Available online at www.sciencedirect.com
ScienceDirect
Computers and Mathematics with Applications 53 (2007) 1854–1866
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

Error analysis of variable stepsize Runge–Kutta methods for a class of multiply-stiff singular perturbation problems☆

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Received 7 November 2005; received in revised form 15 June 2006; accepted 28 June 2006

Abstract

In this paper, we present some results on the error behavior of variable stepsize stiffly-accurate Runge–Kutta methods applied to a class of multiply-stiff initial value problems of ordinary differential equations in singular perturbation form, under some weak assumptions on the coefficients of the considered methods. It is shown that the obtained convergence results hold for stiffly-accurate Runge–Kutta methods which are not algebraically stable or diagonally stable. Some results on the existence and uniqueness of the solution of Runge–Kutta equations are also presented.

Keywords: Singular perturbation problems; Runge–Kutta methods; Variable stepsize; Error; Multiple-stiffness

1. Introduction

Runge–Kutta methods (RKMs) are an important class of the numerical methods for solving the initial value problems (IVPs) in singular perturbation form as a special class of stiff IVPs. It is well known that they can't be satisfactorily covered by B-theory because of their very special structures. Some authors (cf. [1–9]) have presented some convergence results for RKMs and their variations (such as Rosenbrock methods, partitioned linearly implicit RKMs) applied to singly-stiff singular perturbation problems (SSPPs) and multiply-stiff singular perturbation problems (MSPPs). The corresponding reduced problems of MSPPs become a class of stiff differential–algebraic equations (SDAEs). Some practical examples of MSPPs can be found in [9].

So far, the results obtained in [1–8] are mainly under fixed stepsizes or the assumptions that RKMs are algebraically stable and diagonally stable. But some well known families of RKMs used for numerical solutions of stiff systems, like the Lobatto IIIA (cf. [10,11]) and Lobatto IIIC with stage number $s \geq 3$ (cf. [12]), etc., are not algebraically stable or diagonally stable. In this paper, we present some results on the error behavior of variable stepsize stiffly-accurate RKMs applied to a class of multiply-stiff IVPs of ordinary differential equations in singular perturbation form under some weak assumptions on the coefficients of the considered methods. It is shown that the obtained convergence results can hold for stiffly-accurate RKMs which are not algebraically stable or diagonally stable. Some results on the

☆ This work is supported by projects from NSF of China (No. 10571147) and the Scientific Research Fund of Hunan Provincial Education Department (No. 04A057).
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0898-1221/$ - see front matter
doi:10.1016/j.camwa.2006.06.009

existence and uniqueness of the solution of the corresponding Runge–Kutta equations are also presented. The results obtained in the present paper can be considered as a partial extension of the corresponding results for stiff problems in [13].

2. Problems and methods

Consider the singular perturbation problem (SPP)

$$\begin{cases} x'(t) = f(t, x, y), & t \in [0, T], \\ \epsilon y'(t) = g(t, x, y), & 0 < \epsilon \ll 1 \end{cases} \tag{2.1}$$

with initial values $(x(0), y(0)) \in \check{G}$ admitting a smooth solution $(x(t), y(t))$, where $\check{G}$ is an appropriate, convex, and open region on $R^M \times R^N$, and the maps $f : [0, T] \times \check{G} \to R^M$ and $g : [0, T] \times \check{G} \to R^N$ are sufficiently smooth and satisfy the following assumptions H0–H3, as in [7–9,13] etc.:
H0:

$$\langle g(t, x, y_1) - g(t, x, y_2), y_1 - y_2 \rangle \le -\|y_1 - y_2\|^2, \quad \forall t \in [0, T],\ \forall (x, y_1), (x, y_2) \in \check{G}, \tag{2.2a}$$

$$\|f(t, x, y_1) - f(t, x, y_2)\| \le L_1 \|y_1 - y_2\|, \quad \forall t \in [0, T],\ \forall (x, y_1), (x, y_2) \in \check{G}, \tag{2.2b}$$

$$\|g(t, x_1, y) - g(t, x_2, y)\| \le L_2 \|x_1 - x_2\|, \quad \forall t \in [0, T],\ \forall (x_1, y), (x_2, y) \in \check{G} \tag{2.2c}$$

with moderately-sized constants L_1 and L_2, where, throughout this paper, $\langle ., . \rangle$ is the standard inner product in real Euclidean spaces R^M, R^N and R^s with the corresponding norm $\|.\|$, the matrix norm used in the following text is subject to $\|.\|$, and $\mu(.)$ denotes the logarithmic norm with respect to $\langle ., . \rangle$.
H1: All derivatives of the exact solution $(x(t), y(t))$ up to a sufficiently high order are bounded independently of the stiffness of the problem, i.e.

$$\|x^{(j)}(t)\| \le \hat{M}_j, \qquad \|y^{(j)}(t)\| \le \hat{N}_j, \quad j = 1, 2, \ldots, l,\ t \in [0, T] \tag{2.2d}$$

with constants $\hat{M}_j$, $\hat{N}_j$ of moderate size and sufficiently large l.
H2: The Jacobian matrix of f with respect to the x-variable $f_x(t, x, y) := \frac{\partial f(t,x,y)}{\partial x}$ along the exact solution $(x(t), y(t))$ satisfies

$$\mu(f_x(t, x(t), y(t))) \le 0, \quad t \in [0, T]. \tag{2.2e}$$

H3: There exist positive constants δ_j $(j = 1, 2, 3)$ and matrices $E_j = E_j(t, \Delta t, \Delta x, \Delta y) \in R^{M \times M}$ $(j = 1, 2)$ such that for all $(t, x(t), y(t))$ and $(t + \Delta t, x(t) + \Delta x, y(t) + \Delta y) \in [0, T] \times \check{G}$ with

$$\begin{aligned} &|\Delta t| \le \delta_1, \qquad \|\Delta x\| \le \delta_2, \qquad \|\Delta y\| \le \delta_3, \\ &f_x(t + \Delta t, x(t) + \Delta x, y(t) + \Delta y) - f_x(t, x(t), y(t)) \\ &\quad = f_x(t, x(t), y(t)) E_1(t, \Delta t, \Delta x, \Delta y) + E_2(t, \Delta t, \Delta x, \Delta y) \end{aligned} \tag{2.2f}$$

with

$$\|E_j(t, \Delta t, \Delta x, \Delta y)\| \le \mu_j |\Delta t| + \lambda_j \|\Delta x\| + \zeta_j \|\Delta y\|, \quad j = 1, 2.$$

Here $\lambda_j, \mu_j, \zeta_j$ are constants of moderate size, and the constants $\delta_1, \delta_2, \delta_3$ are supposed to be independent of the stiffness of the problem. The class of all IVPs (2.1) statisfying the assumptions H0–H3 for some moderate values of $\hat{M}_j, \hat{N}_j$ $(j = 1, 2, \ldots, l)$, L_j $(j = 1, 2)$, δ_j $(j = 1, 2, 3)$, $\mu_j, \lambda_j, \zeta_j$ $(j = 1, 2)$ will be denoted by P_ϵ. The assumption H3 was firstly introduced by [13–15]. More comments about the above assumptions H1–H3 can be found in [13].

The problem (2.1) is a MSPP, and the problem considered in [1] is a SSPP. The right-side functions of the SSPP in [1] satisfy the Lipschitz conditions with moderately-sized Lipschitz constants, and the stiffness of the SSPP is only caused by the small parameter ϵ. The right-side functions of the MSPP (2.1) satisfy the Lipschitz conditions with moderately-sized Lipschitz constants except f_x, and the function f in (2.1) is stiff. The problem (2.1) is obtained by adding an equation with ϵ to the problem in [13]. In general, the one-sided Lipschitz constant of the problem (2.1) is large and of magnitude $\frac{1}{\epsilon}$. Therefore, the above three classes of problems are essentially different.

A s-stage RKM (A, b, c) with

$$A = [a_{ij}] \in R^{s\times s}, \qquad b^{\mathrm{T}} = (b_1, b_2, \ldots, b_s), \qquad c^{\mathrm{T}} = (c_1, c_2, \ldots, c_s),$$

where $c_i = \sum_{j=1}^{s} a_{ij}$ $(i = 1, 2, \ldots, s)$, applied to the problem (2.1) reads

$$X_{ni} = x_n + h_n \sum_{j=1}^{s} a_{ij} f(t_n + c_j h_n, X_{nj}, Y_{nj}), \quad i = 1, 2, \ldots, s, \tag{2.3a}$$

$$\epsilon Y_{ni} = \epsilon y_n + h_n \sum_{j=1}^{s} a_{ij} g(t_n + c_j h_n, X_{nj}, Y_{nj}), \quad i = 1, 2, \ldots, s, \tag{2.3b}$$

$$x_{n+1} = x_n + h_n \sum_{i=1}^{s} b_i f(t_n + c_i h_n, X_{ni}, Y_{ni}), \tag{2.3c}$$

$$\epsilon y_{n+1} = \epsilon y_n + h_n \sum_{i=1}^{s} b_i g(t_n + c_i h_n, X_{ni}, Y_{ni}) \tag{2.3d}$$

with the starting values x_0 and y_0, where x_n, y_n, X_{ni}, and Y_{ni} are approximations to the exact solutions $x(t_n)$, $y(t_n)$, $x(t_n + c_i h_n)$, and $y(t_n + c_i h_n)$ respectively; the used grid is $\{t_j\}_{j=0}^{\breve{N}}$ with

$$0 = t_0 < t_1 < \cdots < t_{\breve{N}} \le T, \qquad h_i = t_{i+1} - t_i \quad (i = 0, 1, \ldots, \breve{N} - 1).$$

For any positive integer k, l and $k \times l$ matrix H, let I_l denote an $l \times l$ unit matrix and $\bar{H} = H \otimes I_M$, $\widetilde{H} = H \otimes I_N$, and let $\otimes$ denote the Kronecker product of two matrices. Then the method (2.3) can be written in more compact form

$$X_n = e \otimes x_n + h_n \bar{A} F(t_n, X_n, Y_n), \tag{2.4a}$$
$$\epsilon Y_n = \epsilon e \otimes y_n + h_n \widetilde{A} G(t_n, X_n, Y_n), \tag{2.4b}$$
$$x_{n+1} = x_n + h_n \bar{b}^{\mathrm{T}} F(t_n, X_n, Y_n), \tag{2.4c}$$
$$\epsilon y_{n+1} = \epsilon y_n + h_n \bar{b}^{\mathrm{T}} G(t_n, X_n, Y_n), \tag{2.4d}$$

where $e = (1, 1, \ldots, 1)^{\mathrm{T}} \in R^s$,

$$X_n = (X_{n1}^{\mathrm{T}}, X_{n2}^{\mathrm{T}}, \ldots, X_{ns}^{\mathrm{T}})^{\mathrm{T}} \in R^{Ms}, \qquad Y_n = (Y_{n1}^{\mathrm{T}}, Y_{n2}^{\mathrm{T}}, \ldots, Y_{ns}^{\mathrm{T}})^{\mathrm{T}} \in R^{Ns}, \tag{2.5a}$$

$$\begin{aligned} F(t_n, X_n, Y_n) \;=\; & (f(t_n + c_1 h_n, X_{n1}, Y_{n1})^{\mathrm{T}}, f(t_n + c_2 h_n, X_{n2}, Y_{n2})^{\mathrm{T}}, \\ & \ldots, f(t_n + c_s h_n, X_{ns}, Y_{ns})^{\mathrm{T}})^{\mathrm{T}} \in R^{Ms}, \end{aligned} \tag{2.5b}$$

$$\begin{aligned} G(t_n, X_n, Y_n) \;=\; & (g(t_n + c_1 h_n, X_{n1}, Y_{n1})^{\mathrm{T}}, g(t_n + c_2 h_n, X_{n2}, Y_{n2})^{\mathrm{T}}, \\ & \ldots, g(t_n + c_s h_n, X_{ns}, Y_{ns})^{\mathrm{T}})^{\mathrm{T}} \in R^{Ns}. \end{aligned} \tag{2.5c}$$

Now we introduce the Butcher simplifying assumptions

$$B(p) : i b^{\mathrm{T}} c^{i-1} = 1, \quad i = 1, 2, \ldots, p,$$
$$C(q) : i A c^{i-1} = c^i, \quad i = 1, 2, \ldots, q,$$

where $c^i = (c_1^i, c_2^i, \ldots, c_s^i)^{\mathrm{T}}$. If the method (A, b, c) satisfies $B(q)$ and $C(q)$, then it is of stage order q.

For the spaces of stage vectors R^{Ms} and R^{Ns}, we define the inner product $\langle .,. \rangle$ and the corresponding norm $\|.\|$ (also cf. [13]) by

$$\langle \hat{V}, \hat{W} \rangle := \frac{1}{s} \sum_{i=1}^{s} \langle \hat{V}_i, \hat{W}_i \rangle, \qquad \|\hat{V}\| := \sqrt{\langle \hat{V}, \hat{V} \rangle},$$

where $\hat{V} = (\hat{V}_1^{\mathrm{T}}, \hat{V}_2^{\mathrm{T}}, \ldots, \hat{V}_s^{\mathrm{T}})^{\mathrm{T}}$, $\hat{W} = (\hat{W}_1^{\mathrm{T}}, \hat{W}_2^{\mathrm{T}}, \ldots, \hat{W}_s^{\mathrm{T}})^{\mathrm{T}}$ with

$$\hat{V}_i, \hat{W}_i \in R^M \quad \text{or} \quad \hat{V}_i, \hat{W}_i \in R^N, \quad i = 1, 2, \ldots, s.$$

Throughout this paper, the constants symbolized in the $O(\cdots)$terms are independent of the stiffness of the considered problem.

For the method (2.4) (i.e. (2.3)), we will use the following stability assumptions M1, M2 (also cf. [13,14]):
M1: The RKM (A, b, c) is A-stable, i.e. the rational function $R(z) := 1 + zb^{\mathrm{T}}(I_s - zA)^{-1}e$ satisfies $|R(z)| \le 1$ for all complex z with $\mathrm{Re}\, z \le 0$.
M2: The matrix $I_s - zA$ is regular for all $\mathrm{Re}\, z \le 0$, and there exists a constant K such that

$$\sup_{\mathrm{Re}\, z \le 0} \|(I_s - zA)^{-1}\| \le K < +\infty.$$

The assumption M3 in [13,14] can be obtained from M2 if A is invertible, or if there exists a vector d such that $b^{\mathrm{T}} = d^{\mathrm{T}}A$. For example, Lobatto IIIA and Lobatto IIIC are stiffly accurate and satisfy $b^{\mathrm{T}} = e_s^{\mathrm{T}}A$. Moreover, the condition that the eigenvalues of A have positive real parts implies the assumption M2, and that A is invertible (cf. [16]). We also use the grid assumption (cf. [13]):
M4: There is a positive constant L of moderate size independent of the grid such that

$$\sum_{j=0}^{\check{N}-1} \hat{h}_j \le LT,$$

where $\hat{h}_j = \max_{0 \le i \le j} h_i$, $j = 0, 1, \ldots, \check{N} - 1$.

More comments about M1, M2, and M3 can be found in [13]. Let

$$h_{\min} = \min\{h_j : 0 \le j \le \check{N} - 1\}, \qquad R_i(z) = e_i^{\mathrm{T}}(I_s - zA)^{-1}e, \quad i = 1, 2, \ldots, s,$$

where e_i is the unit vector in R^s, $\mathrm{Re}\, z \le 0$. Then

$$R_i(z) = 1 + ze_i^{\mathrm{T}}A(I_s - zA)^{-1}e = 1 + za_i^{\mathrm{T}}(I_s - zA)^{-1}e, \quad i = 1, 2, \ldots, s,$$

where a_i^{T} is the i-row of A. Especially, $R_s(z) = R(z)$ when $a_s^{\mathrm{T}} = b^{\mathrm{T}}$. Therefore, as pointed out in [13], $R_i(z)$ $(i = 1, 2, \ldots, s)$ can be considered as the stability functions associated to the stage (2.4a) and (2.4b), of the RK formula (2.4), and the assumption M2 yields that $R_i(z)$ $(i = 1, 2, \ldots, s)$ are analytic for all $\mathrm{Re}\, z \le 0$, and are uniformly bounded:

$$\sup_{\mathrm{Re}\, z \le 0} |R_i(z)| = r_i < +\infty.$$

3. Convergence results

Now we introduce some notations (in part, also see [8]). Let

$$\Delta x_n = x_n - x(t_n), \qquad \Delta y_n = y_n - y(t_n), \qquad \check{X}_{ni} = x(t_n + c_i h_n), \qquad \check{Y}_{ni} = y(t_n + c_i h_n),$$
$$\Delta X_{ni} = X_{ni} - \check{X}_{ni}, \qquad \Delta Y_{ni} = Y_{ni} - \check{Y}_{ni},$$
$$\Delta f_{ni} = f(t_n + c_i h_n, X_{ni}, Y_{ni}) - f(t_n + c_i h_n, \check{X}_{ni}, \check{Y}_{ni}),$$
$$\Delta g_{ni} = g(t_n + c_i h_n, X_{ni}, Y_{ni}) - g(t_n + c_i h_n, \check{X}_{ni}, \check{Y}_{ni}),$$
$$\check{X}_n = (\check{X}_{n1}^{\mathrm{T}}, \check{X}_{n2}^{\mathrm{T}}, \ldots, \check{X}_{ns}^{\mathrm{T}})^{\mathrm{T}} \in R^{Ms},$$
$$\check{Y}_n = (\check{Y}_{n1}^{\mathrm{T}}, \check{Y}_{n2}^{\mathrm{T}}, \ldots, \check{Y}_{ns}^{\mathrm{T}})^{\mathrm{T}} \in R^{Ns},$$
$$\check{F}(t_n, \check{X}_n, \check{Y}_n) = (f(t_n + c_1 h_n, \check{X}_{n1}, \check{Y}_{n1})^{\mathrm{T}}, f(t_n + c_2 h_n, \check{X}_{n2}, \check{Y}_{n2})^{\mathrm{T}}, \ldots, f(t_n + c_s h_n, \check{X}_{ns}, \check{Y}_{ns})^{\mathrm{T}})^{\mathrm{T}} \in R^{Ms},$$
$$\check{G}(t_n, \check{X}_n, \check{Y}_n) = (g(t_n + c_1 h_n, \check{X}_{n1}, \check{Y}_{n1})^{\mathrm{T}}, g(t_n + c_2 h_n, \check{X}_{n2}, \check{Y}_{n2})^{\mathrm{T}}, \ldots, g(t_n + c_s h_n, \check{X}_{ns}, \check{Y}_{ns})^{\mathrm{T}})^{\mathrm{T}} \in R^{Ns},$$
$$\Delta X_n = X_n - \check{X}_n = (\Delta X_{n1}^{\mathrm{T}}, \Delta X_{n2}^{\mathrm{T}}, \ldots, \Delta X_{ns}^{\mathrm{T}})^{\mathrm{T}},$$
$$\Delta Y_n = Y_n - \check{Y}_n = (\Delta Y_{n1}^{\mathrm{T}}, \Delta Y_{n2}^{\mathrm{T}}, \ldots, \Delta Y_{ns}^{\mathrm{T}})^{\mathrm{T}},$$

$$\Delta F = F - \check{F}, \qquad \Delta G = G - \check{G}.$$

Conditions $B(q)$, $C(q)$, and H_1 imply

$$\check{X}_n = e \otimes x(t_n) + h_n \bar{A}\check{F} + w_n^x, \tag{3.1a}$$

$$\check{Y}_n = e \otimes y(t_n) + \frac{h_n}{\epsilon}\widetilde{A}\check{G} + w_n^y, \tag{3.1b}$$

$$x(t_{n+1}) = x(t_n) + h_n \bar{b}^{\mathrm{T}}\check{F} + w_{n0}^x, \tag{3.1c}$$

$$y(t_{n+1}) = y(t_n) + \frac{h_n}{\epsilon}\bar{b}^{\mathrm{T}}\check{G} + w_{n0}^y, \tag{3.1d}$$

where

$$\max\{\|w_n^x\|, \|w_n^y\|, \|w_{n0}^x\|, \|w_{n0}^y\|\} \le W_1 h_n^{q+1}. \tag{3.1e}$$

It follows from (2.4) and (3.1) that

$$\Delta X_n = e \otimes \Delta x_n + h_n \bar{A}\Delta F - w_n^x, \tag{3.2a}$$

$$\Delta Y_n = e \otimes \Delta y_n + \frac{h_n}{\epsilon}\widetilde{A}\Delta G - w_n^y, \tag{3.2b}$$

$$\Delta x_{n+1} = \Delta x_n + h_n \bar{b}^{\mathrm{T}}\Delta F - w_{n0}^x, \tag{3.2c}$$

$$\Delta y_{n+1} = \Delta y_n + \frac{h_n}{\epsilon}\bar{b}^{\mathrm{T}}\Delta G - w_{n0}^y. \tag{3.2d}$$

Since the eigenvalues of A have positive real parts, A is invertible and we can compute ΔF and ΔG from (3.2a) and (3.2b)

$$\Delta F = \frac{1}{h_n}\bar{A}^{-1}(\Delta X - e \otimes \Delta x_n + w_n^x), \tag{3.3a}$$

$$\Delta G = \frac{\epsilon}{h_n}\widetilde{A}^{-1}(\Delta Y - e \otimes \Delta y_n + w_n^y). \tag{3.3b}$$

Moreover, it follows from (3.2) and (3.3) that

$$\Delta x_{n+1} = \alpha \Delta x_n + \bar{b}^{\mathrm{T}}\bar{A}^{-1}\Delta X_n + \bar{b}^{\mathrm{T}}\bar{A}^{-1}w_n^x - w_{n0}^x, \tag{3.4a}$$

$$\Delta y_{n+1} = \alpha \Delta y_n + \widetilde{b}^{\mathrm{T}}\widetilde{A}^{-1}\Delta Y_n + \widetilde{b}^{\mathrm{T}}\widetilde{A}^{-1}w_n^y - w_{n0}^y, \tag{3.4b}$$

where $\alpha = 1 - b^{\mathrm{T}}A^{-1}e$. We can obtain easily

$$\Delta F = F_X \Delta X_n + F_Y \Delta Y_n, \qquad \Delta G = G_X \Delta X_n + G_Y \Delta Y_n, \tag{3.5}$$

where

$$F_X = \mathrm{blockdiag}(U_{n1}^F, U_{n2}^F, \ldots, U_{ns}^F), \qquad F_Y = \mathrm{blockdiag}(V_{n1}^F, V_{n2}^F, \ldots, V_{ns}^F),$$
$$G_X = \mathrm{blockdiag}(U_{n1}^G, U_{n2}^G, \ldots, U_{ns}^G), \qquad G_Y = \mathrm{blockdiag}(V_{n1}^G, V_{n2}^G, \ldots, V_{ns}^G),$$

where, for $i = 1, 2, \ldots, s$,

$$U_{ni}^F = \int_0^1 f_x(t_n + c_i h_n, \check{X}_{ni} + \theta \Delta X_{ni}, \check{Y}_{ni})\mathrm{d}\theta,$$

$$V_{ni}^F = \int_0^1 f_y(t_n + c_i h_n, \check{X}_{ni} + \Delta X_{ni}, \check{Y}_{ni} + \theta \Delta Y_{ni})\mathrm{d}\theta,$$

$$U_{ni}^G = \int_0^1 g_x(t_n + c_i h_n, \check{X}_{ni} + \theta \Delta X_{ni}, \check{Y}_{ni})\mathrm{d}\theta,$$

$$V_{ni}^G = \int_0^1 g_y(t_n + c_i h_n, \check{X}_{ni} + \Delta X_{ni}, \check{Y}_{ni} + \theta \Delta Y_{ni})\mathrm{d}\theta,$$
$$\Delta f_{ni} = U_{ni}^F \Delta X_{ni} + V_{ni}^F \Delta Y_{ni}, \ \Delta g_{ni} = U_{ni}^G \Delta X_{ni} + V_{ni}^G \Delta Y_{ni}.$$

For (3.5) and (3.2b), we have

$$\Delta Y_n = \frac{h_n}{\epsilon}\left(\widetilde{I}_s - \frac{h_n}{\epsilon}\widetilde{A} G_Y\right)^{-1}\left(\frac{\epsilon}{h_n} e \otimes \Delta y_n + \widetilde{A} G_X \Delta X_n - \frac{\epsilon}{h_n} w_n^y\right). \tag{3.6}$$

Since (2.2a) holds and the eigenvalues of A have positive real parts, the matrix-valued version of a theorem of von Neumann (cf. [1–3,17]) yields, for $\epsilon \le C_0 \check{h}_n$

$$\left\| \frac{h_n}{\epsilon}\left(\widetilde{I}_s - \frac{h_n}{\epsilon}\widetilde{A} G_Y\right)^{-1} \right\| \le W_2, \quad n = 0, 1, \ldots, \check{N} - 1, \tag{3.7}$$

where $\check{h}_n = \min_{0 \le i \le n} h_i$, that the constants W_2, C_0 are independent of the stiffness of the IVP (2.1).

Lemma 3.1. *Let $J_n = f_x(t_n, x(t_n), y(t_n))$ and suppose that the IVP (2.1) satisfies (2.2b) and the assumptions* H1–H3*; then there exist $\check{E}_{i,j} \in R^{M \times M}$ $(j = 1, 2)$ such that*

$$\Delta f_{ni} = J_n \Delta X_{ni} + (J_n \check{E}_{i,1} + \check{E}_{i,2}) \Delta X_{ni} + \check{E}_{i,3} \Delta Y_{ni}, \tag{3.8}$$

where $\check{E}_{i,3} = V_{ni}^F \in R^{M \times N}$,

$$\|\check{E}_{i,3}\| \le L_1, \qquad \|\check{E}_{i,j}\| \le K_1 h_n + K_2 \|\Delta X_{ni}\|, \quad j = 1, 2$$

with K_1, K_2, L_1 independent of the stiffness.

Proof. The proof of Lemma 3.1 can be easily given by some modifications of Lemma 4.1 in [13]. ¶

Let

$$\check{E}_j = \mathrm{diag}(\check{E}_{1,j}, \ldots, \check{E}_{s,j}) \in R^{Ms \times Ms} \quad (j = 1, 2), \qquad \check{E}_3 = \mathrm{diag}(\check{E}_{1,3}, \ldots, \check{E}_{s,3}) \in R^{Ms \times Ns}.$$

It follows from the assumptions M2, H2, and a generalized version of von Neumann's theorem given in [17] that

$$\|R_i(h_n J_n)\| \le r_i \quad (i = 1, 2, \ldots, s), \qquad \|(I_{Ms} - (A \otimes h_n J_n))^{-1}\| \le K. \tag{3.8$'$}$$

Thus, we have

Lemma 3.2. *The inequality*

$$\|\Delta X_{ni}\| \le r_i \|\Delta x_n\| + (K+1)\|\check{E}_1 \Delta X_n\| + h_n K \|A\| \|\check{E}_2 \Delta X_n\| + h_n K \|A\| \|\check{E}_3 \Delta Y_n\| + K \|w_n^x\| \tag{3.9}$$

holds, where K is the constant in the assumption M2.

Proof. The proof of Lemma 3.2 can be easily given by some modifications of Lemma 4.2 in [13]. ¶

Let

$$\delta_{ni}^x = \|\Delta X_{ni}\|, \qquad \delta_{ni}^y = \|\Delta Y_{ni}\|, \quad i = 1, 2, \ldots, s; \qquad \delta_n^x = \max_{1 \le i \le s} \delta_{ni}^x, \qquad \delta_n^y = \max_{1 \le i \le s} \delta_{ni}^y.$$

Then $\delta_n^x \ge \|\Delta X_n\|$, $\delta_n^y \ge \|\Delta Y_n\|$, and

$$\|\check{E}_j\| \le K_1 h_n + K_2 \delta_n^x \quad (j = 1, 2), \qquad \|\check{E}_3\| \le L_1. \tag{3.10}$$

Substituting (3.10) into (3.9) for $h_n \le h_1^*$, $i = 1, 2, \ldots, s$, it follows that

$$\delta_{ni}^x \le r_i \|\Delta x_n\| + h_n K_3 \delta_n^x + K_4 (\delta_n^x)^2 + K \|w_n^x\| + h_n \widetilde{K}_4 \delta_n^y, \tag{3.11}$$

where $\widetilde{K}_4 = K\|A\|L_1$, $K_3 = (K+1)K_1 + K\|A\|K_1h_1^*$, $K_4 = (K+1)K_2 + K\|A\|K_2h_1^*$, h_1^*, $\widetilde{K}_4$ and K_3 are independent of the stiffness of the considered problems.

As we will see in Section 4, there is $h_2^* > 0$ independent of the stiffness such that the nonlinear algebraic system (2.4a) and (2.4b) possess a unique solution $X_{ni} = X_{ni}(h_n)$, $Y_{ni} = Y_{ni}(h_n)$ $(i = 1, 2, \ldots, s)$ which depends continuously on h_n for $h_n \le h_2^*$. This implies that $(\Delta X_{ni}, \Delta Y_{ni})$ $(i = 1, 2, \ldots, s)$ satisfying (3.2a) and (3.2b) are all defined and continuous for $h_n \le h_2^*$. Moreover,

$$\Delta X_{ni}(0) = \Delta x_n, \qquad \Delta Y_{ni}(0) = \Delta y_n, \tag{3.12a}$$

$$\delta_{ni}^x(0) = \delta_n^x(0) = \|\Delta x_n\|, \qquad \delta_{ni}^y(0) = \delta_n^y(0) = \|\Delta y_n\|. \tag{3.12b}$$

Theorem 3.1. *Assume the method (A, b, c) is stiffly-accurate and of stage order $q \ge 2$, and satisfies the assumption* M1 *and the condition that the eigenvalues of A have positive real parts. Then, when this method is applied to the problem P_ϵ, the following global error estimates hold for $\epsilon \le C_0 \min\{\check{h}_n^2, \check{h}_n\}$, $0 \le h_n \le \bar{h}_0$, $x_0 - x(t_0) = 0$, and $y_0 - y(t_0) = 0$*

$$\|x_n - x(t_n)\| \le C_1\hat{h}_n^q, \qquad \|y_n - y(t_n)\| \le C_2\hat{h}_n^q$$

with respect to the grids that satisfy the assumption M4*, where the constants $\bar{h}_0, C_i$ $(i = 0, 1, 2)$ are independent of the stiffness of the considered problem.*

Proof. When the method (A, b, c) is stiffly accurate, $a_s^{\mathrm{T}} = b^{\mathrm{T}}$ and $\alpha = 0$, and we have

$$x_{n+1} = X_{ns}, \qquad y_{n+1} = Y_{ns}, \qquad c_s = 1, \qquad \Delta x_{n+1} = \Delta X_{ns}, \qquad \Delta y_{n+1} = \Delta Y_{ns},$$

and

$$\|\Delta x_{n+1}\| = \|\Delta X_{ns}\| = \delta_{ns}^x, \qquad \|\Delta y_{n+1}\| = \|\Delta Y_{ns}\| = \delta_{ns}^y. \tag{3.13}$$

It follows from (3.11) and (3.1e) that

$$\delta_{ni}^x \le r_i\|\Delta x_n\| + h_nK_3\delta_n^x + K_4(\delta_n^x)^2 + h_n\widetilde{K}_4\delta_n^y + K_5h_n^{q+1}, \quad i = 1, 2, \ldots, s, \tag{3.14}$$

$$\delta_n^x \le \beta\|\Delta x_n\| + h_nK_3\delta_n^x + K_4(\delta_n^x)^2 + h_n\widetilde{K}_4\delta_n^y + K_5h_n^{q+1}, \tag{3.15}$$

where

$$h_n \in [0, h_3^*], \qquad h_3^* = \min\{h_1^*, h_2^*\}, \qquad \beta = \max_{1\le i\le s} r_i \ge 1, \qquad K_5 = KW_1.$$

For $i = s$, from $r_s = 1$, (3.14) and (3.13) we have

$$\|\Delta x_{n+1}\| \le \|\Delta x_n\| + h_nK_3\delta_n^x + K_4(\delta_n^x)^2 + h_n\widetilde{K}_4\delta_n^y + K_5h_n^{q+1}, \quad h_n \in [0, h_3^*]. \tag{3.16}$$

On the other hand, it follows from (3.7) and (3.6) multiplied by $e_i^{\mathrm{T}} \otimes I_M$ that

$$\delta_n^y \le K_6\left(\frac{\epsilon}{h_n}\|\Delta y_n\| + \delta_n^x + \epsilon h_n^q\right), \quad h_n \in [0, h_3^*], \epsilon \le C_0\check{h}_n \tag{3.17}$$

with K_6 independent of the stiffness. Inserting (3.17) into (3.16) yields

$$\delta_n^x \le \beta\|\Delta x_n\| + h_n\hat{K}_3\delta_n^x + K_4(\delta_n^x)^2 + \hat{K}_4\epsilon\|\Delta y_n\| + \hat{K}_5h_n^{q+1}, \quad h_n \in [0, h_3^*] \tag{3.18}$$

where $\epsilon \le C_0\check{h}_n$, $\hat{K}_3 = K_3 + \widetilde{K}_4K_6$, $\hat{K}_4 = \widetilde{K}_4K_6$, $\hat{K}_5 = K_5 + \widetilde{K}_4K_6C_0h_3^*$. Let

$$\gamma(h_n) = \beta\|\Delta x_n\| + \hat{K}_4\epsilon\|\Delta y_n\| + \hat{K}_5h_n^{q+1},$$

where $\gamma(h_n)$ is a positive continuous function, and

$$\gamma(0) = \beta\|\Delta x_n\| + \hat{K}_4\epsilon\|\Delta y_n\|,$$
$$\delta_n^x(0) = \|\Delta x_n\| < \gamma(0) < 2\gamma(0),$$

and (3.18) yields

$$\delta_n^x \leq \gamma(h_n) + h_n \hat{K}_3 \delta_n^x + K_4(\delta_n^x)^2. \tag{3.19a}$$

Moreover, applying Lemma 4.4 in [13] to (3.19a) we have

$$\delta_n^x \leq 2\gamma(h_n) = 2\beta\|\Delta x_n\| + 2\hat{K}_4\epsilon\|\Delta y_n\| + 2\hat{K}_5 h_n^{q+1} \tag{3.19b}$$

for $0 \leq h_n \leq h_4^*$, $h_4^* = \min\{h_3^*, \frac{1}{4\hat{K}_3}\}$. It follows from (3.16), (3.17) and (3.19b) that

$$\begin{aligned} \|\Delta x_{n+1}\| \leq\ & (1 + K_7 h_n)\|\Delta x_n\| + K_8\epsilon\|\Delta y_n\| + K_9\epsilon\|\Delta x_n\|\|\Delta y_n\| \\ & + K_{10}\|\Delta x_n\|^2 + K_{11}\epsilon^2\|\Delta y_n\|^2 + K_{12}h_n^{q+1} + K_{13}\epsilon h_n^{q+1}, \end{aligned} \tag{3.20}$$

where $h_n \in [0, h_4^*]$, K_i $(i = 7, 8, \ldots, 13)$ are positive constants independent of the stiffness.

Now we prove the following formulae by induction:

$$\|\Delta x_n\| \leq \hat{h}_n, \qquad \|\Delta y_n\| = O(\hat{h}_n), \quad n \geq 0 \tag{3.21}$$

for $0 \leq \hat{h}_n \leq h_7^*$. In fact, we have first assumed that

$$\|\Delta x_0\| = 0, \qquad \|\Delta y_0\| = 0.$$

We also assume that (3.21) holds for all $j \leq n$, then $\|\Delta y_n\| \leq K_{15}\hat{h}_n$, and (3.20) yields

$$\|\Delta x_{n+1}\| \leq (1 + K_7 h_n)\|\Delta x_n\| + K_{10}\|\Delta x_n\|^2 + \hat{K}_{12}h_n^{q+1} + K_8 K_{15}\hat{h}_n\epsilon + K_9 K_{15}\hat{h}_n^2\epsilon + K_{11}K_{15}^2\hat{h}_n^2\epsilon^2, \tag{3.22}$$

where $q \geq 2$ and

$$\hat{K}_{12} = K_{12} + \epsilon K_{13} \leq K_{12} + K_{13}C_0(h_4^*)^2, \quad \epsilon \leq C_0\check{h}_n^2.$$

Thus,

$$\|\Delta x_{n+1}\| \leq (1 + K_7\hat{h}_n)\|\Delta x_n\| + K_{10}\|\Delta x_n\|^2 + K_{14}\hat{h}_n^3, \quad \hat{h}_n \in [0, h_4^*], \tag{3.23}$$

where

$$K_{14} = K_8 K_{15} C_0 + \hat{K}_{12} h_4^{*q-2} + K_9 K_{15} C_0 h_4^* + K_{11} K_{15}^2 C_0^2 h_4^{*3}.$$

By means of Lemma 4.4 in [13], (3.23) yields

$$\|\Delta x_{n+1}\| \leq \frac{K_{14}}{K_7 + K_{10}}(e^{(K_7+K_{10})LT} - 1)\hat{h}_n^2 \leq \hat{h}_n, \quad n \geq 0,$$

whenever

$$\hat{h}_n \leq h_5^* = \min\left\{h_4^*, \left(\frac{K_{14}}{K_7 + K_{10}}(e^{(K_7+K_{10})LT} - 1)\right)^{-1}\right\}, \quad (n \geq 0).$$

It follows from (3.17) and (3.19b) that

$$\begin{aligned} \delta_n^y \leq\ & K_6\left(\frac{\epsilon}{h_n}\|\Delta y_n\| + 2\beta\|\Delta x_n\| + 2\hat{K}_4\epsilon\|\Delta y_n\| + 2\hat{K}_5 h_n^{q+1} + \epsilon h_n^q\right) \\ =\ & K_6\left(\frac{\epsilon}{h_n} + 2\hat{K}_4\epsilon\right)\|\Delta y_n\| + 2\beta K_6\|\Delta x_n\| + 2K_6\hat{K}_5\hat{h}_n^{q+1} + K_6\epsilon\hat{h}_n^q. \end{aligned} \tag{3.24}$$

Inserting (3.24) into (3.4b) yields

$$\begin{aligned} \|\Delta y_{n+1}\| \leq\ & \left(|\alpha| + K_{16}\epsilon\left(1 + \frac{1}{h_n}\right)\right)\|\Delta y_n\| + K_{17}\hat{h}_n \\ =\ & (|\alpha| + K_{16}\hat{\epsilon})\|\Delta y_n\| + K_{17}\hat{h}_n, \quad \hat{h}_n \leq h_5^*, \end{aligned} \tag{3.25}$$

where K_{16}, K_{17} are independent of the stiffness and $\hat{\epsilon} = \epsilon(1+\frac{1}{\check{h}_n})$. Since $\epsilon \le C_0\check{h}_n^2, \hat{\epsilon} \le C_4\check{h}_n$, here $C_4 = C_0(1+h_5^*)$. Therefore, there exists $h_6^* > 0$ such that

$$|\alpha| + K_{16}C_4\check{h}_n \le K_{16}C_4h_6^* < 1 \quad \text{for } \check{h}_n \in [0, h_6^*],$$

and

$$\begin{aligned}\|\Delta y_{n+1}\| &\le (|\alpha| + K_{16}C_4h_6^*)\|\Delta y_n\| + K_{17}\hat{h}_n \\ &\le \sum_{i=0}^{n}(K_{16}C_4h_6^*)^i K_{17}\hat{h}_n \\ &\le \frac{K_{17}}{1 - K_{16}C_4h_6^*}\hat{h}_n \\ &= O(\hat{h}_n), \quad \hat{h}_n \in [0, h_7^*].\end{aligned}$$

where $h_7^* = \min\{h_5^*, h_6^*\}$.

Now we obtain the global error estimate results. It follows from (3.20) and (3.21) that

$$\|\Delta x_{n+1}\| \le (1 + K_{19}\hat{h}_n)\|\Delta x_n\| + K_{20}\epsilon\|\Delta y_n\| + K_{12}\hat{h}_n^{q+1} + K_{13}\epsilon\hat{h}_n^{q+1}, \tag{3.26}$$

where $K_{19} = K_7 + K_{10}$, $K_{20} = K_8 + K_9h_7^* + K_{11}K_{15}C_0h_7^{*2}$.
It follows from (3.4b), (3.17) and (3.19b) that

$$\begin{aligned}\|\Delta y_{n+1}\| &\le \left(|\alpha| + K_{21}\epsilon\left(1 + \frac{1}{h_n}\right)\right)\|\Delta y_n\| + K_{22}\|\Delta x_n\| + K_{23}h_n^{q+1} + K_{24}\epsilon h_n^q \\ &\le (|\alpha| + K_{21}\hat{\epsilon})\|\Delta y_n\| + K_{22}\|\Delta x_n\| + K_{23}\hat{h}_n^{q+1} + K_{24}\epsilon\hat{h}_n^q,\end{aligned} \tag{3.27}$$

where $K_{21}, K_{22}, K_{23}, K_{24}$ are independent of the stiffness. (3.25) and (3.27) yield

$$\begin{pmatrix}\|\Delta x_{n+1}\| \\ \|\Delta y_{n+1}\|\end{pmatrix} \le \begin{pmatrix}1 + K_{19}\hat{h}_n & K_{20}\epsilon \\ K_{22} & |\alpha| + K_{21}\hat{\epsilon}\end{pmatrix}\begin{pmatrix}\|\Delta x_n\| \\ \|\Delta y_n\|\end{pmatrix} + \Psi\begin{pmatrix}\hat{h}_n \\ 1\end{pmatrix},$$

where $\Psi = O(\hat{h}_n^q) + O(\epsilon\hat{h}_n^q)$. By means of the same technique used in the proof of [1, pp. 432–433, Lemma 2.9], we easily obtain the conclusion of Theorem 3.1. ¶

Remark. The assumption M1 , and the invertibility of the matrix A, imply in general that the eigenvalues of A have positive real parts. Otherwise, the stability function would have to be reducible (cf. [1, p. 431], [3]). Therefore, Radau IIA methods with $s \ge 2$ and Lobatto IIIC methods with $s \ge 3$ can satisfy the assumptions of Theorem 3.1, and are of $q = s$ and $q = s - 1$ respectively.

The corresponding reduced equations of (2.1) with $\epsilon = 0$ is a SDAE

$$x'(t) = f(t, x, y), \quad t \in [0, T], \tag{3.28a}$$

$$0 = g(t, x, y) \tag{3.28b}$$

whose initial values $x(0)$ and $y(0)$ are consistent if $0 = g(0, x(0), y(0))$. Moreover, if the Jacobian $g_y(t, x, y)$ is invertible and bounded, then the problem (3.28) is of index 1, and the Eq. (3.28b) then possesses a unique solution $y = \Omega(x)$. Inserting it into (3.28a) yields

$$x'(t) = f(t, x, \Omega(x)). \tag{3.29}$$

We obtain from (2.4b)

$$h_nG = \epsilon\widetilde{A}^{-1}(Y_n - e \otimes y_n). \tag{3.30}$$

Insert (3.30) into (2.4d) and let $\epsilon = 0$ in (2.4). Then

$$X_n = e \otimes x_n + h_n\bar{A}F(t_n, X_n, Y_n), \tag{3.31a}$$

$$0 = G(t_n, X_n, Y_n), \tag{3.31b}$$

$$x_{n+1} = x_n + h_n \bar{b}^{\mathrm{T}} F(t_n, X_n, Y_n), \tag{3.31c}$$

$$y_{n+1} = \alpha y_n + \widetilde{b}^{\mathrm{T}} \widetilde{A}^{-1} Y_n. \tag{3.31d}$$

Theorem 3.2. *Suppose that the method* (A, b, c) *is stiffly accurate and of stage order* $q \geq 2$*, and satisfies the assumption* M1 *and the condition that the eigenvalues of* A *have positive real parts. If the problem* (3.28) *satisfies* (2.2b) *and* (2.2c) *and the assumptions* H1–H3*,* g_y *is invertible and bounded, and the initial values are consistent, then the numerical solution of* (3.31) *has global error*

$$x_n - x(t_n) = O(\hat{h}_n^q), \qquad y_n - y(t_n) = O(\hat{h}_n^q)$$

when $x_0 - x(t_0) = 0$*,* $y_0 - y(t_0) = 0$*,* $h \leq \bar{h}_0$*.*

Proof. Because (3.31a)–(3.31c) are independent of y_n and do not change if (3.31d) is replaced by $0 = g(t_{n+1}, x_{n+1}, y_{n+1})$, $x_n - x(t_n) = O(\hat{h}_n^q)$ follows from the fact that (3.29) is a stiff ordinary differential equation which can be covered by [13]. The remaining proof is completely similar to that of Theorem 2.2 in [8], with some modifications; for example, we can obtain

$$\|\Delta X_n\| \leq 2\beta \|\Delta x_n\| + 2\hat{K}_5 h_n^{q+1}$$

by means of the similar process of giving (3.19b), hence $\Delta X_n = O(\hat{h}_n^q)$. ¶

4. Existence and uniqueness of the solution of RK equations

Theorem 4.1. *If the IVP* (2.1) *satisfies the assumptions* H0–H3*, and the RK method satisfies the condition that the eigenvalues of* A *have positive real parts, then there exist* $h^* > 0$ *and* $\delta > 0$ *independent of the stiffness such that the system*

$$X = e_s \otimes x_n + h \bar{A} F(t_n, X, Y), \tag{4.1a}$$

$$\epsilon Y = \epsilon e_s \otimes y_n + h \widetilde{A} G(t_n, X, Y), \tag{4.1b}$$

where $X = (X_1^{\mathrm{T}}, X_2^{\mathrm{T}}, \ldots, X_s^{\mathrm{T}})^{\mathrm{T}} \in R^{Ms}$, $Y = (Y_1^{\mathrm{T}}, Y_2^{\mathrm{T}}, \ldots, Y_s^{\mathrm{T}})^{\mathrm{T}} \in R^{Ns}$*, possess a unique solutions for* $0 \leq h \leq h^*$ *and*

$$\|x_n - x(t_n)\| \leq \delta, \qquad \|y_n - y(t_n)\| \leq \delta.$$

Proof. The part idea of the proof is similar to that of Theorem 5.1 in [13]. (4.1) is obtained from (2.4a) and (2.4b) by omitting the subscript "n" of X_n, Y_n, h_n. Moreover, the other corresponding notations will also be given from Section 3 in the same way.

The conditions $B(q)$ and $C(q)$ imply

$$\check{X} = e \otimes x(t_n) + h \bar{A} F(t_n, \check{X}, \check{Y}) + w_n^x, \tag{4.2a}$$

$$\epsilon \check{Y} = \epsilon e \otimes y(t_n) + h \widetilde{A} G(t_n, \check{X}, \check{Y}) + w_n^y, \tag{4.2b}$$

where $\|w_n^x\| \leq W_1 h^{q+1}$, $\|w_n^y\| \leq W_1 h^{q+1}$. Subtracting (4.2) from (4.1) we have

$$\Delta X = e \otimes \Delta x_n + h \bar{A}(F(t_n, \check{X} + \Delta X, \check{Y} + \Delta Y) - F(t_n, \check{X}, \check{Y})) - w_n^x, \tag{4.3a}$$

$$\epsilon \Delta Y = \epsilon e \otimes \Delta y_n + h \widetilde{A}(G(t_n, \check{X} + \Delta X, \check{Y} + \Delta Y) - G(t_n, \check{X}, \check{Y})) - w_n^y. \tag{4.3b}$$

Let

$$\begin{aligned}\Phi(\Delta X) = {} & (I_{Ms} - h(A \otimes J_n))^{-1}[e \otimes \Delta x_n + h(A \otimes I_M)(-(I_s \otimes J_n)\Delta X \\ & + F(t_n, \check{X} + \Delta X, \check{Y} + \Delta Y) - F(t_n, \check{X}, \check{Y})) - w_n^x].\end{aligned} \tag{4.4}$$

Let $\Delta X, \Delta \bar{X}, \Delta Y, \Delta \bar{Y}$ such that

$$\|\Delta X\| \leq \rho, \qquad \|\Delta \bar{X}\| \leq \rho, \qquad \|\Delta Y\| \leq \rho, \qquad \|\Delta \bar{Y}\| \leq \rho.$$

(4.4) yields

$$\Phi(\Delta X) - \Phi(\Delta\bar{X}) = (I_{Ms} - h(A \otimes J_n))^{-1} h(A \otimes I_M)[-(I_s \otimes J_n)(\Delta X - \Delta\bar{X}) + F(t_n, \check{X} + \Delta X, \check{Y} + \Delta Y) - F(t_n, \check{X} + \Delta\bar{X}, \check{Y} + \Delta\bar{Y})]. \quad (4.5)$$

The j-subvector component of the last bracket can be written as

$$\begin{aligned} &-J_n(\Delta X_j - \Delta\bar{X}_j) + f(t_n + c_j h, x(t_n + c_j h) + \Delta X_j, y(t_n + c_j h) + \Delta Y_j) \\ &\quad - f(t_n + c_j h, x(t_n + c_j h) + \Delta\bar{X}_j, y(t_n + c_j h) + \Delta\bar{Y}_j) \\ &= \int_0^1 (-J_n + f_x(t_n + c_j h, \check{X}_j + \theta\Delta X_j + (1-\theta)\Delta\bar{X}_j, \check{Y}_j + \Delta Y_j))\mathrm{d}\theta(\Delta X_j - \Delta\bar{X}_j) \\ &\quad + \int_0^1 f_y(t_n + c_j h, \check{X}_j + \Delta\bar{X}_j, \check{Y}_j + \theta\Delta Y_j + (1-\theta)\Delta\bar{Y}_j)\mathrm{d}\theta(\Delta Y_j - \Delta\bar{Y}_j), \end{aligned}$$

which can be written in the form

$$(J_n\check{E}_{1j} + \check{E}_{2j})(\Delta X_j - \Delta\bar{X}_j) + \check{E}_{3j}(\Delta Y_j - \Delta\bar{Y}_j) \quad (4.6)$$

by the assumption H3 as in Lemma 3.1, and we have $\|\check{E}_{3j}\| \le L_1$,

$$\|\check{E}_{ij}\| \le (\mu_i + (\lambda_i + \zeta_i)\hat{M}_1)|c_j|h + (\lambda_i + \zeta_i)\rho = (\lambda_i + \zeta_i)\rho + O(h), \quad i = 1, 2.$$

Let $\check{E}_i = \mathrm{diag}(\check{E}_{i1}, \check{E}_{i2}, \ldots, \check{E}_{is})$ $(i = 1, 2, 3)$. Then (4.5) and (4.6) yield

$$\begin{aligned} &\Phi(\Delta X) - \Phi(\Delta\bar{X}) \\ &= (I_{Ms} - h(A \otimes J_n))^{-1}(A \otimes hI_M)[((I_s \otimes J_n)\check{E}_1 + \check{E}_2)(\Delta X - \Delta\bar{X}) + \check{E}_3(\Delta Y - \Delta\bar{Y})] \\ &= [(I_{Ms} - h(A \otimes J_n))^{-1}(A \otimes hJ_n)\check{E}_1 + (I_{Ms} - h(A \otimes J_n))^{-1}(A \otimes hI_M)\check{E}_2](\Delta X - \Delta\bar{X}) \\ &\quad + (I_{Ms} - h(A \otimes J_n))^{-1}(A \otimes hI_M)\check{E}_3(\Delta Y - \Delta\bar{Y}). \end{aligned} \quad (4.7)$$

Thus, from (3.8′) we have

$$\|\Phi(\Delta X) - \Phi(\Delta\bar{X})\| \le ((1 + K)(\lambda_1 + \zeta_1)\rho + O(h))\|\Delta X - \Delta\bar{X}\| + hL_1K\|A\|\|\Delta Y - \Delta\bar{Y}\|. \quad (4.8)$$

On the other hand, (4.3b) and

$$\Delta\bar{Y} = e \otimes \Delta y_n + \frac{h}{\epsilon}\widetilde{A}(G(t_n, \check{X} + \Delta\bar{X}, \check{Y} + \Delta\bar{Y}) - G(t_n, \check{X}, \check{Y})) - w_n^y$$

imply that

$$\begin{aligned} \Delta Y - \Delta\bar{Y} &= \frac{h}{\epsilon}\widetilde{A}(G(t_n, \check{X} + \Delta X, \check{Y} + \Delta Y) - G(t_n, \check{X} + \Delta\bar{X}, \check{Y} + \Delta\bar{Y})) \\ &= \frac{h}{\epsilon}\widetilde{A}[\hat{G}_X(\Delta X - \Delta\bar{X}) + \hat{G}_Y(\Delta Y - \Delta\bar{Y})], \end{aligned} \quad (4.9)$$

where $\hat{G}_X$ and $\hat{G}_Y$ can be given by the similar way to G_X and G_Y in (3.5). Moreover,

$$\Delta Y - \Delta\bar{Y} = \frac{h}{\epsilon}\left(I - \frac{h}{\epsilon}\widetilde{A}\hat{G}_Y\right)^{-1}\widetilde{A}\hat{G}_X(\Delta X - \Delta\bar{X}). \quad (4.10)$$

It follows from (4.10) and (3.7) that

$$\|\Delta Y - \Delta\bar{Y}\| \le L_3\|\Delta X - \Delta\bar{X}\|, \quad (4.11)$$

where $L_3 = W_2\|A\|L_2$. (4.11) and (4.8) yield

$$\|\Phi(\Delta X) - \Phi(\Delta\bar{X})\| \le ((1 + K)(\lambda_1 + \zeta_1)\rho + O(h))\|\Delta X - \Delta\bar{X}\|.$$

The above formula implies that Φ is contractive provided that $\widetilde{h}_1$ and ρ satisfy

$$(1 + K)(\lambda_1 + \zeta_1)\rho + O(\widetilde{h}_1) \le \lambda = \frac{1}{2}.$$

For $\Delta X = 0$, we have

$$\begin{aligned}\|\Phi(0)\| &= \|(I_{Ms} - h(A \otimes J_n))^{-1}\| \\ &\quad \times \|(e \otimes \Delta x_n + h(A \otimes I_M)(F(t_n, \check{X}, \check{Y} + \Delta Y) - F(t_n, \check{X}, \check{Y})) - w_n^x)\| \\ &\le K(\delta + \|A\| L_1 h \|\Delta Y\| + W_1 h^{q+1}),\end{aligned} \tag{4.12}$$

and (4.3b) and (3.7) (or (3.6) and (3.7)) yield

$$\Delta Y = \frac{h}{\epsilon}\left(I_{Ns} - \frac{h}{\epsilon}\widetilde{A}G_Y\right)^{-1}\left(\frac{\epsilon}{h} e \otimes \Delta y_n - \frac{\epsilon}{h} w_n^y\right), \tag{4.13a}$$

$$\|\Delta Y\| \le W_2\left(\frac{\epsilon}{h}\delta + W_1 \epsilon h^q\right), \quad \epsilon \le C_0 h. \tag{4.13b}$$

It follows from (4.12) and (4.13) that

$$\begin{aligned}\|\Phi(0)\| &\le K(\delta + W_2\|A\| L_1(\epsilon\delta + W_1 \epsilon h^{q+1}) + W_1 h^{q+1}) \\ &\le K_1\delta + K_2 h^{q+1}, \quad h \le h_1^*,\end{aligned} \tag{4.14}$$

where $K_1 = K(1 + W_2\|A\| L_1 C_0 \widetilde{h}_1)$, $K_2 = K W_1 (W_2 \|A\| L_1 C_0 \widetilde{h}_1 + 1)$.

We may choose $h^* \le \widetilde{h}_1$ and δ such that

$$K_1\delta + K_2 h^* \le \frac{\rho}{2} = (1 - \lambda)\rho.$$

Hence, by the contractive mapping theorem (cf. [18]), $\Delta X = \Phi(\Delta X)$ equivalent to (4.1) possesses a locally unique solution for X.

For (4.1), since X is locally unique, we can consider (4.1b) as a nonlinear system about Y. By (3.7), we can show that the Jacobian of (4.1b) $\frac{\epsilon}{h} I_{Ns} - \widetilde{A} G_Y (\epsilon \le C_0 h)$ has a bounded inverse. This implies that the system (4.1) possesses a locally unique solution (X, Y). ¶

Acknowledgements

The author is indebted to the referees for their useful comments and valuable suggestions.

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