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ScienceDirect
Expo. Math. 26 (2008) 269–277
EXPOSITIONES MATHEMATICAE
www.elsevier.de/exmath

An extension of the Gleason–Kahane–Żelazko theorem: A possible approach to Kaplansky's problem[☆]

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Received 4 January 2007

Abstract

Let $\mathscr{A}$ and $\mathscr{B}$ be unital Banach algebras with $\mathscr{B}$ semisimple. Is every surjective unital linear invertibility preserving map $\phi : \mathscr{A} \to \mathscr{B}$ a Jordan homomorphism? This is a famous open question, often called "Kaplansky's problem" in the literature. The Gleason–Kahane–Żelazko theorem gives an affirmative answer in the special case when $\mathscr{B} = \mathbb{C}$. We obtain an improvement of this theorem. Our result implies that in order to answer the question in the affirmative it is enough to show that $\phi(x^2)$ and $\phi(x)$ commute for every $x \in \mathscr{A}$. In this way we obtain a new proof of the Marcus–Purves theorem.

MSC 2000: 47B48; 16R50

Keywords: Linear preserver; Invertibility; Commutativity

1. Introduction and statement of the main result

Let $\mathscr{A}$ and $\mathscr{B}$ be complex unital algebras and let $\phi : \mathscr{A} \to \mathscr{B}$ be a linear map. We say that ϕ is *unital* if $\phi(1) = 1$ and that ϕ *preserves invertibility* if $\phi(x) \in \mathscr{B}$ is invertible for every invertible $x \in \mathscr{A}$. Note that if ϕ preserves invertibility, then $x \mapsto \phi(1)^{-1}\phi(x)$, $x \in \mathscr{A}$,

☆ Partially supported by grant from ARRS.

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0723-0869/$ - see front matter
doi:10.1016/j.exmath.2007.11.004

is a unital invertibility preserving map. Thus, when studying invertibility preservers, there is no loss of generality in assuming that they are unital.

The famous Gleason–Kahane–Żelazko theorem [16,18,22] (we will refer to it as the GKZ theorem) states that every unital invertibility preserving linear functional φ on a unital Banach algebra $\mathscr{A}$ is multiplicative. In the case of functionals the invertibility preserving property simply means that $\varphi(x) \neq 0$ for every invertible $x \in \mathscr{A}$. Let M_n denote the algebra of all $n \times n$ complex matrices. In 1959 Marcus and Purves [20] proved that every linear unital invertibility preserving map on M_n is either multiplicative, or anti-multiplicative. Recall that a map $\phi : \mathscr{A} \to \mathscr{B}$ is anti-multiplicative if $\phi(xy) = \phi(y)\phi(x)$ for all $x, y \in \mathscr{A}$. It should be mentioned here that an analogous result for singularity preserving bijective linear maps on matrix algebras had been proved already in 1949 by Dieudonné [13], and that even much earlier, in 1897, Frobenius [14] had obtained a similar result for determinant preserving linear maps.

A linear map $\phi : \mathscr{A} \to \mathscr{B}$ is called a *Jordan homomorphism* if $\phi(x^2) = \phi(x)^2$ for every $x \in \mathscr{A}$. It turns out that a linear map on M_n is a Jordan homomorphism if and only if it is either multiplicative, or anti-multiplicative. Motivated by the above results, in 1970 Kaplansky [19] asked which conditions imply that every unital linear invertibility preserving map $\phi : \mathscr{A} \to \mathscr{B}$ is a Jordan homomorphism. A lot of work has been done on this problem (see the survey paper [3]). Based on several partial positive results and some counterexamples (for a more detailed explanation we refer to [12]) the following conjecture has been posed by Aupetit.

Conjecture 1.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be semisimple unital Banach algebras and let $\phi : \mathscr{A} \to \mathscr{B}$ be a surjective unital invertibility preserving linear map. Then ϕ is a Jordan homomorphism.*

This problem is still open even for C^*-algebras (see [17]). The best partial solutions are due to Aupetit [4] and Sourour [21]. Aupetit confirmed the conjecture for von Neumann algebras, and Sourour for algebras of bounded linear operators acting on Banach spaces. The proofs are based on the reduction of Kaplansky's problem to the problem of characterizing linear maps preserving idempotents or operators of rank one. Unfortunately, these kinds of approaches obviously can not work in the general case. To the best of our knowledge, settling Conjecture 1.1 in its full generality seems to be out of reach at present; that is, none of the approaches that are known from the literature can work out in the general situation.

Let $\mathscr{A}$ and $\mathscr{B}$ be arbitrary algebras and let $\phi : \mathscr{A} \to \mathscr{B}$ be a linear map. We shall say that ϕ *locally preserves commutativity* if

$$[\phi(x^2), \phi(x)] = 0 \quad \text{for all } x \in \mathscr{A}. \tag{1}$$

Here, $[x, y]$ denotes the Lie product $xy - yx$. Such maps were studied extensively in ring theory. The first result was obtained in 1993 by the first author [9], and this was followed by a series of papers by different authors. For history and details we refer the reader to the last papers in the series [7,10]. Let us only mention here that one can regard (1) as a special *functional identity*, so the results on maps that locally preserve commutativity are usually obtained as applications of the general theory of functional identities (see [11]). The standard goal is to show that ϕ is a linear combination of a Jordan homomorphism and a map having the range in the center $Z_{\mathscr{B}}$ of $\mathscr{B}$. Assuming the surjectivity of ϕ, this can be shown under rather mild assumptions on $\mathscr{A}$ and $\mathscr{B}$. On the other hand, some special algebras must obviously be excluded. For instance, if every element in $\mathscr{A}$ is algebraic of degree $\leqslant 2$ (as for example in M_2), then every linear map sending 1 in $Z_{\mathscr{B}}$ obviously locally preserves commutativity; so one cannot expect any reasonable conclusion in this case.

The purpose of this note is to "glue" the topic of invertibility preservers with the topic of preservers of local commutativity, and in this way obtain the following result.

Theorem 1.2. *Let $\mathscr{A}$ and $\mathscr{B}$ be unital Banach algebras with $\mathscr{B}$ semisimple, and let $\phi : \mathscr{A} \to \mathscr{B}$ be a unital surjective linear map that preserves invertibility and locally preserves commutativity. Then ϕ is a Jordan homomorphism.*

Of course, if $\mathscr{B} = \mathbb{C}$, then the assumptions that ϕ locally preserves commutativity and is surjective are automatically fulfilled. Thus, Theorem 1.2 is a noncommutative generalization of the GKZ theorem. On the other hand, in view of the heavy algebraic and analytic machineries that are at our disposal, it is admittedly not so surprising that Theorem 1.2 can be proved. Still the proof has turned out to be nontrivial, and we believe it is of some interest also because of the diversity of the tools that are used. However, our main goal in this paper is to point out a possible new approach to the general Kaplansky's problem. Namely, in order to confirm Conjecture 1.1 one has to show, in principle, that $\phi(x^2)$ is equal to the square of $\phi(x)$ for every $x \in \mathscr{A}$. Theorem 1.2 now tells that it is enough to establish a much weaker assertion that $\phi(x^2)$ and $\phi(x)$ always commute. That is, Conjecture 1.1 is now equivalent to

Conjecture 1.3. *Let $\mathscr{A}$ and $\mathscr{B}$ be semisimple unital Banach algebras and let $\phi : \mathscr{A} \to \mathscr{B}$ be a surjective unital invertibility preserving linear map. Then ϕ locally preserves commutativity.*

At the end of the note we will confirm Conjecture 1.3 for the case when $\mathscr{A} = \mathscr{B} = \mathscr{M}_n$, and thereby obtain a new proof of the aforementioned Marcus-Purves theorem. Unfortunately it is not clear how to extend the method of our proof to more general algebras. But on the other hand this is, to the best of our knowledge, the first proof of the Marcus–Purves theorem that does not depend on rank or idempotents. It is based on the concept of commutativity, which is, unlike the concepts of a rank and an idempotent, non-trivial in every algebra in the sense that we can always find a lot of commuting pairs while it is possible that 0 and 1 are the only idempotents in an algebra and it is also possible that there are no non-zero finite rank elements with respect to any natural definition of the rank. So this at least gives an indication that the proposed new approach could be effective.

On the other hand, one might take the opposite point of view and try to disprove Conjecture 1.1. Then the message of this note is that one has to search for a counterexample among maps that do not locally preserve commutativity.

2. Proof of the main theorem

We begin with some auxiliary results. First we record an elementary lemma.

Lemma 2.1. *Let $\mathscr{A}$ be a ring and let $\mathscr{G}$ be a torsion-free additive group. Assume that $B : \mathscr{A} \times \mathscr{A} \to \mathscr{G}$ is a biadditive map such that*

$$B(x, x) = B(x^2, x) = 0$$

for all $x \in \mathscr{A}$. Then

$$B(x^n, x^m) = 0$$

for all $x \in \mathscr{A}$ and all positive integers n, m.

Proof. Linearizing $B(x, x) = 0$ we get that B is skew-symmetric, that is,

$$B(x, y) = -B(y, x)$$

for all $x, y \in \mathscr{A}$, and linearizing $B(x^2, x) = 0$ we see that B satisfies

$$B(x \circ y, z) + B(z \circ x, y) + B(y \circ z, x) = 0 \tag{2}$$

for all $x, y, z \in \mathscr{A}$. Here, $u \circ v$ denotes $uv + vu$. Fix an integer $N \geqslant 4$ and let us show that

$$a_k = B(x^{N-k}, x^k)$$

is equal to 0 for every $k = 1, \ldots, N - 1$. Setting $y = x^k$ and $z = x^{N-k-1}$ in (2) we get

$$a_{N-k-1} + a_k + a_1 = 0, \quad k = 1, \ldots, N - 2.$$

As $a_{N-m} = -a_m$ by skew-symmetry of B, we can rewrite this as

$$a_{k+1} = a_k + a_1, \quad k = 1, \ldots, N - 2.$$

This clearly implies that

$$a_k = ka_1, \quad k = 1, \ldots, N - 1.$$

However, as $a_{N-1} = -a_1$, we have $-a_1 = (N - 1)a_1$. Thus, $Na_1 = 0$, and hence $a_1 = 0$ for $\mathscr{G}$ is torsion-free. Accordingly, $a_k = 0$ for each $k = 1, \ldots, N - 1$. □

The next result is much deeper. It is one of the important by-products of the general theory of functional identities.

Lemma 2.2. *Let $\mathscr{V}$ be linear space over a field $\mathbb{F}$ with characteristic not 2, let $\mathscr{B}$ be a prime $\mathbb{F}$-algebra with extended centroid $\mathscr{C}$, let $\phi : \mathscr{V} \to \mathscr{B}$ be a surjective linear map, and let $F : \mathscr{V} \times \mathscr{V} \to \mathscr{B}$ be a bilinear map such that $[F(x, x), \phi(x)] = 0$ for all $x \in \mathscr{V}$. Suppose that $\mathscr{B}$ contains an element that is not algebraic over $\mathscr{C}$ of degree $\leqslant 2$. Then there exist $\lambda \in \mathscr{C}$, a linear map $\mu : \mathscr{V} \to \mathscr{C}$, and a bilinear map $\nu : \mathscr{V} \times \mathscr{V} \to \mathscr{C}$ such that*

$$F(x, x) = \lambda\phi(x)^2 + \mu(x)\phi(x) + \nu(x, x) \quad \text{for all } x \in \mathscr{V}.$$

Although this lemma can be considered as a “folklore” result among specialists in functional identities, it is not so easy to find a direct reference. Anyway, from

[11, Theorem 5.11 and Corollary 4.15] the assertion of the lemma follows, however, with μ (resp. ν) only additive (resp. biadditive). Using a standard argument one can easily show that μ (resp. ν) is actually linear (resp. bilinear).

Lemma 2.2 involves the concept of the extended centroid of a prime $\mathbb{F}$-algebra. We refer the reader to [8] for a full account on this concept. Let us only mention here that in general the extended centroid is a certain field extension of the base field $\mathbb{F}$. Fortunately, in the only case we will be interested in, the extended centroid coincides with the base field:

Lemma 2.3. The extended centroid of a primitive complex Banach algebra is $\mathbb{C}$.

This lemma is another folklore result. It can be deduced, for example, from [8, Corollary 4.1.2].

We now have enough information to start the proof of Theorem 1.2. Actually, we will use several other results in this proof. But they are either well-known or easy to formulate, so it does not seem necessary to state them explicitly.

By $\sigma(a)$ we will denote the spectrum of the element a.

Proof of Theorem 1.2. . So, let us assume that $\mathscr{A}$ and $\mathscr{B}$ are unital Banach algebras with $\mathscr{B}$ semisimple and $\phi : \mathscr{A} \to \mathscr{B}$ is a unital surjective linear map that preserves invertibility and locally preserves commutativity. We may and we will assume that $\mathscr{B}$ is a primitive algebra. Indeed, if P is a primitive ideal in $\mathscr{B}$, then $\mathscr{B}/\mathscr{P}$ is a primitive Banach algebra and the product of ϕ and the quotient map is a surjective unital invertibility preserver which locally preserves commutativity. If this product is a Jordan homomorphism for every primitive ideal P, then, because the intersection of all primitive ideals is 0 by the semisimplicity of $\mathscr{B}$, the map ϕ itself is a Jordan homomorphism.

If $\mathscr{B}$ is finite dimensional, and hence isomorphic to M_n for some n, then it is exactly the result by Aupetit [1] that gives the desired conclusion. So, we may assume that $\mathscr{B}$ is infinite dimensional (incidentally, our proof that follows works as long as $\dim \mathscr{B} > 16$). Note that this implies that for every $n \geqslant 1$ there exists an element in $\mathscr{B}$ which is not algebraic of degree $\leqslant n$. This is a standard fact that follows easily from the Jacobson density theorem.

Define $F : \mathscr{A} \times \mathscr{A} \to \mathscr{B}$ by $F(x, y) = \phi(xy)$. Clearly, F is bilinear and satisfies $[F(x, x), \phi(x)] = 0$ for every $x \in \mathscr{A}$. Therefore, Lemma 2.2, together with Lemma 2.3, implies that

$$\phi(x^2) = \lambda\phi(x)^2 + \mu(x)\phi(x) + \nu(x, x) \quad \text{for all } x \in \mathscr{A}, \tag{3}$$

where $\lambda \in C = \mathbb{C}$, $\mu : \mathscr{A} \to \mathbb{C}$ is a linear functional, and $\nu : \mathscr{A}^2 \to \mathbb{C}$ is a bilinear map. Of course, our goal is to show that $\lambda = 1$, $\mu = 0$, and $\nu = 0$.

Next we define $B : \mathscr{A} \times \mathscr{A} \to \mathscr{B}$ by $B(x, y) = [\phi(x), \phi(y)]$. Since $B(x, x) = B(x^2, x) = 0$ for every $x \in \mathscr{A}$, Lemma 2.1 implies that

$$B(x^n, x^m) = [\phi(x^n), \phi(x^m)] = 0 \quad \text{for all } x \in \mathscr{A}.$$

That is, ϕ maps every unital subalgebra of $\mathscr{A}$ generated by a single element into a commutative set.

We continue with analytic tools. First of all, by [2, Theorem 5.5.2], ϕ is continuous. Fix $x \in \mathscr{A}$ and denote by $\mathscr{A}_1$ the closed unital subalgebra generated by x, and by $\mathscr{B}_1$ the closed subalgebra of $\mathscr{B}$ generated by $\phi(\mathscr{A}_1)$ and all inverses of invertible elements of $\phi(\mathscr{A}_1)$. By the previous paragraph and the continuity of ϕ, $\mathscr{B}_1$ is a commutative unital Banach algebra. If $u \in \mathscr{A}_1$ is invertible in $\mathscr{A}_1$, then $\phi(u)$ is invertible in $\mathscr{B}$, and in view of the definition of $\mathscr{B}_1$, it is invertible already in $\mathscr{B}_1$. Hence, the restriction of ϕ to $\mathscr{A}_1$ is an invertibility preserving unital linear map between commutative Banach algebras $\mathscr{A}_1$ and $\mathscr{B}_1$. Let $\xi : \mathscr{B}_1 \to \mathbb{C}$ be any non-zero linear multiplicative functional. Then $\xi\,\phi_{|\mathscr{A}_1} : \mathscr{A}_1 \to \mathbb{C}$ is a unital linear functional preserving invertibility. Thus, by the GKZ theorem, $\xi(\phi(x^2) - \phi(x)^2) = 0$ for every linear multiplicative functional ξ on $\mathscr{B}_1$, and therefore $\phi(x^2) - \phi(x)^2$ lies in the Jacobson radical of $\mathscr{B}_1$. In particular, $\phi(x^2) - \phi(x)^2$ is a quasinilpotent in $\mathscr{B}_1$, and hence it is a quasinilpotent in $\mathscr{B}$. Denoting by Q the set of all quasinilpotent elements of $\mathscr{B}$ we thus have $\phi(x^2) - \phi(x)^2 \in Q$ for every $x \in \mathscr{A}$, which together with(3) gives

$$\alpha\phi(x)^2 + \mu(x)\phi(x) + v(x,x) \in Q \quad \text{for all } x \in \mathscr{A}, \tag{4}$$

where $\alpha = \lambda - 1$. As $\mathscr{B}$ is infinite dimensional, it contains elements b whose spectra have cardinality at least three (once again we may apply the Jacobson density theorem to verify this statement). If $\alpha \neq 0$, then (4) shows that $b = \phi(x)$ for some x has the property that $\sigma(p(b)) = \{0\}$ for some polynomial p of degree 2, a contradiction. Therefore, $\alpha = 0$, i. e. $\lambda = 1$. Now (4) reduces to

$$\mu(x)\phi(x) + v(x,x) \in Q \quad \text{for all } x \in \mathscr{A}. \tag{5}$$

Accordingly, for every $x \in \mathscr{A}$ we have either $\mu(x) = 0$, or $|\sigma(\phi(x))| = 1$. Here, $|\cdot|$ denotes the cardinality of the set. Suppose that there exists $x_0 \in \mathscr{A}$ such that $\mu(x_0) \neq 0$. If $y \in \mathscr{A}$ is such that $|\sigma(\phi(y))| \geqslant 2$, then $\mu(y) = 0$, whence $\mu(x_0 + y) \neq 0$, and therefore, $|\sigma(\phi(x_0 + y))| = 1$. By the Jacobson density theorem $\mathscr{B}$ is isomorphic to the algebra of bounded linear operators acting densely on an infinite dimensional Banach space X. Pick linearly independent $\xi_1, \xi_2, \xi_3, \xi_4 \in X$. We will slightly abuse the notation and identify an element in $\mathscr{B}$ by the corresponding operator on X. Now, by the surjectivity of ϕ and density there exists $y \in \mathscr{A}$ satisfying

$$\phi(y)\xi_1 = \xi_1 - \phi(x_0)\xi_1, \quad \phi(y)\xi_2 = -\phi(x_0)\xi_2,$$
$$\phi(y)\xi_3 = \xi_3, \quad \phi(y)\xi_4 = 0.$$

The last two relations show that $|\sigma(\phi(y))| \geqslant 2$. From the first two relations we infer that

$$\phi(x_0 + y)\xi_1 = \xi_1 \quad \text{and} \quad \phi(x_0 + y)\xi_2 = 0$$

and consequently, $|\sigma(\phi(x_0 + y))| \geqslant 2$, a contradiction. Therefore, $\mu = 0$, and so, by (5), $v = 0$ as well. This completes the proof.

3. A new proof of the Marcus–Purves theorem

First we need a preliminary result characterizing the commutativity of certain matrices through the means of the addition and the spectrum. Let $A \in M_n$ be a matrix with n distinct eigenvalues. We will call a linear subspace $\mathscr{W} \subset M_n$ an A-space if

- $\dim \mathscr{W} = \frac{1}{2}n(n-1)$, and
- $\sigma(A+T) = \sigma(A)$ for every $T \in \mathscr{W}$.

Lemma 3.1. *Let $A, B \in M_n$ be two matrices both having n distinct eigenvalues. Then the following are equivalent*:

- $AB = BA$,
- *The set of all A-spaces is equal to the set of all B-spaces.*

Proof. Assume first that A and B commute and that $\mathscr{W}$ is an A-space. Let $a_1, \ldots, a_n$ be the eigenvalues of A and C any member of $\mathscr{W}$. From $\sigma(A + \lambda C) = \sigma(A)$ we conclude that

$$(A + \lambda C - a_1 I) \ldots (A + \lambda C - a_n I) = 0$$

for every $\lambda \in \mathbb{C}$. As all the coefficients of this polynomial in λ must be zero we get in particular that C is a nilpotent. Hence, $\mathscr{W}$ is a space of nilpotent matrices of dimension $(1/2)n(n-1)$, and is therefore simultaneously triangularizable by Gerstenhaber's theorem [15]. We may assume with no loss of generality that $\mathscr{W}$ is the space of all strictly upper triangular matrices. We claim that from $\sigma(A+C) = \sigma(A)$, $C \in \mathscr{W}$, it now follows that A is an upper triangular matrix whose diagonal entries form the set $\{a_1, \ldots, a_n\}$. Indeed, let A_1 be the strictly upper triangular matrix obtained from A by replacing all entries on the diagonal and below the diagonal by zeroes. Then $A - A_1$ is a lower diagonal matrix and because $\sigma(A - A_1) = \sigma(A)$ we conclude that the set of diagonal entries of A is equal to $\sigma(A)$. Suppose now that there exists a_{ij}, the (i, j)-entry of A, such that $a_{ij} \neq 0$ and $i > j$. Assume further that among all nonzero entries of A below the main diagonal a_{ij} is chosen in such a way that the difference $i - j$ is minimal. Then $-A_1 + E_{ji} \in \mathscr{W}$, where E_{ji} is the matrix having all entries equal to zero but the (j, i)-entry which is equal to one. Then $\sigma(A - A_1 + E_{ji}) \neq \sigma(A)$, a contradiction.

Thus, we can find eigenvectors of A that are of the form

$$e_1, e_2 + \mu e_1, \ldots, e_n + \delta e_{n-1} + \cdots + \eta e_1.$$

Here, $e_1, \ldots, e_n$ denote the standard basis vectors of $\mathbb{C}^n$. As A and B are commuting matrices with n distinct eigenvalues, they have the same eigenspaces. It follows that B is an upper triangular matrix as well. Consequently, $\mathscr{W}$ is a B-space, as desired.

To prove the converse assume that the set of all A-spaces is equal to the set of all B-spaces. We have to show that A and B have the same eigenspaces. Assume to the contrary that a certain vector, say e_1, is an eigenvector of A, but is not an eigenvector of B. We may further assume that $e_2, \ldots, e_n$ are eigenvectors of A. Then the set of all strictly upper triangular matrices is an A-space, and hence a B-space. As before we conclude that B is upper triangular, which further yields that e_1 is an eigenvector of B, a contradiction. □

As mentioned in the introduction, our main theorem proposes a possible way to solve Kaplansky's problem in full generality. We will illustrate this by giving a new proof of the Marcus–Purves theorem.

Theorem 3.2. (Marcus and Purves [20]). *Let $\phi : M_n \to M_n$ be a unital linear map preserving invertibility. Then ϕ is a Jordan homomorphism.*

Proof. It is not too difficult to see that ϕ is bijective [20, Lemma 2.3]. From $\phi(I) = I$ and invertibility preserving property we get immediately that

$$\sigma(\phi(A)) \subset \sigma(A) \quad \text{for all } A \in M_n.$$

Indeed, if a complex number λ does not belong to $\sigma(A)$, then $\lambda I - A$ is invertible, and consequently, $\lambda I - \phi(A)$ is invertible, or equivalently, λ does not belong to $\sigma(\phi(A))$. It follows that

$$\sigma(\phi(A)) = \sigma(A)$$

for every $A \in M_n$ with the property that $\phi(A)$ has n distinct eigenvalues. The set of matrices with n distinct eigenvalues is dense in M_n, and by the continuity of spectra the above equality holds true for every $A \in M_n$.

In order to complete the proof we only need to show that

$$[\phi(A^2), \phi(A)] = 0 \quad \text{for all } A \in M_n, \tag{6}$$

since then the conclusion follows directly from Theorem 1.2. In fact, it is enough to prove that (6) holds true on some dense subset of M_n. We choose this set to be the set of all matrices A such that both A and A^2 have n distinct eigenvalues. By Lemma 3.1, for every such A the set of all A-spaces is equal to the set of all A^2-spaces. By the spectrum preserving property and bijectivity the set of all $\phi(A)$-spaces coincides with the set of all $\phi(A^2)$-spaces. Applying Lemma 3.1 once more we get the desired equality (6). □

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