$ or, using the notation of Remark 2.1, $K$ is a (possibly trivial) $p$-group, $\pi(G) = \{2, p\}$ with $p \in \{3, 5\}$, and $H \simeq \text{Sym}(3)$ or $H \simeq D_{10}$ (acting naturally on $\Delta$ which has three or five elements respectively).

**Proof.** We argue by induction on the order of the group. Note that $|G| \neq 1$, whence $|\Omega| \neq 1$ as well. Consider first the case when $K$ is not a $p$-group. Since $K$ is a transitive solvable subgroup of $\text{Sym}(\Sigma)$ and $|K| < |G|$, we can apply the inductive hypothesis and conclude that there exists a $p'$-deranged orbit for the action of $K$ on $P_2(\Sigma)$. So, take $(A, B, C) \in P_2(\Sigma)$ lying in such an orbit, and construct an element of $P(\Omega)$ as follows: define $\Gamma$ to be a subset of $\Omega$ containing no element of $\Delta \cdot g_i$ for every $i \in A$, exactly one element of $\Delta \cdot g_i$ for every $j \in B$, and the whole block $\Delta \cdot g_i$ for every $l \in C$ (recall that $|\Delta| \geq 2$). We claim that $(\Gamma, \Omega \setminus \Gamma)$ lies in a $p'$-deranged orbit for the action of $G$ on $P(\Omega)$.

In fact, let $y$ be a $p'$-element of $K$ which does not fix any element in the $K$-orbit of $(A, B, C)$, and let $g$ be an element of $G$ such that the image of $\phi(g)$ under the top projection is $y$ (we can choose $g$ to be a $p'$-element as well). If $g$ fixes $(\Gamma, \Omega \setminus \Gamma) \cdot x$ for some $x \in \Omega$, then $g^{x^{-1}}$ fixes $(\Gamma, \Omega \setminus \Gamma)$. Now, denoting by $u \in K$ the image of $\phi(x)$ under the top projection, it is not hard to check that $y^{u^{-1}}$ fixes $(A, B, C)$, whence $y$ fixes $(A, B, C) \cdot u$, against our choice of $(A, B, C)$ and $y$. Therefore the claim (i.e. the first option in conclusion (b)) is proved, and conclusion (a) also follows from the observation in the paragraph after Definition 2.5.

For the rest of the proof we shall assume that $K$ is a $p$-group, which clearly implies that $H$ is not a $p$-group. Now, if there exists a $p'$-deranged orbit for the action of $H$ on $P(\Delta)$, then Lemma 2.6 provides a $p'$-deranged orbit for the action of $G$ on $P(\Omega)$, which in turn implies the existence of a $p'$-deranged orbit for the action of $G$ on $P_2(\Omega)$ as well. So we are done in this case, and it remains to treat the situation when there does not exist a $p'$-deranged orbit for the action of $H$ on $P(\Delta)$. In particular, $H$ does not have any regular orbit on $P(\Delta)$. Since, $\Delta$ being a minimal block, the action of $H$ on $\Delta$ is primitive, we are in a position to apply Theorem 5.6 of [6]. Besides the cases $H \simeq \text{Sym}(3)$ and $H \simeq D_{10}$ (acting naturally on sets of three and five elements, respectively), we have hence the following cases to consider:

(a) $\Delta = \{1, 2, 3, 4\}$ and $H \in \{\text{Alt}(4), \text{Sym}(4)\}$. Then the $H$-orbits on $P(\Delta)$ containing $(\{1, 2, 3\}, \{4\})$ and $(\{1, 2\}, \{3, 4\})$ are a $3'$-deranged orbit and a $2'$-deranged orbit, respectively.

(b) $\Delta = \{1, 2, 3, 4, 5\}$ and $H$ is the Frobenius group of order 20. Then the $H$-orbit on $P(\Delta)$ containing $(\{1, 2, 3\}, \{4, 5\})$ is both a $2'$-deranged and a $5'$-deranged orbit.

(c) $\Delta = \{1, 2, 3, 4, 5, 6, 7\}$ and $H$ is the Frobenius group of order 42. As the 3-elements of $H$ are products of two 3-cycles, the stabilizer of $(\{1, 2\}, \{3, 4, 5, 6, 7\})$ in $H$ has order 2. Thus, the corresponding $H$-orbit on $P(\Delta)$ is $p'$-deranged for all prime divisors $p$ of $|H|$. 

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(d) \( \Delta = \{1, 2, 3, 4, 5, 6, 7, 8\} \) and \( H = \Delta G(2^3) \) is the affine semilinear group on the field with eight elements (see [6, p. 38]). Now, as the stabilizer in \( H \) of \( \langle (1), \{2, 3, 4, 5, 6, 7, 8\}\rangle \) has odd order and the stabilizer of \( \langle (1), 2, \{3, 4, 5, 6, 7, 8\}\rangle \) has order coprime to 7, we get that \( H \) has a \( p' \)-deranged orbit on \( \mathcal{P}(\Delta) \) for all prime divisors \( p \) of \( |H| \).

(e) \( \Delta = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \) and \( H \) is the semidirect product of the regular normal subgroup \( N = C_3 \times C_3 \) with \( D_8 \), \( SD_{16} \) (the semidihedral group of order 16), \( SL(2, 3) \) or \( GL(2, 3) \). The stabilizer in \( H \) of \( \langle (1), 2, 3, 4, 5, 6, 7, 8, 9\rangle \) intersects \( N \) trivially. Moreover, the stabilizer in \( H \) of \( \langle (1, 2), \{3, 4, 5, 6, 7, 8, 9\}\rangle \) does not contain any element of order 4. In fact, an element \( x \) of order 4 fixes exactly one element in \( \Delta \), because \( x^2 \) acts on \( \Delta \) as the involution of \( SL(2, 3) \) acts on \( N \) (i.e. as the inversion map). It follows that \( H \) has a \( p' \)-deranged orbit on \( \mathcal{P}(\Delta) \) for all prime divisors \( p \) of \( |H| \).

Therefore, in each of the above cases, \( H \) has \( p' \)-deranged orbits on \( \mathcal{P}(\Delta) \) for every prime divisor \( p \) of \( |H| \), and the proof is complete. \( \square \)

On the other hand we observe that, if \( H \in \{\text{Sym}(3), D_{10}\} \) acts naturally on a set \( \Delta \) of three or five elements respectively, then the stabilizer in \( H \) of every element of \( \mathcal{P}(\Delta) \) contains a Sylow 2-subgroup of \( H \). So, in these cases, \( H \) has no \( 3' \)-deranged orbit, or respectively \( 5' \)-deranged orbit, on \( \mathcal{P}(\Delta) \).

We conclude this preliminary section with an easy lemma.

**Lemma 2.9.** Let \( G \) be a group, \( p \) a prime number, and \( V \) a faithful irreducible \( G \)-module over a finite field. If there does not exist any \( p' \)-deranged orbit for the action of \( G \) on \( V \), then the Fitting subgroup \( F(G) \) of \( G \) is a \( p \)-group.

**Proof.** Set \( Z := Z(G)F(q^n) \) and observe that, if \( V \) is a module over a field of characteristic \( q \), then \( q \) does not divide \( |Z| \); in fact, \( V \) is a faithful completely reducible \( F(G) \)-module, whence \( F(q^n) = 1 \). Now, by Brodkey’s theorem there exists a regular orbit for the action of \( Z \) on \( V \), and we can choose an element \( v \in V \) lying in such an orbit. If there exists a nontrivial element \( z \in Z \), then \( z \) is a \( p' \)-element of \( G \) which does not centralize any \( G \)-conjugate of \( v \). Therefore, the \( G \)-orbit of \( v \) is in fact a \( p' \)-deranged orbit, a contradiction. The conclusion is that \( Z = 1 \), which yields the claim. \( \square \)

3. Linear actions with no \( p' \)-deranged orbits

This section is devoted to the proof of Theorem C. A key step towards that result is the analysis of the “primitive case”, which is carried out in the following theorem. Recall that, if \( V \) is a finite vector space of order \( q^n \), \( q \) a prime, \( \Gamma(V) \) denotes a subgroup of \( \text{Aut}(V) \) isomorphic to the semilinear group \( \Gamma(q^n) \), obtained by identifying \( V \) with \( G(q^n) \).

**Theorem 3.1.** Let \( G \) be a solvable group, \( p \geq 5 \) and \( q \) prime numbers, and \( V \) a faithful primitive \( G \)-module of order \( q^n \). Assume that \( G \) is not a \( p \)-group, and that there are no \( p' \)-deranged orbits for the action of \( G \) on \( V \). Then the following conclusions hold:

- (a) The group \( G \) is isomorphic to a subgroup of the semilinear group \( \Gamma(V) \), acting naturally on \( V \).
- (b) \( G \) is a Frobenius group with cyclic kernel of order \( p^n \) and Frobenius complement of prime order \( r \), with \( r \nmid p \). Moreover, \( n \mid p - 1 \) and, for every \( v \in V \setminus \{0\} \), we have \( |C_G(v)| = r \). Finally, \( p^n = \frac{q^n - 1}{q^r - 1} \).

**Remark 3.2.** We point out that part (a) of Theorem 3.1 is not true for \( p = 2 \) (see Example 3.3 below), but it could be proved that it holds for \( p = 3 \). However, we decided not to include the details of the proof here, since in any case Theorem C is not valid for \( p = 3 \) (see Example 3.5).

As regards part (b) of Theorem 3.1, it fails both for \( p = 2 \) and for \( p = 3 \) (see Examples 3.3 and 3.4).
Example 3.3. Consider the action of $G = \text{SL}(2, 3)$ on the natural module $V$. Then, for every $v \in V$, $C_v(G)$ contains a Sylow 3-subgroup of $G$. Hence, as $G$ is a $[2, 3]$-group, there is no $2'$-deranged orbit for the action of $G$ on $V$.

Example 3.4. Let $G$ be a subgroup of order $2 \cdot 3^3 = 54$ of $\Gamma(2^6)$, acting on $V = GF(2^6)$. For every $v \in V$, we have that $C_v(G)$ contains a Sylow 2-subgroup of $G$. But $G$ is not a Frobenius group. (Moreover, $n = 6$ does not divide $p - 1 = 2$, and $|C_v(G)|$ is either 2 or 6, depending on the choice of $v \in V \setminus \{0\}$.)

Proof of Theorem 3.1(a). By Lemma 2.9, $F = F(G)$ is a $p$-group. As $G$ acts faithfully and primitively on $V$, we get

$$F = EU$$

where $E = \Omega_1(F)$ is either cyclic of prime order or an extraspecial $p$-group, $U = Z(F)$ is cyclic and $Z = Z(E) = E \cap U$ has order $p$ (see for instance [6, Lemma 0.5 and Theorem 1.9]). Set $e = \sqrt{|E : Z|}$.

Since $G$ is not a $p$-group and it has no $p'$-deranged orbit on $V$, then in particular $G$ has no regular orbit on $V$. Hence, by Theorem 3.1 of [8], either $e \leq 9$ or $e = 16$. We recall that $e = 1$ if and only if $G$ is isomorphic to a subgroup of the semilinear group $\Gamma(V)$ acting naturally on $V$ (see [6, Corollary 2.3(a)]). For a proof by contradiction, assume $e \neq 1$. Hence, as $e$ is a power of the prime $p \geq 5$, we have that either $e = p = 5$ or $e = p = 7$.

Let $N/F$ be a chief factor of $G$. Then $N/F$ is an elementary abelian $r$-group for some prime $r \neq p$. Let $A = C_G(U)$. If $A > F$, choose $N/F \leq A/F$. In this case we have $A/F \leq \text{Sp}(2, p) = \text{SL}(2, p)$ (see for instance [8, Theorem 2.2(4), (5)]). Recall that, when $p$ is odd, $\text{SL}(2, p)$ has a unique involution; moreover, for $p \in \{5, 7\}$, the $(2, 2)'$-part of $|\text{SL}(2, p)|$ is 3. We conclude that $N/F$ is cyclic (of order 2 or 3). On the other hand, if $A = F$, then again $N/F$ is cyclic because $G/F = G/A \leq \text{Aut}(U)$ and $U$ is a cyclic group of odd prime order.

We claim that, in both cases $e = 5$ and $e = 7$, there exist $v \in V$ and $x \in N$ of order $r$ such that $x$ does not fix any element in the $G$-orbit of $v$. In other words, we prove the existence of a $p'$-deranged orbit for the action of $G$ on $V$, against the assumption.

Let $R$ be a Sylow $r$-subgroup of $N$. So $R$ is cyclic of order $r$. As $N \leq G$, the number of $G$-conjugates of $R$ is $|N : N_N(R)| = |F : C_F(R)|$, and it is easily seen that our claim follows if

$$|F : C_F(R)| |C_V(R)| < |V|$$

holds.

Clearly, $|F : C_F(R)| \leq |F| = p^2|U|$. If $N \leq A$, then $|F : C_F(R)| \leq p^2$. Moreover, $|C_V(R)| \leq |V|^{\beta}$ where $\beta = 1/2$ if $r \neq 2$ or $N$ is not contained in $A$, and $\beta = 2/3$ if $r = 2$ and $N \leq A$ (see [8, Lemma 2.4]).

Finally, let $W$ be an irreducible submodule of $V_U$. Then $|U|$ divides $|W| - 1$, because $U$ acts fixed point freely on $W$ (see [8, Theorem 2.2(6)]) and $|V| = w^a$, where $w = |W|^{1/3}$ for some positive integer $b$ (see [8, Theorem 2.2(7)]). Observe that, if $e = 5$, then $p = 5$ divides $|W| - 1$, whence $w \geq 11$; similarly, if $e = 7$, then we get $w \geq 8$.

Assume $r = 2$ and $N \leq A$. Then for $e = 5$ the inequality (1) is satisfied, because $w \geq 11$ and hence we have

$$|F : C_F(R)| |C_V(R)| \leq 5^2 w^{10} < w^5 = |V|.$$ 

Similarly, one proves that (1) holds for $r = 2$ and $e = 7$.

When $r \neq 2$ or $N$ is not contained in $A$, the inequality (1) also holds because, as $|U| < |W| \leq w$,

$$e^2 |U| w^2 < e^2 w^{5/2 + 1} < w^e$$

holds both for $e = 5$ and for $e = 7$.

Therefore, $e = 1$ and $G$ is isomorphic to a subgroup of $\Gamma(V)$. This concludes the proof of (a). □
Proof of Theorem 3.1(b). Denote by $I_0 = I_0(V)$ the subgroup of $\Gamma(V)$ consisting of the multiplication maps (see [6, p. 38]), and set $G_0 = G \cap I_0$; we know that $G/G_0 \simeq G/I_0$ is cyclic. Now, if $R$ is a Hall $p'$-subgroup of $G$, we get $R \cap G_0 = 1$ because Lemma 2.9 yields that $G_0 \leq F(G)$ is a (cyclic) $p$-group. This implies that $R \simeq G_0R/G_0$ is cyclic, and also that $r := |R|$ divides $|\Gamma(V) : I_0| = n$.

Next, we observe that $C_V(R) \cap C_V(R^g) = \{0\}$ for every $g \in G$ such that $R^g \neq R$. In fact, we can assume $g \in G_0$, and we can choose an element $x \in R \setminus R^g$: now, if $v \in V \setminus \{0\}$ is centralized by both $R$ and $R^g$, we get $[x, g] \in C_V(v) \cap G_0 = 1$, whence $x = x^g \in R^g$, a contradiction. Since $R$ is cyclic, the assumption on $p'$-deranged orbits implies that the centralizer of every nontrivial element of $V$ contains one (and hence only one) Hall $p'$-subgroup of $G$. Thus, $V \setminus \{0\}$ is partitioned by the sets $C_V(R) \setminus \{0\}$ for $R \in \text{Hall}_{p'}(G)$. It follows that $q^n - 1 = h(C_V(R) - 1) = h(q^{n/r} - 1)$, where $h$ is the cardinality of the set of Hall $p'$-subgroups of $G$ and the second equality follows from Lemma 3(ii) of [2].

By coprimality, $G_0 = C_{G_0}(R) \times [G_0, R]$, so $h = |[G_0, R]|$ is a power of $p$. Hence

$$h = p^a = \frac{q^n - 1}{q^{n/r} - 1}. \quad (2)$$

Moreover, $G_0$ being a cyclic $p$-group not centralized by $R$, we have $C_{G_0}(R) = 1$ and $[G_0, R] = G_0$.

Note that there exists a Zsigmondy prime divisor of $q^n - 1$ (see [6, 6.2]). In fact, if $n = 2$ and $q$ is a Mersenne prime, then $p^2 = q + 1$ is a power of 2, a contradiction. If $n = 6$ and $q = 2$, then $r = 2$ (as $2^6 - 1$ and $(2^5 - 1)/(2^3 - 1)$ are not prime powers) and $p = 3$, which is not the case. Therefore, $p$ is the (unique) Zsigmondy prime divisor of $q^n - 1$. In particular, $n$ divides $p - 1$. As $G_0$ is isomorphic to a subgroup of the cyclic group $\Gamma(V)/I_0$ of order $n$, we see that $p \mid |G_0/G_0|$, whence $|G_0/G_0| = r$. Therefore we get $G = G_0R$, and now it is clear that the centralizer of every nontrivial element of $V$ is a Hall $p'$-subgroup of $G$.

Now, write $n = mr$ and $r = r_1r_2$, where $r_1$ and $r_2$ are positive integers. Then $(q^m - 1)/(q^n - 1)$ is a divisor of $(q^n - 1)/(q^{n/r} - 1) = p^a$, whence either $r_1 = 1$ or $p$ divides $q^{n/r} - 1$, which yields $r_1 = r$. We conclude that $r$ is a prime number.

Finally, $R$ acts fixed point freely on $G_0 = [G_0, R]$, so that $G$ is a Frobenius group with cyclic kernel $G_0$ of order $p^a$ and Frobenius complement of order $r$. \hfill \Box

We are now ready to prove Theorem C, which we state again.

Theorem C. Let $G$ be a solvable group, $p \geq 5$ and $q$ prime numbers, and $V$ a faithful irreducible $G$-module over $GF(q)$. Assume that there are no $p'$-deranged orbits for the action of $G$ on $V$. Then the following conclusions hold:

(a) Either $G$ is a $p$-group, or there exist $H \subseteq \Gamma(q^n)$ (for a suitable $n \in \mathbb{N}$) and a (possibly trivial) $p$-group $K$ such that $G$ is isomorphic to a subgroup of $H \times K$. Moreover, $H$ is a Frobenius group with cyclic kernel of $p$-power order and Frobenius complement of prime order $r$.

(b) The Hall $p'$-subgroups of $G$ are elementary abelian $r$-groups and $I_p'(G) \leq 1$.

Proof. Assuming that $G$ is not a $p$-group (otherwise there is nothing to prove), we start by proving (a).

Choose a subgroup $T$ of $G$ and a primitive submodule $W$ of $V_T$ such that $V = W^G$ (possibly $T = G$). Set $|W| = q^d$. Denoting by $H$ the factor group $T/C_T(W)$, we first observe that, by Lemma 2.7, there does not exist any $p'$-deranged orbit for the action of $H$ on $W$. Therefore, by Lemma 2.9, $F(H)$ is a $p$-group; moreover, if $H \neq F(H)$, Theorem 3.1 yields that $H \leq \Gamma(W) = \Gamma(q^n)$ and that $H$ is a Frobenius group with cyclic kernel of $p$-power order and Frobenius complement of prime order $r$.

In what follows we shall keep in mind Remark 2.1 and Remark 2.3; in particular, recall that $G$ can be identified with a subgroup $\phi(G)$ of $H \times K$, where $K$ is a transitive subgroup of $Sym(\Sigma)$. Our first goal is to show that $K$ is a $p$-group and, for a proof by contradiction, we shall assume the contrary.
If there exists a subset $A$ of $\Sigma$ such that $(A, \Sigma \setminus A)$ lies in a $p'$-deranged orbit for the action of $K$ on $\mathcal{P}(\Sigma)$, then take a nonzero element $w \in W$, and consider the element $v$ of $W^{\oplus s}$ whose $i$th component is $w$ if $i \in A$, whereas it is $0$ if $i \notin A$. We claim that $v$ lies in a $p'$-deranged orbit for the action of $\phi(G)$ on $W^{\oplus s}$. In fact, let $k \in K$ be a $p'$-element which does not fix any element in the $K$-orbit of $(A, \Sigma \setminus A)$, and let $x \in \phi(G)$ be a preimage of $k$ along the top projection of $\phi(G)$ onto $K$ (note that $x$ can be chosen to be a $p'$-element as well). Now, it is easy to see that $x$ does not fix any element in the $\phi(G)$-orbit of $v$. Our claim is proved, yielding a contradiction. We conclude that there does not exist any $p'$-deranged orbit for the action of $K$ on $\mathcal{P}(\Sigma)$.

As $K$ is assumed not to be a $p'$-group, we are in a position to apply Lemma 2.8 with $K$ in place of $G$ and $\Sigma$ in place of $\Omega$, getting that $p$ is $5$ (because our assumptions imply $p \neq 3$) and that there exists a $p'$-deranged orbit for the action of $K$ on $\mathcal{P}_2(\Sigma)$. Let $(A, B, C)$ be an element of $\mathcal{P}_2(\Sigma)$ lying in such an orbit. Assume that $H$ is not transitive on $W \setminus \{0\}$, and choose two elements $w, z$ of $W \setminus \{0\}$ lying in distinct $H$-orbits. Set now $v$ to be the element of $W^{\oplus s}$ whose $i$th component is $w$ if $i \in A$, it is $z$ if $i \in B$, and it is $0$ if $i \in C$. As in the previous paragraph, it is not difficult to see that $v$ lies in a $p'$-deranged orbit for the action of $\phi(G)$ on $W^{\oplus s}$, again contradicting our assumptions.

Therefore, the action of $H$ on $W \setminus \{0\}$ must be transitive. As we already observed, $F(H)$ is a $p$-group. Now, recall that we set $|W| = q^n$: if $H = F(H)$, then $q^n - 1$ is clearly a power of $p$. We claim that the same holds also when $H \neq F(H)$.

In fact, in that case Theorem 3.1 applies to the action of $H$ on $W$. In particular, denoting the group $H \cap \Gamma_0(W)$ by $H_0$, we get $H = H_0C_p(w)$ for some $w \in W \setminus \{0\}$. It follows that the elements of $W \setminus \{0\}$ are transitively permuted by $H_0$ as well and, since $|H_0|$ is a $p'$-power, our claim is proved.

In this situation, by Proposition 3.1 of [6], the prime $p$ must be either 2 or a Mersenne prime, which is not the case because we know that $p = 5$. We reached the final contradiction.

Our argument so far shows that $K$ is a $p$-group. As $G$ is not a $p'$-group and $G$ is isomorphic to a subgroup of $H : K$, we conclude that $H$ is not a $p$-group. Therefore, as already observed, Theorem 3.1 applies to the action of $H$ on $W$, and all the conclusions in (a) follow.

As for (b), this is an immediate consequence of (a). □

The following example shows that Theorem C fails for $p = 3$.

**Example 3.5.** Consider $G = \text{GL}(2, 2) : \text{Sym}(3)$ acting imprimitively (and irreducibly) on the vector space $V$ of dimension 6 over GF(2). There is no $3'$-deranged orbit for the action of $G$ on $V$, because $C_G(v)$ contains a Sylow 2-subgroup of $G$ (which is a Hall 3'-subgroup of $G$) for every $v \in V$. But the Sylow 2-subgroups of $G$ are nonabelian and $l_2(G) = 2$.

4. Brauer character tables with no zeros

As an application of the results in the previous section, we can now derive Theorem A, which we state again.

**Theorem A.** Let $G$ be a group and $p > 5$ a prime number. Assume that $O_p(G) = 1$. If the $p'$-Brauer character table of $G$ does not contain any zero, then the Hall $p'$-subgroups of $G/F(G)$ are abelian of squarefree exponent and the $p'$-length of $G/F(G)$ is at most 1.

**Proof.** Observe that our assumption on the Brauer character table is obviously inherited by factor groups. In view of this fact, it will be enough to prove Theorem A in the case when $\Phi(G) = 1$; this extra assumption ensures that $F := F(G)$ is a completely reducible $G$-module (possibly in mixed characteristic), and that every abelian normal subgroup of $G$ has a complement in $G$ (see [3, III.4.4]).

Let $V$ be a minimal normal subgroup of $G$. As $V \leq F$, we have that $V$ is abelian, and it can be viewed as a simple $G$-module in characteristic different from $p$. Denoting by $L$ a complement for $V$ in $G$, we get $C_G(V) \leq VL = G$, and we can consider the factor group $\overline{G} = G/C_G(V)$. Now, adopting the bar convention for the natural homomorphism of $G$ onto $\overline{G}$, we have that $W := \overline{V}$ is a faithful irreducible $\overline{L}$-module (note that $W \cong V$), and therefore also $\overline{W} := \text{Irr}(W)$ is such.
Take $\mu \in \hat{W}$, and observe that $\mu$ lies in $\text{IBr}_p(W)$ as $p$ does not divide $|W|$ (recall that we are assuming $O_p(G) = 1$). If $\phi \in \text{IBr}_p(G)$ lies over $\mu$, then $\phi$ is induced from $I_G^C(\mu)$ (see [7, (8.9)]), and therefore it vanishes on every element outside $\bigcup_{x \in G} I_G^C(\mu^x)$. In particular, since the hypothesis about the Brauer character table is inherited by elementary abelian $p'$-subgroups of $G$, every $p'$-element of $\hat{L}$ lies in $\bigcup_{x \in \hat{L}} C^G_F(\mu^x)$. In other words, there does not exist any $p'$-deranged orbit for the action of $L$ on $\hat{W}$, and we can apply Theorem C: since $G/C_G(V)$ acts on $V$ as $\hat{L}$ acts on $\hat{W}$, we get that the Hall $p'$-subgroups of $G/C_G(V)$ are elementary abelian $r'$-groups for a suitable prime $r$ and that $l_{p'}(G/C_G(V)) \leq 1$.

Writing $F = V_1 \times \cdots \times V_n$ where the $V_i$ are minimal normal subgroups of $G$, and observing that $F = \bigcap_{i=1}^n C_G(V_i)$, the result now follows because $G/F$ can be regarded as a subgroup of $G/C_G(V_1) \times \cdots \times G/C_G(V_n)$. $\square$

Corollary B, that was stated in the Introduction, is an immediate consequence of Theorem A.

**Proof of Corollary B.** An application of Theorem A to the factor group $G/O_p(G)$ yields the desired conclusions. $\square$

As mentioned in the Introduction, the bounds of Corollary B are sharp, as shown by the following example.

**Example 4.1.** Let $H$ be the subgroup of order $2^5 \cdot 5$ of the affine semilinear group $A\Gamma(2^4)$, and let $G = H : C_5$ be the wreath product of $H$ with a cyclic group of order 5. As the 5-Brauer character table of $H$ contains just odd integers, it is easily seen that the 5-Brauer character table of $G$ contains no zeros. In fact, the restriction of any irreducible 5-Brauer character of $G$ to the base group $N$ of $G$ is the sum of an odd number of irreducible characters of $N$, and their values are odd integers. Now, we get $O_5(G) = 1$,

$$l_5(G) = 2 \quad \text{and} \quad l_5'(G) = l_2(G) = 2.$$  

Finally, we prove Proposition D.

**Proof of Proposition D.** Since $p$ is an odd prime and the Brauer character table of $G$ for the prime $p$ contains no zeros, $G$ is solvable by Theorem 1.3 in [4]. Let $\phi$ be in $\text{IBr}_p(G)$ and, for a proof by contradiction, assume that $p$ is not a divisor of $\phi(1)$. By the Fong–Swan Theorem (see for instance [7, (10.1)]), there exists $\chi \in \text{Irr}(G)$ whose restriction to the $p'$-elements of $G$ coincides with $\phi$. In particular, $\chi(1)$ is a $p'$-number. But now Theorem A in [5] yields that $\chi$ (whence $\phi$) must vanish on some $p'$-element of $G$, against the hypothesis. $\square$

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**References**