Linear Algebra and its Applications 437 (2012) 2026–2037

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Invariant adjacency matrices of configuration graphs

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ARTICLE INFO

Article history:
Received 6 June 2011
Accepted 18 May 2012
Available online 21 June 2012

Submitted by R.A. Brualdi

AMS classification:
05B20
05B30
05C12
15B36
51E30

Keywords:
Configuration graph
Neighbourhood geometry
Adjacency matrix

ABSTRACT

Some graphs Γ have the following property $\mathcal{P}$: *the configuration graph (i.e. the non-collinearity graph) of the neighbourhood geometry of Γ is isomorphic to Γ*. For instance, the ubiquitous Petersen graph satisfies $\mathcal{P}$. The purpose of this paper is to reveal repercussions of property $\mathcal{P}$ on adjacency matrices A for Γ. This will be achieved in terms of invariance under (powers of) the following mapping Θ: denote by I and J the identity matrix and the all one matrix, respectively, both of order $n = k^2+1$, and define $\Theta(A) := (k-1)I+J-A^2$. If $k = 3$ and A is an adjacency matrix for the Petersen graph, it is well known that $\Theta(A) = A$. In 1960, Hoffman and Singleton showed that for arbitrary k the matrix equation $\Theta(A) = A$ has only very few solutions, namely for $k = 2, 3, 7$ and possibly for $k = 57$.

We prove that the property $\mathcal{P}$ implies the existence of an integer $m \geq 1$ such that $\Theta^m(A) = A$. We determine a standard form for such matrices which is invariant under the action of Θ. In particular, we characterize all solutions, on A, to $\Theta^m(A) = A$ for $k = 3$ and $k = 4$, where A is an adjacency matrix of some k-regular graph, solving a conjecture posed by the authors.

1. Introduction

The material in this paper is based on some unpublished features of earlier work [1]. At that time, our objective was to construct k-regular graphs Γ of order $k^2 + 1$ which have property $\mathcal{P}$ that *the*

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1 This research was carried out by M. Abreu, D. Labbate and V. Napolitano within the activity of INdAM-GNSAGA and supported by the Italian Ministry MIUR.

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http://dx.doi.org/10.1016/j.laa.2012.05.029

configuration graph of the neighbourhood geometry of Γ is isomorphic to Γ and to classify all those graphs for $k = 3$ and 4. Particular attention was given to the two "new" graphs which are not in the list of Hoffman and Singleton [10], namely the so-called Terwilliger graph T_2 of order 10 and the graph $G_{17}(T_1')$ of order 17 whose adjacency matrices are solutions for $\Theta^3(A) = A$ and $\Theta^2(A) = A$, respectively. An essential step in this classification, in terms of graph theory, was the conjecture (reformulated here as Conjecture 1) that any such graph should have a centre with radius 2, see [1, p. 119].

A look at the celebrated paper by Hoffman and Singleton [10] makes clear that the existence of graphs satisfying property $\mathcal{P}$ depends on some rather deep and sophisticated problems in linear algebra of (0, 1)-matrices. In [1], however, adjacency matrices play only a secondary rôle, for a twofold reason. First, the graphs could easily be constructed in graph theoretic terms "around their centres of radius 2". Secondly, we did not wish to burden that paper with additional rather technical notions, which at that time would only have given equivalent descriptions of the graphs.

The current paper is designed as a follow up of [1] in terms of adjacency matrices of graphs, since this approach has allowed us to prove Conjecture 1 (cf. Sections 2 and 3), to find an invariant form for adjacency matrices of graphs satisfying property $\mathcal{P}$ under the action of powers of Θ (cf. Section 4), and to complete the characterization of such graphs for $k = 4$ (cf. Section 5).

2. Preliminaries

For basic notions from graph theory and incidence geometry, we refer to [4,6], respectively. We consider undirected graphs without loops or multiple edges.

According to Dembowski [6, p. 9], a *partial plane* is an incidence structure [6, p. 1] such that any two distinct points are incident with at most one line. We will refer to a partial plane also as a *linear* incidence structure. A *configuration $\mathcal{C}$ of type n_k* or *n_k-configuration* is a linear incidence structure consisting of n points and n lines such that each point and line is incident with k lines and points, respectively. We will mainly be concerned with *self-polar n_k*-configurations without *absolute elements*, see [6, p. 9].

A word of warning: there is some confusion in the literature about configurations. Seen as special cases in the wider class of (v_r, b_k)-configurations, n_k-configurations have also been named *self-dual* configurations in [5] and, in a long standing tradition, *symmetric* configurations (e.g. in [1,2,7–9]). Unfortunately, these names are not related with *self-dual* and *self-polar* configurations according to [6, p. 9] or with *symmetric* incidence matrices as understood in Linear Algebra.

The *configuration graph* $\Gamma(\mathcal{C})$ of an n_k-configuration $\mathcal{C}$, known also as *non-collinearity graph*, is defined as the image of the following mapping Γ: the vertices of $\Gamma(\mathcal{C})$ are the points of $\mathcal{C}$ and any two vertices are joined by an edge if they are not incident with some line of $\mathcal{C}$ [9]. Let $\delta := n - k^2 + k - 1$ be the *deficiency* of an n_k-configuration [8,13]. Note that δ indicates the number of points not joined with an arbitrary point; hence finite projective planes are characterized by deficiency 0. Thus the configuration graph $\Gamma(\mathcal{C})$ is a δ-regular graph on n vertices. In particular, when $n = k^2 + 1$, the configuration graph of an n_k-configuration is k-regular.

On the other hand, Lefèvre-Percsy et al. [12] introduced a mapping $\mathcal{N}$ which associates with each graph G its *neighbourhood geometry* $\mathcal{N}(G) = (P, B, |)$: let P and B be two copies of $V(G)$, whose elements are called *points* and *blocks*, respectively; a point $x \in P$ is incident with a block $b \in B$ (in symbols $x|b$) if and only if x and b, seen as vertices in G, are adjacent. If the graph G is k-regular on n vertices, its neighbourhood geometry $\mathcal{N}(G)$ is an incidence structure with n points and blocks, and each point and block of $\mathcal{N}(G)$ is incident with exactly k blocks and points, respectively.

It is easy to check that the Petersen graph is the configuration graph of the Desargues configuration [7,9], and, conversely, that the Desargues configuration is the neighbourhood geometry of the Petersen graph. Thus, the Desargues configuration and the Petersen graph are invariant under the compositions $\mathcal{N} \circ \Gamma$ and $\Gamma \circ \mathcal{N}$, respectively.

In general, the mappings Γ and $\mathcal{N}$ are not mutually inverse and need not even be composable. In fact, the incidence structure $\mathcal{N}(G)$ turns out to be linear if and only if G is 4-cycle-free, while the configuration graph $\Gamma(\mathcal{C})$ is not 4-cycle-free in general, see [1] for details. In that paper, we have investigated possible compositions of the mappings Γ and $\mathcal{N}$ and determined configurations and graphs fixed or isomorphic to their images under these compositions, called $\Gamma \circ \mathcal{N}$*-admissible* or $\mathcal{N} \circ \Gamma$*-admissible*, respectively.

Let G be a graph and A_G be an adjacency matrix of G. Then, A_G is also an incidence matrix of the neighbourhood geometry $\mathcal{N}(G)$. On the other hand, given an n_k-configuration $\mathcal{C}$ and an incidence matrix $A_{\mathcal{C}}$ of $\mathcal{C}$, an adjacency matrix of $\Gamma(\mathcal{C})$ is

$$A_{\Gamma(\mathcal{C})} = (k-1)I_n + J_n - A_{\mathcal{C}}A_{\mathcal{C}}^{\mathrm{T}},$$

where I_n and J_n denote the identity matrix and the all one matrix of order n (cf. e.g. [1, Lemma 3.1]). In the special case that $\mathcal{C}$ is a *self-polar configuration without absolute points* [6, p. 9], then $\mathcal{C}$ admits a *symmetric incidence matrix with trace zero* and

$$A_{\Gamma(\mathcal{C})} = (k-1)I_n + J_n - A_{\mathcal{C}}^2.$$

Let $\mathfrak{D}_k$ be the class of symmetric (0, 1)-matrices of order k^2+1 with trace zero, containing exactly k elements 1 in each row and column, without submatrices of order 2 all of whose entries are 1 (i.e. J_2*-free*). Then every adjacency matrix of a C_4-free k-regular graph of order k^2+1 belongs to $\mathfrak{D}_k$. On the other hand, each self-polar $(k^2+1)_k$-configuration without absolute points admits incidence matrices in $\mathfrak{D}_k$. In [1], we defined the mapping

$$\Theta \ : \begin{cases} \mathfrak{D}_k & \longrightarrow \quad (0,1) - \text{matrices} \\ A & \longmapsto \quad \Theta(A) = (k-1)I_{k^2+1} + J_{k^2+1} - A^2 \end{cases}$$

which describes the action of Γ in terms of incidence matrices of self-polar $(k^2+1)_k$-configurations without absolute points. Since applying $\mathcal{N}$ has no effect on the adjacency matrix A, the mapping Θ also describes the action of $\mathcal{N} \circ \Gamma$ and $\Gamma \circ \mathcal{N}$.

The famous equation solved by Hoffman and Singleton [10] for square matrices of order n can be written, in these terms, as $\Theta(A) = A$. In our context, their result characterizes self-polar configurations without absolute points and C_4-free graphs for which $\mathcal{N} \circ \Gamma(\mathcal{C}) = \mathcal{C}$ and $\Gamma \circ \mathcal{N}(G) = G$.

Recall that a vertex v of a graph G is said to be a *centre of G with radius* 2 if the distance $d_G(v, w) \leqslant 2$ for each $w \in V(G)$. In [1] we found configurations $\mathcal{C}$ and graphs G for which $\mathcal{N} \circ \Gamma(\mathcal{C}) \cong \mathcal{C}$ and $\Gamma \circ \mathcal{N}(G) \cong G$ for $k = 3$, for $k = 4$ in the case the graphs involved admitted a centre with radius 2, and we also formulated the following:

Conjecture 1 [1, p. 119]. *Let G be a $\Gamma \circ \mathcal{N}$-admissible k-regular graph, with $k = 2, 3, 4$, such that $\Gamma \circ \mathcal{N}(G) \cong G$. Then G admits a centre with radius 2.*

With these preliminaries, we can describe more in details how this paper is organized. We prove that the isomorphisms $\mathcal{N} \circ \Gamma(\mathcal{C}) \cong \mathcal{C}$ and $\Gamma \circ \mathcal{N}(G) \cong G$ imply that there exists an integer $m \geq 1$ such that $\Theta^m(A) = A$ (cf. Lemma 2). This allows us to prove Conjecture 1 (cf. Proposition 8 and Theorem 10), and motivates the study of solutions to the *generalized Hoffman–Singleton* matrix equation

$$(gHS) \qquad \Theta^m(A) = A$$

for matrices $A \in \mathfrak{D}_k$, for some $m > 1$. In particular, we find solutions A for $k = 3, 4$ (cf. Theorems 15 and 16 respectively). In Section 4, we determine a structure for such matrices called *HS-form* which is invariant under the action of Θ. Finally, in Section 5, we use these results to completely characterize configurations $\mathcal{C}$ and graphs G for which $\mathcal{N} \circ \Gamma(\mathcal{C}) \cong \mathcal{C}$ and $\Gamma \circ \mathcal{N}(G) \cong G$ for $k = 3, 4$.

3. Solutions to (*gHS*) and graphs

In this section we study some properties of equation (*gHS*), we relate the actions of $\Gamma \circ \mathcal{N}$ and Θ^m and we prove Conjecture 1 in Theorem 10.

Lemma 2. *Let G be a $\Gamma \circ \mathcal{N}$-admissible graph such that $\Gamma \circ \mathcal{N}(G) \cong G$. Then there exists an integer $m \geq 1$ such that $\Theta^m(A_G) = A_G$.*

Proof. By hypothesis, the matrix $A_{\Gamma \circ \mathcal{N}(G)} = \Theta(A_G)$ is p-equivalent to A_G, which means that there exists a permutation matrix Q such that $A_G = Q\Theta(A_G)Q^{-1}$. Thus

$$Q\Theta(A_G)Q^{-1} = Q((k-1)I_{k^2+1} + J_{k^2+1} - A_G^2)Q^{-1}$$
$$= (k-1)I_{k^2+1} + J_{k^2+1} - (QA_GQ^{-1})^2 = \Theta(QA_GQ^{-1}).$$

Hence, by induction, $A_G = Q^m\Theta^m(A_G)Q^{-m}$ for all $m \geq 1$. In particular when m is the order of Q, we have $\Theta^m(A_G) = A_G$, as desired. □

Analogously, we have the following Lemma:

Lemma 3. *Let $\mathcal{C}$ be a $\mathcal{N} \circ \Gamma$-admissible configuration such that $\mathcal{N} \circ \Gamma(\mathcal{C}) \cong \mathcal{C}$. Then, there exists an integer $m \geq 1$ such that $\Theta^m(A_{\mathcal{C}}) = A_{\mathcal{C}}$.* □

The following technical lemmas are necessary for the proof of Proposition 8.

Lemma 4. *Let A be a symmetric integer matrix of order $k^2 + 1$ with row and column sum equal to k. Then the entries on the main diagonal of $\Theta(A)$ are $d_{ii} \leq 0$ and equality holds if and only if A is a $(0, 1)$-matrix.*

Proof. Each entry on the main diagonal of A^2, say b_{ii}, is the product of the ith row of A with itself. Hence $b_{ii} = \sum_{j=1}^{k^2+1} a_{ij}^2$ is a sum of squares. Collect the positive summands of $\sum_{j=1}^{k^2+1} a_{ij} = k$ and write it as

$$\sum_{j=1}^{k^2+1} a_{ij} = \sum_{s \in S} p_s - \sum_{t \in T} q_t,$$

where $p_s \geq 1$ and $q_t \geq 0$ are non-negative integers for suitable (possibly empty) index sets S and T, respectively. Since $p_s^2 \geq p_s$ and $q_t^2 \geq q_t$, one has

$$b_{ii} = \sum_{j=1}^{k^2+1} a_{ij}^2 = \sum_{s \in S} p_s^2 + \sum_{t \in T} q_t^2 \geq \sum_{s \in S} p_s - \sum_{t \in T} q_t = \sum_{j=1}^{k^2+1} a_{ij} = k.$$

Thus the summands $(k-1)I_{k^2+1}, J_{k^2+1}$, and $-A^2$ contribute $k-1$, 1 and $-b_{ii}$, respectively, to each entry d_{ii} on the main diagonal of $\Theta(A)$. This implies $d_{ii} = k - 1 + 1 - b_{ii} = k - b_{ii} \leq 0$.

Equality holds if and only if

$$\sum_{s \in S} p_s^2 - \sum_{s \in S} p_s = -\sum_{t \in t} q_t^2 - \sum_{t \in T} q_t.$$

Since the left-hand side $\sum_{s \in S}(p_s^2 - p_s)$ is either zero or positive, whereas the right-hand side $-\sum_{t \in T}(q_t^2 + q_t)$ is either zero or negative, both sides must be zero. Clearly, the right-hand side is zero if and only if $q_t = 0$ for all $t \in T$, whereas the left-hand side is zero if and only if $p_s^2 = p_s$ and thus $p_s = 1$ for all $s \in S$. Hence b_{ii} assumes its minimum value, namely k, if and only if the ith row of A consists of k entries 1 and $k^2 - k + 1$ entries 0. □

Lemma 5. *Let A be a symmetric $(0, 1)$-matrix of order $k^2 + 1$, with a zero main diagonal, containing exactly k elements 1 in each row and column, which is a solution of $\Theta^m(A) = A$, for some $m \geqslant 1$. Then A is J_2-free.*

Proof. By hypothesis $\Theta^m(A) = A$, for some $m \geq 1$. If A contained entries 1 in positions (i, j), (i, t), (s, j), and (s, t) for $i, j, s, t \in \{1, \ldots, k^2+1\}$ with $i \neq s$ and $j \neq t$, then the entry $(A^2)_{is} = \sum_{r=1}^{n} a_{ir}a_{sr}$ would have at least two summands 1, namely for $r = j$ and $r = t$. This would imply $(A^2)_{is} \geq 2$ and $(\Theta(A))_{is} \leq -1$, hence $\Theta(A) \neq A$. The only remaining option would be $\Theta^m(A) = A$ for some $m \geq 2$. On the other hand, $(\Theta(A))_{is} \leq -1$ means $\Theta(A)$ is not a $(0, 1)$-matrix. Then Lemma 4 implies that each entry on the main diagonal of $\Theta^2(A)$ is negative, hence $\Theta^2(A)$ is not a $(0, 1)$-matrix. Induction on m shows that $\Theta^m(A)$ is not a $(0, 1)$-matrix for all $m \geq 2$, a contradiction. □

Lemma 6. *Let $A \in \mathfrak{D}_k$ be a solution of (gHS) for some $m \geq 1$. Then any p-equivalent matrix B is also a solution of (gHS).*

Proof. Suppose $B = S^{-1}AS$ for some permutation matrix S. Then

$$\Theta(S^{-1}AS) = (k-1)\, I_{k^2+1} + J_{k^2+1} - (S^{-1}AS)^2$$

$$= S^{-1}((k-1)\, I_{k^2+1} + J_{k^2+1} - A^2)S = S^{-1}\Theta(A)S$$

and hence

$$\Theta^m(S^{-1}AS) = \Theta(\Theta^{m-1}(S^{-1}AS)) = \Theta(S^{-1}\Theta^{m-1}(A)S)$$

$$= S^{-1}\Theta(\Theta^{m-1}(A)S) = S^{-1}\Theta^m(A)S$$

by induction on $m \geq 1$. In particular, $\Theta^m(B) = B$ if $\Theta^m(A) = A$, for some $m \geq 1$. □

In the sequel we will use the following result:

Proposition 7 [1, Proposition 2.7]. *Let G be a C_4-free, k-regular graph of order $k^2 + 1$, with $k \geq 2$. Then G has diameter $diam(G) \leq 3$. In particular, $diam(G) = 2$ if and only if G has girth 5.* □

In general, a graph G with $diam(G) = 3$ need not admit a centre with radius 2, but we can prove the following:

Proposition 8. *Let G be a k-regular graph of order $k^2 + 1$, with $k = 2, 3, 4$. If an adjacency matrix A_G of G is a solution of (gHS), for some $m \geq 1$, then G admits a centre with radius 2.*

Proof. Firstly, we prove the claim: *a vertex $v \in V(G)$ does not lie in a 3-cycle of G if and only if v is a centre of G with radius 2.* To see this, let $v_1, \ldots, v_k$ denote the k vertices at distance 1 from v. Then v does not lie in a 3-cycle of G if and only if we encounter further $k-1$ vertices v_{ij}, $j = 1, \ldots, k-1$, at distance 1 from each v_i. (Since A_G is J_2-free by Lemma 5 and hence G is C_4-free, the vertices v_{ij} turn out to be distinct in pairs.) This, in turn, holds true if and only if there are no vertices at distance 3 from v since $\{v, v_i, v_{ij} \mid i = 1, \ldots k, j = 1, \ldots, k-1\}$ is all of $V(G)$.

Secondly, a short calculation verifies that $k^2 + 1 \not\equiv 0 \pmod 3$ for every integer k. Hence $V(G)$ cannot be partitioned into vertex-disjoint 3-cycles.

If $\kappa = 2$, then G is a 5-cycle and every vertex is a centre with radius 2. If $k = 3$ and hence $|V(G)| = 10$, then G contains at most three disjoint 3-cycles in G and the remaining vertex is a centre with radius 2.

Now let $k = 4$ and suppose that G has no centre with radius 2. Then the above claim implies that G contains at least one vertex v_0 lying in two different 3-cycles of G, say $v_0v_1v_2v_0$ and $v_0v_3v_4v_0$. With respect to a labelling of the vertices which starts with $v_0, v_1, v_2, v_3, v_4, \ldots$, the first five rows and columns of the corresponding adjacency matrix B and $\Theta(B)$ read

$$\begin{array}{c|cccccc} & v_0 & v_1 & v_2 & v_3 & v_4 & \dots \\ \hline v_0 & 0 & 1 & 1 & 1 & 1 & \dots \\ v_1 & 1 & 0 & 1 & 0 & 0 & \dots \\ v_2 & 1 & 1 & 0 & 0 & 0 & \dots \\ v_3 & 1 & 0 & 0 & 0 & 1 & \dots \\ v_4 & 1 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

respectively. Denote by T the submatrix of order 5×12 of $\Theta(B)$ made up by the first five rows and the 6th through the 17th columns. Since B is p-equivalent to A and A is a solution of (gHS), by Lemma 6, the image $\Theta(B)$ is also a solution of (gHS) and by Lemma 5, $\Theta(B)$ is an adjacency matrix of a C_4-free, 4-regular graph on 17 vertices. Thus T must contain four entries 1 in each row. This is not compatible with $\Theta(B)$ being J_2-free. In fact, in the best case we would need 13 columns to fit four entries 1 into each row without producing a submatrix of T isomorphic to J_2, e.g.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & & & & & & & & & \\ 1 & & & & 1 & 1 & 1 & & & & & & \\ 1 & & & & & & & 1 & 1 & 1 & & & \\ 1 & & & & & & & & & & 1 & 1 & 1 \\ & 1 & & & 1 & & & 1 & & & 1 & & \end{pmatrix}.$$

This contradiction shows that G has a centre with radius 2. $\square$

Corollary 9. *Let G be a k-regular graph on k^2+1 vertices where $k = 2, 3, 4$ and let v_0 be a centre with radius 2. If G admits an adjacency matrix A which is a solution of (gHS), then the vertex set of G is*

$$\{v_0, v_1, \dots, v_k, v_{1,1}, \dots, v_{1,k-1}, v_{2,1}, \dots, v_{2,k-1}, \dots, v_{k,1}, \dots, v_{k,k-1}\},$$

where, v_i and v_{ij} denote the k neighbours of v_0 and the $k-1$ neighbours of each v_i other than v_0, respectively, for $i = 1, \dots, k$ and $j = 1, \dots, k-1$. $\square$

The following shows that Conjecture 1 holds true.

Theorem 10. *Let G be a $\Gamma \circ \mathcal{N}$-admissible k-regular graph, with $k = 2, 3, 4$, such that $\Gamma \circ \mathcal{N}(G) \cong G$. Then G admits a centre with radius 2.*

Proof. It follows immediately from Lemma 2, equation (gHS) and Proposition 8. $\square$

4. Standard forms

Motivated by Corollary 9, in this section we find a structure for a standard representative, called *HS-form*, within each class of p-equivalency in $\mathfrak{D}_k$, which is invariant under the action of Θ.

Denote by $\mathbf{0}_\nu$ and $\mathbf{1}_\nu$ row vectors of dimension ν all of whose entries are 0 and 1, respectively. Let $\mathbf{0}_{k,k}$ be a copy of the zero matrix of order k and $K = I_k \otimes \mathbf{1}_{k-1}$ the Kronecker product of matrices. With respect to the labelling mentioned in Corollary 9, the adjacency matrix of a k-regular graph on k^2+1 vertices G admitting a centre with radius 2 gets the *standard form*

$$S(P) \quad := \quad \begin{pmatrix} 0 & \mathbf{1}_k & \mathbf{0}_{k^2-k} \\ \mathbf{1}_k^{\mathrm{T}} & \mathbf{0}_{k,k} & K \\ \mathbf{0}_{k^2-k}^{\mathrm{T}} & K^{\mathrm{T}} & P \end{pmatrix},$$

where P is a symmetric $(0, 1)$-matrix of order $k^2 - k$ with all row and column sums equal to $k - 1$. We regard P as a block matrix $P = (P_{ij})_{1 \le i,j, \le k}$ of order k with square blocks P_{ij} of order $k - 1$. The matrix P is the adjacency matrix of a $(k-1)$-regular subgraph H of G, which we call the *periphery* of G.

Theorem 11. *Let A be a solution of (gHS) and suppose $k \le 4$. Then*

(i) A is p-equivalent to a standard form $S(P)$ where each block P_{ij} is a $(0, 1)$-matrix which has at most one entry 1 in each row and column.
(ii) If there exists a zero block in some row and column of P, then all other blocks of P in that row and column are permutation matrices.
(iii) If there exists a zero block on the main diagonal of P, then we can write $S(P)$ in such a way that $P_{11} = \mathbf{0}_{k-1,k-1}$ and $P_{1,i} = P_{i,1} = I_{k-1}$ for all $i = 2, \ldots, k$.

Proof. (i) For $k \le 4$, Corollary 9 guarantees that A has a p-equivalent standard form $S(P)$, which is a solution of (gHS). Suppose that there are two entries 1 in one and the same row of some block $P_{i,j}$, say in positions (s, t) and (s, u), for some $s, t, u \in \{1, \ldots, \kappa - 1\}$. Then in $S(P)$, we encounter entries 1 in the following four positions:

$$(1+j, 1+k+(j-1)(k-1)+t)$$
$$(1+j, 1+k+(j-1)(k-1)+u)$$
$$(1+k+(i-1)(k-1)+s, 1+k+(j-1)(k-1)+t)$$
$$(1+k+(i-1)(k-1)+s, 1+k+(j-1)(k-1)+u),$$

thus obtaining a J_2 submatrix of $S(P)$, which contradicts Lemma 5. By symmetry, an analogous reasoning works if there were two entries 1 in one and the same column of some block P_{ij}.

(ii) follows for arithmetic reasons: P is at once a block matrix of order k and a $(0, 1)$-matrix with all row and column sums equal to $k - 1$.

(iii) is the result of a suitable relabelling of the rows and columns of $S(P)$. Consider the vertices v_0, v_i, v_{ij} introduced in Corollary 9: if $P_{ii} = \mathbf{0}_{k-1,k-1}$, then first exchange the roles of v_1 and v_i; secondly, for $i = 2, \ldots, k$, relabel the vertices within each set $\{v_{ij} \mid j = 1, \ldots, k-1\}$ such that $v_{1,j}$ is adjacent with v_{ij} for all $j = 1, \ldots, k$. □

Definition 12. The standard form $S(P)$ is said to be a *Hoffman–Singleton form*, or an *HS-form* for short, denoted by $S_{HS}(P)$, if all the diagonal blocks P_{ii} of P are zero blocks.

Proposition 13. *Let G be a k-regular C_4-free graph on k^2+1 vertices, admitting a centre v_0 with radius 2, and let $S(P)$ be a standard form of A_G with respect to v_0. If $S(P)$ is in HS-form, then $\Theta(S(P)) = S(P')$, where $S(P')$ is also in HS-form, for some block matrix P'.*

Proof. By definition of Θ, we have that $\Theta(S(P)) = (k-1)I_{k^2+1} + J_{k^2+1} - S(P)^2$. We need to show that $\Theta(S(P))$ is a block matrix $(S_{i,j})$,

$$\Theta(S(P)) = (S_{i,j}) := \begin{pmatrix} 0 & \mathbf{1}_k & \mathbf{0}_{k^2-k} \\ \mathbf{1}_k^{\mathrm{T}} & \mathbf{0}_{k,k} & K \\ \mathbf{0}^{\mathrm{T}}_{k^2-k} & K^{\mathrm{T}} & P' \end{pmatrix}$$

for some matrix P' with zero blocks on the main diagonal. In fact,

$$\begin{aligned} S_{1,1} &= k - (0|\mathbf{1}_k|\mathbf{0}_{k^2-k})(0|\mathbf{1}_k^{\mathrm{T}}|\mathbf{0}^{\mathrm{T}}_{k^2-k})^{\mathrm{T}} = 0; \\ S_{1,2} &= \mathbf{1}_k - (0|\mathbf{1}_k|\mathbf{0}_{k^2-k})(\mathbf{1}_k|\mathbf{0}_{k,k}|K^{\mathrm{T}})^{\mathrm{T}} = \mathbf{1}_k; \\ S_{1,3} &= \mathbf{1}_{k^2-1} - (0|\mathbf{1}_k|\mathbf{0}_{k^2-k})(\mathbf{0}_{k^2-k}|K|P)^{\mathrm{T}} = \mathbf{1}_{k^2-1} - \mathbf{1}_{k^2-1} = \mathbf{0}_{k^2-k}; \\ S_{2,2} &= (k-1)I_k + J_k - (\mathbf{1}_k^{\mathrm{T}}|\mathbf{0}_{k,k}|K)(\mathbf{1}_k^{\mathrm{T}}|\mathbf{0}_{k,k}|K)^{\mathrm{T}} \\ &= (k-1)I_k + J_k - ((k-1)I_k + J_k) = \mathbf{0}_{k,k}. \end{aligned}$$

Since $P_{i,i} = \mathbf{0}_{k-1,k-1}$ and $P_{i,m}$ is a permutation matrix for each $i \neq m$, each column of KP is $(KP)_{(i-1)(k-1)+j} = K_i P_j = \mathbf{1}_k^{\mathrm{T}} - e_i$, for some $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, k-1\}$, where e_i is the ith unitary column vector. This implies that $KP = (J_k - I_k) \otimes \mathbf{1}_{k-1}$. On the other hand $J_{k,k^2-k} = J_k \otimes \mathbf{1}_{k-1}$, and hence by the properties of the Kronecker product,

$$\begin{aligned} S_{2,3} &= J_{k,k^2-k} - (\mathbf{1}_k^{\mathrm{T}}|\mathbf{0}_{k,k}|K)(\mathbf{0}_{k^2-k}|K|P)^{\mathrm{T}} = J_{k,k^2-k} - KP \\ &= J_k \otimes \mathbf{1}_{k-1} - (J_k - I_k) \otimes \mathbf{1}_{k-1} = I_k \otimes \mathbf{1}_{k-1} = K. \end{aligned}$$

We also have that $S_{2,1} = S_{1,2}^{\mathrm{T}}$, $S_{3,1} = S_{1,3}^{\mathrm{T}}$ and $S_{3,2} = S_{2,3}^{\mathrm{T}}$, since $\Theta(S(P))$ is a symmetric matrix, This implies that $\Theta(S(P)) = S(P')$ for some matrix P', i.e. $\Theta(S(P))$ is in standard form. To show that $\Theta(S(P))$ is actually in *HS*-form, we need to prove that P' has zero blocks on the main diagonal.

$$\begin{aligned} P' := S_{3,3} &= (k^2-k)I_{k^2-k} + J_{k^2-k} - (\mathbf{0}_{k^2-k}|K^{\mathrm{T}}|P)(\mathbf{0}_{k^2-k}|K|P)\mathrm{T} \\ &= (k^2-k)I_{k^2-k} + J_{k^2-k} - K^{\mathrm{T}}K - P^2. \end{aligned}$$

First we note that $K^{\mathrm{T}}K = I_k \otimes J_{k-1}$ and that this implies that $(K^{\mathrm{T}}K)_{i,i} = J_{k-1}$. We also note that $(P^2)_{i,i} = (k-1)I_{k-1}$ because $P_{i,i} = \mathbf{0}_{k-1,k-1}$ and $P_{i,j}$ is a permutation matrix for each $i \neq j$. Finally, consider the i-th diagonal block of P':

$$\begin{aligned} P'_{i,i} &= (k-1)I_{k-1} + J_{k-1} - (K^{\mathrm{T}}K)_{i,i} - (P^2)_{i,i} \\ &= (k-1)I_{k-1} + J_{k-1} - J_{k-1} - (k-1)I_{k-1} = \mathbf{0}_{k-1,k-1}. \quad \square \end{aligned}$$

Note that, in the previous proof, there is nothing we can say about the other blocks of P' in general. However, by Lemma 11(i), if $\Theta^m(S(P)) = S(P')$ for some $m \geq 1$, they must be permutation matrices.

5. Classification

In this last section we classify the admissible configurations $\mathcal{C}$ and graphs G for which $\mathcal{N} \circ \Gamma(\mathcal{C}) \cong \mathcal{C}$ and $\Gamma \circ \mathcal{N}(G) \cong G$ for $k = 4$ proving Theorem 16.

Hoffman and Singleton [10] classified all κ-regular graphs on $\kappa^2 + 1$ vertices with girth 5. Their result obtained by eigenvalue techniques, in the notation of this paper can be read as follows:

Theorem 14. [10] *Let $A \in \mathfrak{D}_k$ be a solution of the Hoffman–Singleton equation $\Theta(A) = A$. Then one of the following statements holds:*

$$\begin{array}{rll} (i) & k = 2 & \text{and}A\text{is an adjacency matrix for the 5-cycle;} \\ (ii) & k = 3 & \text{and}A\text{is an adjacency matrix for the Petersen graph as well} \\ & & \text{as an incidence matrix for the Desargues configuration;} \\ (iii) & k = 7 & \text{and}A\text{is an adjacency matrix for Hoffman–Singleton's graph;} \\ (iv) & k = 57 & \text{(no graph or configuration is known).} \end{array} \qquad \square$$

An *HS*-form for $k = 7$, i.e. for an adjacency matrix of the Hoffman–Singleton graph, is presented in [10, Figure 3]. The following two matrices are *HS*-forms for adjacency matrices of the 5-cycle and the Petersen graph, respectively:

$$\left(\begin{array}{c|cc|c|c} & 1 & 1 & & \\ \hline 1 & & & 1 & \\ 1 & & & & 1 \\ \hline & 1 & & & 1 \\ \hline & & 1 & 1 & \end{array}\right) \qquad \left(\begin{array}{c|ccc|cc|cc|cc} & 1 & 1 & 1 & & & & & & \\ \hline 1 & & & & 1 & 1 & & & & \\ 1 & & & & & & 1 & 1 & & \\ 1 & & & & & & & & 1 & 1 \\ \hline & 1 & & & & & 1 & & 1 & \\ & 1 & & & & & & 1 & & 1 \\ \hline & & 1 & & 1 & & & & & 1 \\ & & 1 & & & 1 & & & 1 & \\ \hline & & & 1 & 1 & & & 1 & & \\ & & & 1 & & 1 & 1 & & & \end{array}\right).$$

Recall that a *Terwilliger graph* is a non-complete graph G such that, for any two vertices $v_1, v_2 \in V(G)$ at distance 2 from each other, the induced subgraph $G[N_G(v_1) \cap N_G(v_2)]$ is a clique of size μ, for some fixed $\mu \geq 0$ (cf. e.g. [3, p. 34]). Each C_4-free graph is a Terwilliger graph with $\mu = 1$, since vertices at distance two must have exactly one common neighbour. In particular, $(\Gamma \circ \mathcal{N})$-admissible graphs are Terwilliger graphs with $\mu = 1$. In [1, Lemma 4.1] we have proved that there exist exactly three cubic graphs on 10 vertices which are C_4-free, namely the Petersen graph and the Terwilliger graphs T_1 and T_2 (Fig. 1).

These graphs admit the following adjacency matrices in *HS*-form and standard form respectively:

It is important to note that even though A_1 is an *HS*-form, $\Theta^m(A_1) \neq A_1$ for all $m \geq 1$, while $\Theta^m(A_1) = \Theta(A_1)$ for all $m \geq 1$. In fact, $\Theta(A_1)$ is an adjacency matrix for the Petersen graph. On the other hand, the graph T_2 does not admit an *HS*-form, but with the standard form A_2 it can be checked (by an arithmetic calculation on matrices) that $\Theta^3(A_2) = A_2$.

Kantor [11] denoted the 10 configurations of type 10_3 by $10_3A, \ldots, 10_3I$, 10_3K. The Desargues configuration corresponds to 10_3B. It is part of mathematical folklore that these ten configurations

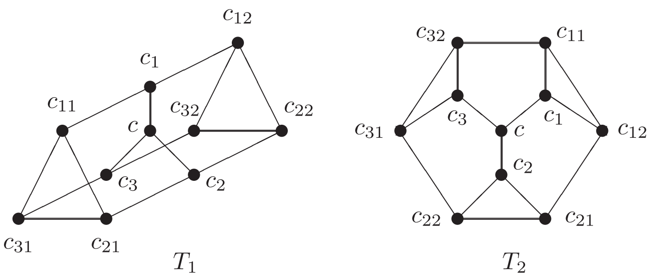

Fig. 1. The Terwilliger graphs T_1 and T_2.

	c	c_1	c_2	c_3	c_{11}	c_{12}	c_{21}	c_{22}	c_{31}	c_{32}
c		1	1	1						
c_1	1				1	1				
c_2	1						1	1		
c_3	1								1	1
c_{11}		1					1		1	
c_{12}		1						1		1
c_{21}			1		1				1	
c_{22}			1			1				1
c_{31}				1	1		1			
c_{33}				1		1		1		

A_1

	c	c_1	c_2	c_3	c_{11}	c_{12}	c_{21}	c_{22}	c_{31}	c_{32}
c		1	1	1						
c_1	1				1	1				
c_2	1						1	1		
c_3	1								1	1
c_{11}		1				1				1
c_{12}		1			1		1			
c_{21}			1			1		1		
c_{22}			1				1		1	
c_{31}				1				1		1
c_{33}				1	1				1	

A_2

Fig. 2. Adjacency matrices A_1 for T_1 in *HS*-form and A_2 for T_2 in standard form.

yield seven pairwise non-isomorphic configuration graphs, namely the Petersen graph, T_1, and T_2, as well as four graphs containing 4-cycles.

Theorem 15. *Let $A \in \mathfrak{D}_3$. Then the following are equivalent:*

(i) A is an incidence matrix of the 10_3 configuration 10_3F.
(ii) A is an adjacency matrix of the Terwilliger graph T_2.
(iii) A is p-equivalent to the standard form A_2.
(iv) $\Theta^3(A) = A$ but $\Theta(A) \neq A$, i.e. A is a proper solution of (gHS) with $m = 3$.

Proof. $(i) \Longrightarrow (ii)$ Since $\Gamma(10_3F) = T_2$ and $\mathcal{N}(T_2) \cong 10_3F$ (cf. e.g. [1]), $\Theta(A) = A_{T_2}$ is p-equivalent to A. Hence A is also an adjacency matrix for T_2.
$(ii) \Longrightarrow (iii)$ It is immediate to see that A_2 is an adjacency matrix for T_2 (cf. Figs. 1 and 2), thus A is p-equivalent to A_2.
$(iii) \Longrightarrow (iv)$ As mentioned above, a simple arithmetic calculation shows that $\Theta^3(A_2) = A_2$ and that $\Theta(A_2) \neq A_2$. Hence (iv) holds, by Lemma 6.
$(iv) \Longrightarrow (i)$ $A \in \mathfrak{D}_3$ implies that it is an incidence matrix for a $(\mathcal{N} \circ \Gamma)$-admissible 10_3-configuration $\mathcal{C}$. In general, given a configuration $\mathcal{C}$, the equation $\Theta^m(A_{\mathcal{C}}) = A_{\mathcal{C}}$ for some $m > 1$, does not necessarily imply that $\mathcal{N} \circ \Gamma(\mathcal{C}) \cong \mathcal{C}$. However, we have proved in [1, Proposition 4.2] that the only $(\mathcal{N} \circ \Gamma)$-admissible 10_3-configuration $\mathcal{C}$ such that $\mathcal{N} \circ \Gamma(\mathcal{C}) \cong \mathcal{C}$ but $\mathcal{N} \circ \Gamma(\mathcal{C}) \neq \mathcal{C}$, is $\mathcal{C} = 10_3F$. Hence (i) holds by Lemma 3. $\square$

In [1], we found a further solution on A for the generalized Hoffman–Singleton equation $\Theta^m(A) = A$, with A a matrix of order $k^2 + 1$ and $k = 4$. It is an incidence matrix for the 17_4-configuration #1971 in Bettens' list [2].

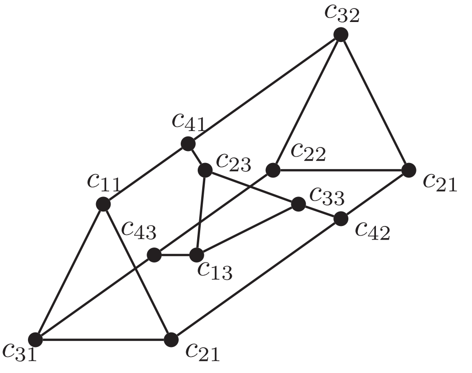

Fig. 3. The graph T_1'.

Theorem 16. *Let $A \in \mathfrak{D}_4$. Then the following are equivalent:*

(i) A is an incidence matrix of the 17_4-configuration #1971 in Bettens' list [2].
(ii) A is p-equivalent to the HS-form $S_{HS}(P)$, where

$$P = \left(\begin{array}{ccc|ccc|ccc|ccc}
 & & & 1 & & & 1 & & & 1 & & \\
 & & & & 1 & & & 1 & & & 1 & \\
 & & & & & 1 & & & 1 & & & 1 \\
\hline
1 & & & & & & 1 & & & & 1 & \\
 & 1 & & & & & & 1 & & & & 1 \\
 & & 1 & & & & & & 1 & 1 & & \\
\hline
1 & & & 1 & & & & & & & & 1 \\
 & 1 & & & 1 & & & & & 1 & & \\
 & & 1 & & & 1 & & & & & 1 & \\
\hline
1 & & & & & 1 & & 1 & & & & \\
 & 1 & & 1 & & & & & 1 & & & \\
 & & 1 & & 1 & & 1 & & & & &
\end{array}\right).$$

(iii) $\Theta^2(A) = A$, i.e A is a solution of (gHS) with $m = 2$.

Proof. $(i) \Longrightarrow (ii)$ In [1] we verified that the configuration graph of the 17_4-configuration #1971 in Bettens' list [2] admits a centre with radius 2, and denoted such graph by $G_{17}(T_1')$, where T_1' (cf. Fig. 3) is its periphery. With this labelling, the periphery graph T_1' admits the matrix P (as in statement (ii) of this theorem) as an adjacency matrix, which implies that $S_{HS}(P)$ is an adjacency matrix of the graph $G_{17}(T_1')$ in *HS*-form. It was also checked in [1] that $\mathcal{N}(G_{17}(T_1'))$ is isomorphic to the 17_4-configuration #1971, thus $S_{HS}(P)$ is an incidence matrix for it. Hence A is p-equivalent to $S_{HS}(P)$.
$(ii) \Longrightarrow (iii)$ It follows from a simple arithmetic check.
$(iii) \Longrightarrow (i)$ $A \in \mathfrak{D}_4$ implies that A is an incidence matrix for an $(\mathcal{N} \circ \Gamma)$-admissible 17_4-configuration $\mathcal{C}$. We have proved in [1, Corollary 4.5] that there are only four C_4-free 4-regular graphs of order 17 which admit a centre with radius 2. Theorem 10 proves that these are the only graphs for which the adjacency matrices can be solutions to (gHS). In [1, Theorem 4.6] we showed that among those four graphs, the only one for which its neighbourhood geometry $\mathcal{C}$ is such that $\mathcal{N} \circ \Gamma(\mathcal{C}) \cong \mathcal{C}$ is the 17_4-configuration #1971. Hence (i) holds by Lemma 3. $\square$

Remark 17. The solutions of (gHS) for $m = 1$ yield graphs which can be seen as association schemes since they are distance regular. However, this is not the case in general. In fact, for $m = 3$, the graph T_2 (from Theorem 15(iii)) cannot be seen as an association scheme since its vertex c is the only centre of radius 2, while all other vertices have exactly two vertices at distance 3.

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