Sharpening Jordan’s inequality and Yang Le inequality, II

Ling Zhu

Department of Mathematics, Zhejiang Gongshang University, Hangzhou 310035, China

Received 30 July 2005; received in revised form 7 November 2005; accepted 10 November 2005

Abstract

In this work, two refined forms of Jordan’s inequality:

(a) \[
\frac{2}{\pi} + \frac{1}{\pi^3} (\pi^2 - 4x^2) + \frac{12 - \pi^2}{16\pi^5} (\pi^2 - 4x^2)^2 \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{1}{\pi^3} (\pi^2 - 4x^2) + \frac{\pi - 3}{\pi^5} (\pi^2 - 4x^2)^2
\]

and

(b) \[
\frac{2}{\pi} + \frac{1}{\pi^3} (\pi^2 - 4x^2) + \frac{4(\pi - 3)}{\pi^3} \left(x - \frac{\pi}{2}\right)^2 \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{1}{\pi^3} (\pi^2 - 4x^2) + \frac{12 - \pi^2}{\pi^5} \left(x - \frac{\pi}{2}\right)^2
\]

are established, where \(x \in (0, \pi/2]\). The applications of the two results above give some new improvement of the Yang Le inequality.

\(\copyright\) 2005 Elsevier Ltd. All rights reserved.

Keywords: Lower and upper bounds; Jordan’s inequality; Yang Le inequality

1. Introduction

The following Theorem is known as Jordan’s inequality [1]:

**Theorem 1.** If \(0 < x \leq \pi/2\), then

\[
\frac{2}{\pi} \leq \frac{\sin x}{x} < 1
\]

with equality if and only if \(x = \pi/2\).

Debnath and Zhao [2] have obtained a new lower bound for the function \(\frac{\sin x}{x}\). Their result reads as follows

\[E-mail address: zhuling0571@163.com.\]
Theorem 2. If \( 0 < x \leq \pi/2 \), then
\[
\frac{\sin x}{x} \geq \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2)
\] (2)
with the equality if and only if \( x = \pi/2 \).

Recently, the author of this work [3] obtained a further result:

Theorem 3. If \( 0 < x \leq \pi/2 \), then
\[
\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{2}{\pi^3}(\pi^2 - 4x^2)
\] (3)
with the equalities if and only if \( x = \pi/2 \). Furthermore, \( \frac{1}{\pi^3} \) and \( \frac{2}{\pi^3} \) are the best constants in (3).

Ozban [4] has given the following fresh inequality.

Theorem 4. If \( 0 < x \leq \pi/2 \), then
\[
\frac{\sin x}{x} \geq \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{4(\pi - 3)}{\pi^3}\left(\frac{x}{\pi} - \frac{1}{2}\right)^2
\] (4)
with the equality if and only if \( x = \pi/2 \).

In this work, we show some new lower and upper bounds for the function \( \frac{\sin x}{x} \) and obtain the following two important results.

Theorem 5. If \( 0 < x \leq \pi/2 \), then
\[
\frac{2}{\pi} + \frac{1}{\pi^3}\left(\pi^2 - 4x^2\right) + \frac{12 - \pi^2}{16\pi^5}\left(\pi^2 - 4x^2\right)^2 \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{\pi - 3}{\pi^5}(\pi^2 - 4x^2)^2
\] (5)
with the equalities if and only if \( x = \pi/2 \). Furthermore, \( \frac{12 - \pi^2}{16\pi^5} \) and \( \frac{\pi - 3}{\pi^5} \) are the best constants in (5).

Theorem 6. If \( 0 < x \leq \pi/2 \), then
\[
\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{4(\pi - 3)}{\pi^3}\left(\frac{x}{\pi} - \frac{1}{2}\right)^2 \leq \frac{\sin x}{x} \leq \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{12 - \pi^2}{\pi^3}\left(\frac{x}{\pi} - \frac{1}{2}\right)^2
\] (6)
with the equalities if and only if \( x = \pi/2 \). Furthermore, \( \frac{4(\pi - 3)}{\pi^3} \) and \( \frac{12 - \pi^2}{\pi^3} \) are the best constants in (6).

In [2] the authors have obtained an improvement of the Yang Le inequality.

Theorem 7. Let \( A_i > 0 \) \((i = 1, \ldots, n)\) with \( \sum_{i=1}^{n} A_i \leq \pi \), let \( 0 \leq \lambda \leq 1 \), \( \delta \equiv (n-1)\sum_{i=1}^{n} \cos^2 \lambda A_i - 2\cos \lambda \pi \sum_{1 \leq i < j \leq n} \cos \lambda A_i \cos \lambda A_j \), and \( n \geq 2 \) be a natural number. Then
\[
N_1(\lambda) \leq N_2(\lambda) \leq \delta \leq M_1(\lambda),
\] (7)
where \( N_1(\lambda) = 4 \left(\frac{n}{2}\right) \lambda^2 \cos^2 \frac{\lambda}{2} \), \( M_1(\lambda) = \left(\frac{n}{2}\right) \lambda^2 \pi^2 \) and \( N_2(\lambda) = \left(\frac{n}{2}\right) \lambda^2 (3 - \frac{\lambda^2}{2}) \cos^2 \frac{\lambda}{2} \pi \).

Refs. [3] and [4] obtained the following two new results using the right inequality of (3) and the inequality in (4) respectively.

Theorem 8. Let \( A_i > 0 \) \((i = 1, \ldots, n)\) with \( \sum_{i=1}^{n} A_i \leq \pi \), let \( 0 \leq \lambda \leq 1 \), \( \delta \equiv (n-1)\sum_{i=1}^{n} \cos^2 \lambda A_i - 2\cos \lambda \pi \sum_{1 \leq i < j \leq n} \cos \lambda A_i \cos \lambda A_j \), and \( n \geq 2 \) be a natural number. Then
\[
N_3(\lambda) \leq \delta \leq M_2(\lambda),
\] (8)
where \( M_2(\lambda) = 4 \left(\frac{n}{2}\right) \lambda^3 + \frac{\lambda(1-\lambda^2)}{2} \pi^2 \) and
\[
N_3(\lambda) = \left(\frac{n}{2}\right) \lambda^2 [3 - \lambda^2 + (\pi - 3)(1 - \lambda^2)] \pi^2 = \left(\frac{n}{2}\right) \lambda^2 [\pi + (6 - 2\pi)\lambda + (\pi - 4)\lambda^2] \cos^2 \frac{\lambda}{2} \pi.
\]
Using inequalities in Theorem 5 and the right inequality of (6), we shall establish and prove the following two improvements of the Yang Le inequality.

**Theorem 9.** Let \( A_i > 0 \) \((i = 1, 2, \ldots, n)\) with \( \sum_{i=1}^{n} A_i \leq \pi \), let \( 0 \leq \lambda \leq 1 \), \( \delta \equiv (n-1) \sum_{k=1}^{n} \cos^2 \lambda A_k - 2 \cos \lambda \pi \sum_{1 \leq i < j \leq n} \cos \lambda A_i \cos \lambda A_j \), and \( n \geq 2 \) be a natural number. Then

\[
N_3(\lambda) \leq \delta \leq \min(M_3(\lambda), M'_3(\lambda)),
\]

where \( N_3(\lambda) = \left( \frac{n}{2} \right) \lambda^2 \left[ 3 - \lambda^2 + \frac{12 - \pi^2}{16} (1 - \lambda^2)^2 \right] \cos^2 \frac{\lambda \pi}{2}, \) and

\[
M_3(\lambda) = \left( \frac{n}{2} \right) \lambda^2 \left[ 3 - \lambda^2 + (\pi - 3)(1 - \lambda^2)^2 \right], \quad M'_3(\lambda) = \left( \frac{n}{2} \right) \lambda^2 \left[ 3 - \lambda^2 + \frac{12 - \pi^2}{4} (1 - \lambda^2)^2 \right].
\]

2. A lemma

**Lemma 1** ([6,7]). Let \( f, g : [a, b] \rightarrow R \) be two continuous functions which are differentiable on \((a, b)\). Further, let \( g' \neq 0 \) on \((a, b)\). If \( f' / g' \) is increasing (or decreasing) on \((a, b)\), then the functions

\[
\frac{f(x) - f(b)}{g(x) - g(b)}
\]

and

\[
\frac{f(x) - f(a)}{g(x) - g(a)}
\]

are also increasing (or decreasing) on \((a, b)\).

3. A short proof of Theorem 5

Let \( f_1(x) = \frac{\sin x}{x} - \frac{\pi^2 - 4x^2}{\pi^2}, \) \( f_2(x) = (\pi^2 - 4x^2)^2, \) \( f_3(x) = -\frac{x \cos x - \sin x}{8x^3} - \frac{1}{\pi^2}, \) \( f_4(x) = 2(\pi^2 - 4x^2); \)

\( f_5(x) = 3(\sin x - x \cos x) - x^2 \sin x, \) \( f_6(x) = 64x^5, \) \( f_7(x) = \sin x - x \cos x, \) \( f_8(x) = 320x^3 \) and \( x \in (0, \pi/2)\). Then we have

\[
\frac{f'_1(x)}{f_1(x)} = \frac{f_3(x)}{f_4(x)}, \quad \frac{f'_2(x)}{f_2(x)} = \frac{f_3(x)}{f_4(x)},
\]

\[
\frac{f'_3(x)}{f_3(x)} = \frac{f_5(x)}{f_6(x)}, \quad \frac{f'_4(x)}{f_4(x)} = \frac{f_7(x)}{f_8(x)}.
\]

We find that \( \frac{f_3(x)}{f_1(x)} = \frac{\sin x}{960x} \) is decreasing on \((0, \pi/2)\), so \( \frac{f'_1(x)}{f_1(x)} = \frac{f_3(x)}{f_8(x)} = \frac{f_3(x) - f_3(0)}{f_8(x) - f_8(0)} \) is decreasing on \((0, \pi/2)\) by Lemma 1, and \( \frac{f_3(x)}{f_4(x)} = \frac{f_3(x) - f_3(0)}{f_6(x) - f_6(0)} \) is decreasing on \((0, \pi/2)\) by Lemma 1. Then

\[
\frac{f_3(x)}{f_4(x)} = \frac{f_3(x) - f_3(0)}{f_6(x) - f_6(0)} \]

is decreasing on \((0, \pi/2)\) by Lemma 1. This leads to that \( \frac{f'_1(x)}{f_1(x)} \) is decreasing on \((0, \pi/2)\), and \( f(x) = \frac{\sin x - \frac{\pi}{2} \cdot \frac{x^2 - 4x^2}{\pi^2}}{(\pi^2 - 4x^2)^2} = \frac{f_1(x) - f_1(0)}{f_2(x) - f_2(0)} \) is decreasing on \((0, \pi/2)\) by Lemma 1.

Furthermore, \( \lim_{x \to 0^+} f(x) = -\frac{\pi^2}{8} \) and \( \lim_{x \to \pi/2} f(x) = \frac{12 - \pi^2}{16\pi^2} \), thus \( \frac{12 - \pi^2}{16\pi^2} \) and \( \frac{\pi^2}{8} \) are the best constants in (5).
4. A concise proof of Theorem 6

Let \( g_1(x) = \frac{\sin x}{x} - \frac{2}{\pi} - \frac{\pi^2 - 4x^2}{\pi^2} \), \( g_2(x) = (x - \frac{\pi}{2})^2 \), \( g_3(x) = \frac{8}{\pi}x^3 + x \cos x - \sin x \), \( g_4(x) = x^3 - \frac{\pi}{2}x^2 \), and \( x \in (0, \pi/2) \). Then we have

\[
g'_{ij} = \frac{\frac{8}{\pi}x^3 + x \cos x - \sin x}{g_2(x)} = \frac{1}{2} g_3(x),
\]

\[
k(x) = \frac{g'_2(x)}{g'_4(x)} = \frac{\frac{24}{\pi}x - \sin x}{\frac{8}{\pi}x - \sin x} = \frac{8}{\pi} + \frac{8}{\pi} - \sin x.
\]

We simply compute

\[
k'(x) = \frac{u(x)}{(3x - \pi)^2},
\]

where \( u(x) = 3 \sin x + \pi \cos x - 3x \cos x - \frac{24}{\pi} \) and \( u'(x) = 3(x - \frac{\pi}{2}) \sin x \) for \( x \in (0, \frac{\pi}{2}) \). Then we have: when \( x \in (0, \pi/3) \), \( u'(x) < 0 \), so \( u(x) \) is decreasing on \((0, \pi/3)\); when \( x \in (\pi/3, \pi/2) \), \( u'(x) > 0 \), so \( u(x) \) is increasing on \((\pi/3, \pi/2)\). That is, \( u(\pi/3) = 3\sqrt{3}/2 - 24/\pi^2 > 0 \) is the minimum of the function \( u(x) \) on \((0, \pi/2)\). Therefore \( u(x) > 0 \), \( k'(x) > 0 \) hold on \((0, \pi/2)\) or \( g'_i(x) \) is increasing on \((0, \pi/2)\). Then \( g'_i(x) \) is increasing on \((0, \pi/2)\) by Lemma 1. This leads to that \( g(x) = \frac{\sin x - \frac{2}{\pi}x}{(x - \frac{\pi}{2})^2} = \frac{g_i(x) - g_i(\pi/2)}{g_i'(x) - g_i'(\pi/2)} \) is increasing on \((0, \pi/2)\) by Lemma 1.

Furthermore, \( \lim_{x \to 0^+} g(x) = \frac{4(\pi - 3)}{\pi^3} \) and \( \lim_{x \to 0^-} g(x) = \frac{12 - \pi^2}{\pi^3} \), thus \( \frac{4(\pi - 3)}{\pi^3} \) and \( \frac{12 - \pi^2}{\pi^3} \) are the best constants in (6).

5. The proof of Theorem 9

Let \( H_{ij} = \cos^2 \lambda A_i + \cos^2 \lambda A_j - 2 \cos \lambda \pi \cos \lambda A_i \cos \lambda A_j \). It follows from [5] that

\[
\sin^2 \lambda \pi \leq H_{ij} \leq 4 \sin^2 \frac{\lambda}{2} \pi.
\]

Let \( 1 \leq i < j \leq n \). Taking the sum for all inequalities in (10), we obtain

\[
\sum_{1 \leq i < j \leq n} \sin^2 \lambda \pi \leq \sum_{1 \leq i < j \leq n} H_{ij} \leq \sum_{1 \leq i < j \leq n} 4 \sin^2 \frac{\lambda}{2} \pi.
\]

It follows from the definition of \( H_{ij} \) that

\[
\sum_{1 \leq i < j \leq n} H_{ij} = \sum_{1 \leq i < j \leq n} (\cos^2 \lambda A_i + \cos^2 \lambda A_j - 2 \cos \lambda \pi \cos \lambda A_i \cos \lambda A_j)
\]

\[
= (n - 1) \sum_{k=1}^{n} \cos^2 \lambda A_k - 2 \cos \lambda \pi \sum_{1 \leq i \leq j \leq n} \cos \lambda A_i \cos \lambda A_j \equiv \delta.
\]

Making use of the inequalities in (5) we obtain

\[
\sum_{1 \leq i < j \leq n} 4 \sin^2 \frac{\lambda}{2} \pi \leq \sum_{1 \leq i < j \leq n} \left( \frac{\lambda}{2} \pi \right)^2 \left[ \frac{2}{\pi} + \frac{1 - \lambda^2}{\pi} + \frac{(\pi - 3)(1 - \lambda^2)^2}{\pi} \right]^2
\]

\[
= \left( \begin{array}{c} n \\ 2 \end{array} \right) \lambda^2 [3 - \lambda^2 + (\pi - 3)(1 - \lambda^2)^2]
\]

and
\[
\sum_{1 \leq i < j \leq n} \sin^2 \lambda \pi = 4 \cos^2 \frac{\lambda}{2} \pi \sum_{1 \leq i < j \leq n} \sin^2 \frac{\lambda}{2} \pi \\
\geq 4 \cos^2 \frac{\lambda}{2} \pi \sum_{1 \leq i < j \leq n} \left( \frac{\lambda}{2} \pi \right)^2 \left[ \frac{2}{\pi} + \frac{1 - \lambda^2}{\pi} + \frac{12 - \pi^2}{16\pi}(1 - \lambda^2)^2 \right]^2 \\
= \left( \frac{n}{2} \right) \lambda^2 \left[ 3 - \lambda^2 + \frac{12 - \pi^2}{16}(1 - \lambda^2)^2 \right]^2 \cos^2 \frac{\lambda}{2} \pi. 
\]

Substituting (13) and (14) into (11), we obtain
\[ N_3(\lambda) \leq \delta \leq M_3(\lambda). \]
At the same time, making use of (11), (12), and the right inequality of (6) we can obtain \[ \delta \leq M'_3(\lambda). \]
The proof of Theorem 9 is complete. \( \square \)

References