Jordan-type inequalities for differentiable functions and their applications

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Abstract

In this work, we extend Jordan’s inequality to obtain a new type of inequality involving functions and their higher-order derivatives. The result is then used to obtain some higher accurate inequalities of Jordan type.

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1. Introduction

The celebrated Jordan inequality (see Mitrinović and Vasić [1])

\[
\frac{2}{\pi} \leq \frac{\sin x}{x} < 1,
\]

and its various extensions have many applications in mathematical analysis. In recent years, considerable attention has been given to improve and generalize in many different directions (see [2–13] and references therein) and to apply them in analysis and applied mathematics.

The major objective of this paper is to consider the new improvements of the original Jordan inequality and its generalized version. Using the Taylor polynomial, it is shown that the function \(\frac{\sin x}{x}\) inequality (1) can be extended to general function \(\frac{f(x)}{x}\), where \(f\) is \((n+1)\) times differentiable with the monotonicity property of \((n+1)\) in a given interval. The main result of this work is a very general Jordan-type inequality which contains many earlier results as a special case of this paper. Some applications of the generalized versions of the Jordan inequality are included. It is shown that many earlier results follow as special cases of those presented in this paper.

2. Lemmas

In order to prove the main results in Section 3, we first introduce the following lemmas.
Lemma 1 (see Anderson et al. [14,15]). Let \( f, g : [a, b] \rightarrow \mathbb{R} \) be two continuous functions which are differentiable on \((a, b)\). Further, let \( g' \neq 0 \) on \((a, b)\). If \( f'/g' \) is increasing (or decreasing) on \((a, b)\), then the functions
\[
\frac{f(x) - f(b)}{g(x) - g(b)} \quad \text{and} \quad \frac{f(x) - f(a)}{g(x) - g(a)}
\]
are also increasing (or decreasing) on \((a, b)\).

Lemma 2. Let \( J(x) = \frac{f(x)}{x} (x \neq 0) \). Then for any natural number \( n \), the following identities hold
\[
\begin{align*}
J^{(n-1)}(x) + x J^{(n)}(x) &= f^{(n)}(x), \\
J^{(n)}(x) &= \frac{(-1)^{n-1}n!}{x^{n+1}} \sum_{k=0}^{n} (-1)^{k-1} x^k f^{(k)}(x).
\end{align*}
\]
(2)

Proof. We rewrite \( J(x) = \frac{f(x)}{x} \) in the form
\[
x J(x) = f(x).
\]
(4)

Differentiating both sides of (4) with respect to \( x \), again and again, it follows that
\[
\begin{align*}
J(x) + x J'(x) &= f'(x), \\
2J'(x) + x J''(x) &= f''(x), \\
3J''(x) + x J'''(x) &= f'''(x), \\
& \quad \ldots
\end{align*}
\]

after \( n \) steps, we obtain
\[
J^{(n-1)}(x) + x J^{(n)}(x) = f^{(n)}(x).
\]

The identity (2) is proved.

Multiplying both sides of the identity \( k J^{(k-1)}(x) + x J^{(k)}(x) = f^{(k)}(x) \) by \( (-1)^{k-1} x^{k-1} k!^{-1} \), we obtain
\[
\frac{(-1)^{k-1} x^{k-1} J^{(k-1)}(x)}{(k-1)!} + \frac{(-1)^{k-1} x^k J^{(k)}(x)}{k!} = \frac{(-1)^{k-1} x^{k-1} f^{(k)}(x)}{k!}. \quad k = 1, 2, \ldots, n.
\]
(5)

Taking the sum of all identities in (5), we obtain
\[
\sum_{k=1}^{n} \left[ \frac{(-1)^{k-1} x^{k-1} J^{(k-1)}(x)}{(k-1)!} + \frac{(-1)^{k-1} x^k J^{(k)}(x)}{k!} \right] = \sum_{k=1}^{n} \frac{(-1)^{k-1} x^{k-1} f^{(k)}(x)}{k!}.
\]
Consequently,
\[
J(x) + \frac{(-1)^{n-1} x^n J^{(n)}(x)}{n!} = \sum_{k=1}^{n} \frac{(-1)^{k-1} x^{k-1} f^{(k)}(x)}{k!}.
\]

We then deduce from the above identity that
\[
J^{(n)}(x) = \frac{(-1)^{n-1} n!}{x^n} \left[ -J(x) + \sum_{k=1}^{n} \frac{(-1)^{k-1} x^{k-1} f^{(k)}(x)}{k!} \right] = \frac{(-1)^{n-1} n!}{x^{n+1}} \left[ -f(x) + \sum_{k=1}^{n} \frac{(-1)^{k-1} x^k f^{(k)}(x)}{k!} \right] = \frac{(-1)^{n-1} n!}{x^{n+1}} \sum_{k=0}^{n} \frac{(-1)^{k-1} x^k f^{(k)}(x)}{k!},
\]
which is the desired identity (3). The proof of Lemma 2 is complete. □
3. Main results

**Theorem 1.** Let $f$ be a real-valued $n + 1$-times differentiable function on $[0, \theta]$ with $f(0) = 0$, and let $J(x) = f(x)/x$. If $n$ is a positive even number such that $f^{(n+1)}$ is increasing on $[0, \theta]$, or $n$ is a positive odd number such that $f^{(n+1)}$ is decreasing on $[0, \theta]$. Then, for $0 < x \leq \theta$, we have the inequality

$$\sum_{k=0}^{n-1} \frac{J^{(k)}(\theta)}{k!} (x - \theta)^k + \left( -\frac{1}{\theta} \right)^n \left( f'(0) - \sum_{k=0}^{n-1} \frac{(-\theta)^k J^{(k)}(\theta)}{k!} \right) (x - \theta)^n \leq J(x) \leq \sum_{k=0}^{n} \frac{J^{(k)}(\theta)}{k!} (x - \theta)^k. \tag{6}$$

If $n$ is a positive even number such that $f^{(n+1)}$ is decreasing on $[0, \theta]$, or $n$ is a positive odd number such that $f^{(n+1)}$ is increasing on $[0, \theta]$. Then, for $0 < x \leq \theta$, we have the inequality

$$\sum_{k=0}^{n} \frac{J^{(k)}(\theta)}{k!} (x - \theta)^k \leq J(x) \leq \sum_{k=0}^{n-1} \frac{J^{(k)}(\theta)}{k!} (x - \theta)^k + \left( -\frac{1}{\theta} \right)^n \left( f'(0) - \sum_{k=0}^{n-1} \frac{(-\theta)^k J^{(k)}(\theta)}{k!} \right) (x - \theta)^n. \tag{7}$$

Furthermore, the equalities in (6) or (7) hold if and only if $x = \theta$.

**Proof.** When $x = \theta$, (6) and (7) become the identity. We suppose $0 < x < \theta$ below.

Let

$$h(x) = J(x) - \sum_{k=0}^{n-1} \frac{J^{(k)}(\theta)}{k!} (x - \theta)^k, \quad g(x) = (x - \theta)^n.$$

It is easy to verify that

$$h^{(k)}(\theta) = 0 \quad \text{and} \quad g^{(k)}(\theta) = 0 \quad \text{for} \quad k = 0, 1, 2, \ldots, n - 1. \tag{8}$$

In view of (8), we have the following relational expressions:

$$\frac{h(x)}{g(x)} = \frac{h(x) - h(\theta)}{g(x) - g(\theta)},$$

$$h'(x) = h'(x) - h'(\theta),$$

$$g'(x) = g'(x) - g'(\theta),$$

$$h''(x) = h''(x) - h''(\theta),$$

$$g''(x) = g''(x) - g''(\theta),$$

$$\ldots$$

$$h^{(n-1)}(x) = h^{(n-1)}(x) - h^{(n-1)}(\theta),$$

$$g^{(n-1)}(x) = g^{(n-1)}(x) - g^{(n-1)}(\theta),$$

$$h^{(n)}(x) = h^{(n)}(x),$$

$$g^{(n)}(x) = \frac{1}{n!}.$$ \tag{9}

Let $u(x) = x^{n+1}$, $v(x) = x^{n+1} J^{(n)}(x)$. From identity (3), we find

$$v(x) = x^{n+1} J^{(n)}(x) = (-1)^{n-1} n! \sum_{k=0}^{n} \frac{(-1)^{k-1} x^k f^{(k)}(x)}{k!}.$$

The hypothesis and the above identity show that

$$u(0) = 0, \quad v(0) = 0.$$

Hence, we have

$$J^{(n)}(x) = \frac{v(x)}{u(x)} = \frac{v(x) - v(0)}{u(x) - u(0)}.$$
On the other hand, it follows from identity (2) that
\[
\frac{v'(x)}{u'(x)} = \frac{(n + 1)x^n J^{(n)}(x) + x^{n+1} J^{(n+1)}(x)}{(n + 1)x^n} = \frac{(n + 1)J^{(n)}(x) + x J^{(n+1)}(x)}{n + 1} = \frac{f^{(n+1)}(x)}{n + 1}.
\]

We now proceed to prove the required inequality, we consider the following two cases.

Case (I): If \( f^{(n+1)} \) is increasing on \([0, \theta] \), then \( \frac{v'(x)}{u'(x)} = \frac{1}{n+1} f^{(n+1)}(x) \) is increasing on \((0, \theta) \). It follows from Lemma 1 that \( J^{(n)}(x) = \frac{v(x) - u(x)}{u'(x)} \) is increasing on \((0, \theta) \). Further, by relation chains (9) and Lemma 1, we deduce that \( h^{(n)}(x), h^{(n-1)}(x), \ldots, h'(x), h(x) \) are all increasing on \((0, \theta) \).

Note that
\[
\begin{split}
\lim_{x \to 0^+} h(x) &= \left( -\frac{1}{\theta} \right)^n \left( f'_+(0) - \sum_{k=0}^{n-1} \frac{(-\theta)^k J^{(k)}(\theta)}{k!} \right), \\
\lim_{x \to 0^-} h(x) &= \lim_{x \to 0^-} \frac{h'(x)}{g'(x)} = \lim_{x \to 0^-} \frac{h^{(n)}(x)}{g^{(n)}(x)} = \frac{J^{(n)}(\theta)}{n!}.
\end{split}
\]

We thus obtain
\[
\left( -\frac{1}{\theta} \right)^n \left( f'_+(0) - \sum_{k=0}^{n-1} \frac{(-\theta)^k f^{(k)}(\theta)}{k!} \right) < \frac{h(x)}{g(x)} < \frac{J^{(n)}(\theta)}{n!} \quad \text{for } x \in (0, \theta),
\]
which implies inequality (6) under the assumption that \( n \) is an even number, and implies inequality (7) under the assumption that \( n \) is an odd number.

Case (II): If \( f^{(n+1)} \) is decreasing on \([0, \theta] \), then \( \frac{v'(x)}{u'(x)} = \frac{1}{n+1} f^{(n+1)}(x) \) is decreasing on \((0, \theta) \). Using argument similar to Case I, we can deduce that \( h^{(n)}(x), h^{(n-1)}(x), \ldots, h'(x), h(x) \) are all decreasing on \((0, \theta) \). Hence, we have
\[
\lim_{x \to 0^+} h(x) > \lim_{x \to 0^-} h(x) = \frac{h(x)}{g(x)} > \lim_{x \to 0^-} \frac{h(x)}{g(x)},
\]
that is,
\[
\left( -\frac{1}{\theta} \right)^n \left( f'_+(0) - \sum_{k=0}^{n-1} \frac{(-\theta)^k f^{(k)}(\theta)}{k!} \right) > \frac{h(x)}{g(x)} > \frac{f^{(n)}(\theta)}{n!} \quad \text{for } x \in (0, \theta),
\]
which implies inequality (6) under the assumption that \( n \) is an odd number, and implies inequality (7) under the assumption that \( n \) is an even number.

This completes the proof of Theorem 1.  \( \Box \)

4. Some applications

In this section we show some consequences of Theorem 1, the results presented here provide applications of the generalized Jordan’s inequality.

Proposition 1. Let \( J(x) = \sin x \). Then for \( 0 < x \leq \theta \leq \pi/2 \) with \( n = 4j + 1 \) \((j = 0, 1, 2, \ldots)\), or \( 0 < x \leq \theta \leq \pi \) with \( n = 4j + 2 \) \((j = 0, 1, 2, \ldots)\), the following inequality holds
\[
\sum_{k=0}^{n-1} \frac{J^{(k)}(\theta)}{k!} (x - \theta)^k + \left( -\frac{1}{\theta} \right)^n \left( 1 - \sum_{k=0}^{n-1} \frac{(-\theta)^k J^{(k)}(\theta)}{k!} \right) (x - \theta)^n \leq J(x) \leq \sum_{k=0}^{n} \frac{J^{(k)}(\theta)}{k!} (x - \theta)^k.
\]
For $0 < x \leq \theta \leq \pi/2$ with $n = 4j + 3$ ($j = 0, 1, 2, \ldots$), or $0 < x \leq \theta \leq \pi$ with $n = 4j + 4$ ($j = 0, 1, 2, \ldots$), the following inequality holds
\[
\sum_{k=0}^{n} \frac{f^{(k)}(\theta)}{k!} (x - \theta)^k \leq J(x) \leq \sum_{k=0}^{n-1} \frac{f^{(k)}(\theta)}{k!} (x - \theta)^k + \left( -\frac{1}{\theta} \right)^n \left( 1 - \sum_{k=0}^{n-1} \frac{(-\theta)^k f^{(k)}(\theta)}{k!} \right) (x - \theta)^n.
\] (11)

**Proof.** Consider the function $f(x) = \sin x$, $x \in [0, \theta]$ and $J(x) = \frac{\sin x}{x}$, $x \in (0, \theta]$. Direct calculation gives
\[
f(0) = 0, \quad f'(0) = 1, \quad f^{(n+1)}(x) = \sin \left( x + \frac{n+1}{2} \pi \right) = \begin{cases} 
-\sin x, & n = 4j + 1 \\
-\cos x, & n = 4j + 2 \\
\sin x, & n = 4j + 3 \\
\cos x, & n = 4j + 4.
\end{cases}
\]

By the hypothesis condition of parameter $\theta$ in Proposition 1, it is easy to find that $f^{(n+1)}$ is increasing on $[0, \theta]$ for $n = 4j + 2$ or $n = 4j + 3$ ($j = 0, 1, 2, \ldots$), and $f^{(n+1)}$ is decreasing on $[0, \theta]$ for $n = 4j + 1$ or $n = 4j + 4$ ($j = 0, 1, 2, \ldots$). Now, using Theorem 1 together with the monotonicity of $f^{(n+1)}$, inequalities (10) and (11) follow immediately. \[ \square \]

**Remark 1.** Putting $n = 2$ in Proposition 1 yields the main result of Wu and Debnath [9], that is
\[
\frac{\sin \theta + \theta \cos \theta - \sin \theta}{\theta^2} (x - \theta) + \frac{\theta - 2 \sin \theta + \theta \cos \theta}{\theta^3} (x - \theta)^2 \leq \sin x \leq \frac{\sin \theta + \theta \cos \theta - \sin \theta}{\theta^2} (x - \theta) + \frac{2 \sin \theta - 2 \theta \cos \theta - \theta^2 \sin \theta}{2\theta^3} (x - \theta)^2,
\] (12)
where $0 < x \leq \theta \leq \pi$.

In particular, if we put in Proposition 1 $\theta = \pi/2$ and $n = 2$, we get the results of Zhu [7] and Özban [8]. In fact, Proposition 1 enable us to obtain a large number of inequalities of Jordan type by assigning appropriate values to the parameter $n$. For example, letting $\theta = \pi/2$, and putting $n = 3$ and $n = 4$ in Proposition 1 respectively, we obtain the following new inequalities which are more accurate than those obtained in Zhu [7] and Özban [8].

**Corollary 1.** If $0 < x \leq \pi/2$, then
\[
\frac{2}{\pi^2} \left( x - \frac{\pi}{2} \right) + \frac{8 - \pi^2}{\pi^3} \left( x - \frac{\pi}{2} \right)^2 - \frac{16 - 2\pi^2}{\pi^4} \left( x - \frac{\pi}{2} \right)^3 \leq \frac{\sin x}{x} \leq \frac{2}{\pi^2} \left( x - \frac{\pi}{2} \right) + \frac{8 - \pi^2}{\pi^3} \left( x - \frac{\pi}{2} \right)^2 + \frac{48 - 8\pi - 2\pi^2}{\pi^4} \left( x - \frac{\pi}{2} \right)^3,
\] (13)
\[
\frac{2}{\pi} \left( x - \frac{\pi}{2} \right) + \frac{8 - \pi^2}{\pi^3} \left( x - \frac{\pi}{2} \right)^2 - \frac{16 - 2\pi^2}{\pi^4} \left( x - \frac{\pi}{2} \right)^3 + \frac{384 - 48\pi^2 + \pi^4}{12\pi^5} \left( x - \frac{\pi}{2} \right)^4
\]
\[
\leq \frac{\sin x}{x} \leq \frac{2}{\pi^2} \left( x - \frac{\pi}{2} \right) + \frac{8 - \pi^2}{\pi^3} \left( x - \frac{\pi}{2} \right)^2 + \frac{16 - 2\pi^2}{\pi^4} \left( x - \frac{\pi}{2} \right)^3
\]
\[
- \frac{128 - 16\pi - 8\pi^2}{\pi^5} \left( x - \frac{\pi}{2} \right)^4.
\] (14)

**Proposition 2.** Let $J(x) = \frac{\sin x}{x}$. Then for $0 < x \leq r$ with $n = 2j$ ($j = 1, 2, \ldots$), the following inequality holds
\[
\sum_{k=0}^{n-1} \frac{J^{(k)}(r)}{k!} (x - r)^k \leq J(x) \leq \sum_{k=0}^{n-1} \frac{J^{(k)}(r)}{k!} (x - r)^k + \left( -\frac{1}{r} \right)^n \left( 1 - \sum_{k=0}^{n-1} \frac{(-r)^k J^{(k)}(r)}{k!} \right) (x - r)^n.
\] (15)

For $0 < x \leq r$ with $n = 2j - 1$ ($j = 1, 2, \ldots$), the following inequality holds
\[
\sum_{k=0}^{n} \frac{J^{(k)}(r)}{k!} (x - r)^k \leq J(x) \leq \sum_{k=0}^{n} \frac{J^{(k)}(r)}{k!} (x - r)^k + \left( -\frac{1}{r} \right)^n \left( 1 - \sum_{k=0}^{n-1} \frac{(-r)^k J^{(k)}(r)}{k!} \right) (x - r)^n.
\] (16)
Proof. Let \( f(x) = \sinh x, x \in [0, r] \) and \( J(x) = \frac{\sinh x}{x}, x \in (0, r] \). A simple calculation gives
\[
f(0) = 0, \quad f'(0) = 1, \quad f^{(n+1)}(x) = \begin{cases} \sinh x, & n = 1, 3, \ldots \\ \cosh x, & n = 2, 4, \ldots \end{cases}
\]
It is easy to see that \( f^{(n+1)} \) is increasing on \([0, r]\) for \( n = 1, 2, 3, \ldots \), using Theorem 1 leads us to the desired inequalities (15) and (16).

Taking \( n = 2 \) in Proposition 2 yields the following inequality

**Corollary 2.** If \( 0 < x \leq r \), then
\[
\frac{\sinh r}{r} + \frac{r \cosh r - \sinh r}{r^2} (x - r) + \frac{r - 2 \sinh r + r \cosh r}{r^3} (x - r)^2 \leq \frac{\sinh x}{x} \leq \frac{\sinh r}{r} + \frac{r \cosh r - \sinh r}{r^2} (x - r) + \frac{2 \sinh r - 2r \cosh r + r^2 \sinh r}{2r^3} (x - r)^2. \tag{17}
\]

**Proposition 3.** Let \( J(x) = \frac{\ln(1+x)}{x}. \) Then for \( 0 < x \leq a \) and any natural number \( n \), the following inequality holds true
\[
\sum_{k=0}^{n} \frac{J^{(k)}(a)}{k!} (x - a)^k \leq J(x) \leq \sum_{k=0}^{n-1} \frac{J^{(k)}(a)}{k!} (x - a)^k + \left( \frac{1}{a} \right)^n \left( 1 - \sum_{k=0}^{n-1} \frac{(-a)^k J^{(k)}(a)}{k!} \right) (x - a)^n. \tag{18}
\]

**Proof.** Let \( f(x) = \ln(1 + x), x \in [0, a] \) and \( J(x) = \frac{\ln(1+x)}{x}, x \in (0, a] \). Then one has
\[
f(0) = 0, \quad f'(0) = 1, \quad f^{(n+1)}(x) = (-1)^n n!(1 + x)^{-n-1}.
\]

In view of the fact that \( f^{(n+1)}(x) = (-1)^n n!(1 + x)^{-n-1} \) is increasing on \([0, a]\) for \( n = 1, 3, 5, \ldots \), and \( f^{(n+1)}(x) = (-1)^n n!(1 + x)^{-n-1} \) is decreasing on \([0, a]\) for \( n = 2, 4, 6, \ldots \), applying Theorem 1 yields immediately the inequality (18).

Taking \( n = 2 \) in Proposition 3, we get the following inequality:

**Corollary 3.** If \( 0 < x \leq a \), then
\[
\frac{\ln(1 + a)}{a} - \left( \frac{\ln(1 + a)}{a^2} - \frac{1}{a(1 + a)} \right) (x - a) + \left( \frac{\ln(1 + a)}{a^3} - \frac{2 + 3a}{2a^2(1 + a)^2} \right) (x - a)^2 \leq \frac{\ln(1 + x)}{x} \leq \frac{\ln(1 + a)}{a} - \left( \frac{\ln(1 + a)}{a^2} - \frac{1}{a(1 + a)} \right) (x - a)
\]
\[
- \left( \frac{2 \ln(1 + a)}{a^3} - \frac{2 + a}{a^2(1 + a)} \right) (x - a)^2. \tag{19}
\]

**Remark 2.** Integrating both sides of inequality (19) over \([0, a]\), an estimate for \( \int_0^a \frac{\ln(1+x)}{x} \, dx \) is derived as follows:

**Corollary 4.** Let \( a > 0 \), then
\[
\frac{(6 + 5a) \ln(1 + a)}{6a} - \frac{5a + 6a^2}{6(1 + a)^2} < \int_0^a \frac{\ln(1+x)}{x} \, dx < \frac{(6 - a) \ln(1 + a)}{6a} + \frac{a + 2a^2}{6(1 + a)^2}. \tag{20}
\]
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