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Asymptotic formulas for the diameter of sections of symmetric convex bodies

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Abstract

Sharpening work of the first two authors, for every proportion $\lambda \in (0,1)$ we provide exact quantitative relations between global parameters of n-dimensional symmetric convex bodies and the diameter of their random $\lfloor \lambda n \rfloor$-dimensional sections. Using recent results of Gromov and Vershynin, we obtain an "asymptotic formula" for the diameter of random proportional sections.
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1. Introduction

One of the most important recent developments in asymptotic convex geometry has been the gradual recognition of the fact that lower-dimensional sections and projections

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of high-dimensional convex bodies exhibit an unexpectedly uniform structure. Several questions regarding the asymptotic behaviour of convex bodies can be answered through very precise estimates which depend only on a few "simple parameters" and are exact for every sequence of convex bodies of increasing dimension. We call such exact estimates "asymptotic formulas".

The aim of this article is to provide such asymptotic formulas for the diameter of a random $\lfloor \lambda n \rfloor$-dimensional central section of a symmetric convex body K in $\mathbb{R}^n$, where the proportion $\lambda \in (0,1)$ is arbitrary but fixed and the dimension n tends to infinity. We continue a line of thought which was initiated by the first two authors in [4–6].

In order to give a precise formulation of the problems, we need to introduce some notation. We work on $\mathbb{R}^n$ which is equipped with a Euclidean structure and write $|\cdot|$ for the corresponding Euclidean norm. The Euclidean unit ball and sphere are denoted by B_2^n and S^{n-1}, respectively. We write σ_n for the rotationally invariant probability measure on S^{n-1} and μ_n for the Haar probability measure on $O(n)$. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\nu_{n,k}$. Every symmetric convex body K in $\mathbb{R}^n$ induces the norm $\|x\|_K = \inf\{t > 0 : x \in tK\}$. The polar body $\left\{y \in \mathbb{R}^n : \max_{x\in K} |\langle y, x\rangle| \leqslant 1\right\}$ of K is denoted by K°. We define

$$M(K) = \int_{S^{n-1}} \|x\|_K \,\sigma_n(dx) \quad \text{and} \quad M^*(K) = \int_{S^{n-1}} \max_{y\in K} |\langle x, y\rangle| \,\sigma_n(dx). \tag{1.1}$$

So, $M = M(K)$ is the average of the norm of K on the sphere and $M^* = M^*(K)$ is the mean width of K (in the classical terminology of convexity, the mean width $w(K)$ of K is equal to $2M^*(K)$). Note that $M^* = M(K^\circ)$. We also define a and b as the least positive constants for which $(1/a)|x| \leqslant \|x\|_K \leqslant b|x|$ holds true for every $x \in \mathbb{R}^n$. Thus, a is the circumradius of K—also denoted by $D(K)$—and $1/b$ is the inradius of K—also denoted by $d(K)$.

The approach of [4] was based on the second author's "M^*-estimate" (see [8,9,16,2]) which compares the diameter of proportional sections of a symmetric convex body K in $\mathbb{R}^n$ to its mean width $M^*(K)$. A precise quantitative form of this inequality can be found in [2]: Let K be a symmetric convex body in $\mathbb{R}^n$ and let $\lambda, \varepsilon \in (0,1)$. Then,

$$D(K \cap E) \leqslant \frac{M^*(K)}{(1-\varepsilon)\sqrt{1-\lambda}} \tag{1.2}$$

for all E in a subset $A_{n,k}$ of $G_{n,k}$ of almost full measure, where $k = \lfloor \lambda n \rfloor$ (the proof of (1.2) is based on a more general result of Gordon which will be discussed in Section 2; see Lemma 2.7). A direct consequence of the M^*-estimate is the following (see [4]):

Theorem A (*upper bound for the diameter*). *Let $\varepsilon, \lambda \in (0,1)$. If K is a symmetric convex body in $\mathbb{R}^n$, and if r_1 is the solution of the equation*

$$M^*(K \cap rB_2^n) = (1-\varepsilon)\sqrt{1-\lambda}\, r, \tag{1.3}$$

then $D(K \cap E) \leqslant r_1$ *for all subspaces* E *in a subset* $\mathcal{A}(\lambda)$ *of* $G_{n,\lfloor \lambda n \rfloor}$ *with measure* $\nu_{n,\lfloor \lambda n \rfloor}(\mathcal{A}(\lambda)) \geqslant 1 - c_1 \exp(-c_2 \varepsilon^2 (1-\lambda) n)$, *where* $c_1, c_2 > 0$ *are absolute constants.*

In other words, solving the equation $M^*(K \cap r B_2^n) \asymp \sqrt{1-\lambda}\, r$, we get an upper bound for the diameter of a random $\lfloor \lambda n \rfloor$-dimensional section of K. The main idea in [4] was to see if an analogous (or even the same) equation can be used for a lower bound as well.

The main new ingredient was a "conditional M-estimate": Let K be a symmetric convex body in $\mathbb{R}^n$ with $B_2^n \subseteq K$ and let $\lambda \in (0,1)$. If $M(K) \geqslant 1 - c^{\frac{1}{1-\lambda}}$, then there exists a subset $\mathcal{B}(\lambda)$ of $G_{n,k}$ with $\nu_{n,k}(\mathcal{B}(\lambda)) \geqslant 1 - c^k$, where $k = \lfloor \lambda n \rfloor$, such that $D(K \cap E) \leqslant C^{\frac{\lambda}{1-\lambda}}$ for all $E \in \mathcal{B}(\lambda)$, where $0 < c < 1$ and $C > 1$ are absolute constants, and n is large enough. In Section 2 we give two different arguments which provide better estimates. The first argument uses the M^*-estimate and the second author's "distance lemma"; the second one is based on Gordon's work (see Lemma 2.7) and was kindly communicated to us by R. Vershynin.

Theorem B (*low* **M**-*estimate*). *Let* $\lambda \in (0,1)$ *and let* K *be a symmetric convex body in* $\mathbb{R}^n$ *with* $B_2^n \subseteq K$. *Assume that*

$$M(K) > \sqrt{\lambda} \tag{1.4}$$

and set $\delta = (M^2 - \lambda)/(1 - M^2)$. *Then, a* $\lfloor \lambda n \rfloor$-*dimensional central section* $K \cap E$ *of* K *satisfies*

$$D(K \cap E) \leqslant \frac{c\sqrt{1-\lambda}}{M - \sqrt{\lambda}} \tag{1.5}$$

with probability greater than $1 - c_1 \exp(-c_2 \delta^2 (1-\lambda) n)$, *where* $c, c_1, c_2 > 0$ *are absolute constants.*

This follows from Theorem 2.3, where the following estimates are proved for a random $E \in G_{n,\lfloor \lambda n \rfloor}$:

(i) If $M^2 < \frac{1}{2}$, then $D(K \cap E) \leqslant \frac{cM}{M^2 - \lambda}$.

(ii) If $M^2 \geqslant \frac{1}{2}$, then $D(K \cap E) \leqslant \frac{c\sqrt{1-\lambda}}{M^2 - \lambda}$.

By Dvoretzky's theorem, there exists an absolute constant $c \in (0,1)$ such that if $B_2^n \subseteq K$ then a random $\lfloor c M^2 n \rfloor$-dimensional section $K \cap E$ of K satisfies $\frac{1}{2M} B_2^n \cap E \subseteq K \cap E \subseteq \frac{2}{M} B_2^n \cap E$. The Low M-estimate above provides an isomorphic version of this fact for all dimensions up to the natural bound $k_* =: M^2 n$. After this paper was written, Litvak noted that, in fact, analogous estimates can be recovered from [2]. As Remark 2.9 shows, under additional conditions, modifications of our first method of proof may give information for dimensions greater than k_*.

An interesting application is given in Section 3, where we improve substantially the estimates from [5] on a question about the comparison of local to global parameters of symmetric convex bodies.

Theorem C. *Let $\rho > 0$, let $t \geqslant 2$ be an integer and let $n \geqslant 2(t+1)$. For every symmetric convex body K in $\mathbb{R}^n$, if there exist orthogonal transformations $u_1, \dots, u_t$ such that $u_1(K) \cap \cdots \cap u_t(K) \subseteq \rho B_2^n$ then a random $\lfloor \frac{n}{c_1 t} \rfloor$-dimensional section $K \cap E$ of K satisfies $D(K \cap E) \leqslant c_2 \sqrt{t} \rho$, where $c_1, c_2 > 0$ are absolute constants.*

A qualitative version of the results in [4] reads as follows: There exist two explicit functions $h_1, h_2 : (0,1) \to (0,1)$ such that for every $\lambda \in (\frac{1}{2}, 1)$ and every symmetric convex body K in $\mathbb{R}^n$, the solutions r_i of the equations $M^*(K \cap r B_2^n) = h_i(\lambda) r$ in r $(i = 1, 2)$ determine a confidence interval for the diameter of a random $\lfloor \lambda n \rfloor$-dimensional section of K. The important point is that the functions h_1 and h_2 are universal and that the statement holds true for an arbitrary symmetric convex body K. Another advantage of this statement is that it makes use of the global (hence computationally simple) parameter M^* of the body. The estimates in [4] are not tight and a main disadvantage of the method is the use of Borsuk's theorem, which forces one to study only proportions $\lambda \in (\frac{1}{2}, 1)$. The method of [4] gives no information for small proportions.

In the last two sections we show that the upper estimates given by Theorem A can be complemented by lower estimates for every proportion $\lambda \in (0,1)$: the "equation" $M^*(K \cap r B_2^n) \asymp \sqrt{\frac{2(1-\lambda)}{2-\lambda}} r$ is enough for a lower bound. The main new tool is a recent isoperimetric theorem of Gromov [3]: Assume that $k < n$ are positive integers, n is even and $n - k = 2^m - 1$. For every $\theta > 0$, among all odd continuous functions $f : S^{k-1} \to S^{n-1}$, the θ-extension of the image $f(S^{k-1})$ in S^{n-1} has minimal measure if f is the identity function. Using an application of this result by Vershynin [18], together with precise concentration estimates of Artstein [1], we are able to prove the following.

Theorem D (*lower bound for the diameter*). *Suppose that $\lambda \in (0,1)$ and $\varepsilon > 0$ satisfy $(1+\varepsilon)\sqrt{\frac{2(1-\lambda)}{2-\lambda}} < 1$ and let $n \geqslant n_0(\lambda, \varepsilon) \simeq \frac{1}{(1-\lambda)\varepsilon^2}$. If K is a symmetric convex body in $\mathbb{R}^n$, and if r_2 is the solution of the equation*

$$M^*(K \cap r B_2^n) = (1+\varepsilon)\sqrt{\frac{2(1-\lambda)}{2-\lambda}}\, r, \tag{1.6}$$

then

$$D(K \cap E) \geqslant \frac{\varepsilon\sqrt{1-\lambda}}{3} r_2 \tag{1.7}$$

for every $E \in G_{n,\lfloor \lambda n \rfloor}$.

It should be emphasized that the conclusion of Theorem D holds for every (and not for a random) $E \in G_{n,\lfloor \lambda n \rfloor}$. A striking application of this fact follows by comparison with Theorem A: roughly speaking, for every fixed proportion $\mu \in (0, 1)$ and every $0 < s < 1/(2-\mu)$, the minimal diameter of $\lfloor \mu n \rfloor$-dimensional sections and the random diameter of $\lfloor s\mu n \rfloor$-dimensional sections are comparable up to a constant depending on μ and s. An analogous result is observed by Vershynin [19]. To state the theorem, for every symmetric convex body K in $\mathbb{R}^n$, let $a(\lambda, K)$ denote the minimal (and let $b(\lambda, K)$ denote the "random") circumradius of a $\lfloor \lambda n \rfloor$-dimensional section of K (the precise definitions are given in Section 5).

Theorem E. *Let $0 < \mu < 1$ and $0 < s < 1/(2-\mu)$. There exists $n_0 = n_0(\mu, s)$ such that*

$$\left(\frac{c\mu(1 - s(2-\mu))}{1 - s\mu} \sqrt{1-\mu} \right) b(s\mu, K) \leqslant a(\mu, K) \tag{1.8}$$

for every $n \geqslant n_0$ and every symmetric convex body K in $\mathbb{R}^n$.

Quantitative statements showing that existence implies randomness are still rare in the theory and should have interesting applications. The fact that the smallest and the "random" number of rotations of a convex body whose intersection approximates the Euclidean ball are of the same order (see [13,7]) is such an example. In the local theory, a result of this type appears in [14]: In the language of Theorem E, Proposition 3.2 in [14] states that if most $s\mu n$-dimensional sections of some μn-dimensional projection of a symmetric convex body K have diameter bounded by 1 then most $t\mu n$-dimensional sections of the whole body have diameter bounded by $f(\mu, s, t)$, where $t < s$ and $\mu, s, t \in (0, 1)$.

Note: It is not known whether Gromov's theorem holds true for all positive integers $k < n$. If so, then Theorems D and E would take an optimal form (the precise formulations of the corresponding two conditional statements are given at the end of the paper—see Remark 5.7).

We refer the reader to the books [12,15,17] for notation and background information on asymptotic convex geometry; in particular, the letters c, C, c_1, c_2 etc. denote absolute positive constants which may change from line to line.

2. Low M-estimate

In this section, we give two arguments which prove Theorem B. The first one uses the M^*-estimate and the second author's "distance lemma" (a similar technique was used in [6] in a different setting). The second one was communicated to us by R. Vershynin and is reproduced here with his very kind permission.

First approach (*Distance lemma*). The distance lemma shows that the geometric distance from a symmetric convex body to the Euclidean ball can be estimated if the parameters M and M^* are comparable to $1/b$ and a, respectively.

Lemma 2.1 (*Milman [10]*). *Let T be a symmetric convex body in $\mathbb{R}^n$ with $\rho B_2^n \subseteq T \subseteq rB_2^n$. Assume that*

$$(M^*(T)/r)^2 + (M(T)\rho)^2 = 1 + \kappa \tag{2.1}$$

for some $\kappa > 0$. Then,

$$\frac{r}{\rho} \leqslant \frac{1}{\kappa}. \tag{2.2}$$

If in addition

$$(M^*(T)/r)^2 + \beta(M(T)\rho)^2 \geqslant 1 \tag{2.3}$$

for some constant $\beta \in (0, 1)$, then

$$\frac{r}{\rho} \leqslant \frac{\sqrt{1-\beta}}{1-\sqrt{\beta}} \frac{1}{\sqrt{\kappa}}. \tag{2.4}$$

Combining with the M^*-estimate we get the following technical statement.

Proposition 2.2. *Let $\lambda \in (0, 1)$ and let K be a symmetric convex body in $\mathbb{R}^n$. For every $\delta > 0$ we define r to be the solution of the equation*

$$M^*(K \cap rB_2^n) = \sqrt{\frac{\delta+\lambda}{\delta+1}}\, r. \tag{2.5}$$

Then, for a random $E \in G_{n,\lfloor \lambda n \rfloor}$ and an absolute constant $c > 0$ we have:

(i) *If $0 < \lambda < \frac{1}{2}$ and $0 < \delta < 1 - 2\lambda$, then*

$$D(K^\circ \cap E) \leqslant \frac{c\sqrt{\delta+\lambda}}{\delta} \frac{1}{r}. \tag{2.6}$$

(ii) *If $1 - 2\lambda \leqslant \delta$, then*

$$D(K^\circ \cap E) \leqslant \frac{c}{\sqrt{1-\lambda}} \frac{\delta+1}{\delta} \frac{1}{r}. \tag{2.7}$$

Proof. Let $0 < s < \delta$ be a constant depending on δ which will be suitably chosen. We define $\rho > 0$ by the equation

$$M^*\left(K^\circ \cap \rho^{-1} B_2^n\right) = \sqrt{\frac{1-\lambda}{s+1}} \frac{1}{\rho}. \tag{2.8}$$

Theorem A shows that (with probability greater than $1 - c_1 \exp(-c_2 s^2 (1-\lambda)n)$) a random $E \in G_{n,\lfloor \lambda n \rfloor}$ satisfies

$$D(K^\circ \cap E) \leqslant 1/\rho. \tag{2.9}$$

We may assume that $\rho < r$: if $\rho \geqslant r$ then the result is an immediate consequence of (2.9). We define the convex body $T = \mathrm{co}\big((K \cap rB_2^n) \cup \rho B_2^n\big)$. Since $\rho < r$, we have $\rho B_2^n \subseteq T \subseteq rB_2^n$. Also, by the definition of T we see that $T \supseteq K \cap rB_2^n$ and $T^\circ \supseteq K^\circ \cap \frac{1}{\rho} B_2^n$. Therefore,

$$\begin{aligned}(M^*(T)/r)^2 + (M(T)\rho)^2 &\geqslant \big(M^*(K \cap rB_2^n)/r\big)^2 + \left(M^*\left(K^\circ \cap \rho^{-1} B_2^n\right)\rho\right)^2 \\ &= \frac{\delta+\lambda}{\delta+1} + \frac{1-\lambda}{s+1} \\ &= 1 + \frac{\delta - s}{(\delta+1)(s+1)}(1-\lambda).\end{aligned}$$

We treat the two cases as follows:

(i) We define $\gamma = \frac{s+\lambda}{s+1}\frac{\delta+1}{\delta+\lambda}$. Since $s < \delta$, we have $0 < \gamma < 1$ and

$$\gamma (M^*(T)/r)^2 + (M(T)\rho)^2 \geqslant \frac{s+\lambda}{s+1} + \frac{1-\lambda}{s+1} = 1. \tag{2.10}$$

Applying the distance lemma we get

$$\begin{aligned}\frac{1}{\rho} &\leqslant \frac{\left(\sqrt{(\delta+1)(s+\lambda)} + \sqrt{(s+1)(\delta+\lambda)}\right)\sqrt{(\delta+1)(s+1)}}{(\delta - s)(1-\lambda)} \frac{1}{r} \\ &\leqslant \frac{2(\delta+1)^{3/2}\sqrt{\delta+\lambda}}{(\delta-s)(1-\lambda)} \frac{1}{r}.\end{aligned}$$

Choosing $s = \delta/2$ we get (2.6).

(ii) We define $\beta = \frac{s+1}{\delta+1}$. Since $s < \delta$ we have $0 < \beta < 1$ and

$$(M^*(T)/r)^2 + \beta (M(T)\rho)^2 \geqslant \frac{\delta+\lambda}{\delta+1} + \frac{s+1}{\delta+1}\frac{1-\lambda}{s+1} = 1. \tag{2.11}$$

We can then apply the distance lemma to get

$$\frac{1}{\rho} \leqslant \frac{(\sqrt{\delta+1} + \sqrt{s+1})\sqrt{(\delta+1)(s+1)}}{(\delta - s)\sqrt{1-\lambda}} \frac{1}{r} \leqslant \frac{2(\delta+1)}{\delta - s}\frac{\sqrt{s+1}}{\sqrt{1-\lambda}}\frac{1}{r}. \tag{2.12}$$

We now distinguish two subcases: if $\delta < 1$ we choose $s = \delta/2$, and if $\delta \geqslant 1$ we choose $s = 1/2$. Then, (2.12) proves (2.7). $\square$

Proposition 2.2 leads to the following low M-estimate.

Theorem 2.3. *Let $\lambda \in (0,1)$ and let K be a symmetric convex body in $\mathbb{R}^n$ with $B_2^n \subseteq K$. Assume that*

$$M(K) > \sqrt{\lambda}. \tag{2.13}$$

Then, for a random $E \in G_{n,\lfloor \lambda n \rfloor}$ and an absolute constant $c > 0$ we have:

(i) *If $M^2 < \frac{1}{2}$, then*

$$D(K \cap E) \leqslant \frac{cM}{M^2 - \lambda}. \tag{2.14}$$

(ii) *If $M^2 \geqslant \frac{1}{2}$, then*

$$D(K \cap E) \leqslant \frac{c\sqrt{1-\lambda}}{M^2 - \lambda}. \tag{2.15}$$

Proof. If $M = 1$ then $K = B_2^n$ and there is nothing to prove. So, we assume that $M < 1$ and set $\delta = \frac{M^2-\lambda}{1-M^2}$. Since $B_2^n \subseteq K$, we have

$$M^*(K^\circ \cap B_2^n) = M^*(K^\circ) = \sqrt{\frac{\delta+\lambda}{\delta+1}}. \tag{2.16}$$

Consider the following two cases:

(i) If $M^2 < \frac{1}{2}$ then $\delta < 1 - 2\lambda$ (and $\lambda < M^2 < \frac{1}{2}$). Therefore, Proposition 2.2(i) shows that

$$D(K \cap E) \leqslant \frac{c\sqrt{\delta+\lambda}}{\delta} < \frac{cM\sqrt{1-M^2}}{M^2-\lambda} \tag{2.17}$$

for a random $E \in G_{n,\lfloor \lambda n \rfloor}$. This proves (2.14).

(ii) If $M^2 \geqslant \frac{1}{2}$ then $1 - 2\lambda \leqslant \delta$. In this case, Proposition 2.2(ii) shows that

$$D(K \cap E) \leqslant \frac{c}{\sqrt{1-\lambda}} \frac{\delta+1}{\delta} \tag{2.18}$$

for a random $E \in G_{n,\lfloor \lambda n \rfloor}$. Since

$$\frac{\delta+1}{\delta} = \frac{1-\lambda}{M^2-\lambda}, \tag{2.19}$$

this proves (2.15).

Remark 2.4. From the proof of Proposition 2.2 one can check that the results in Theorem 2.3 hold true for all subspaces E in a subset $\mathcal{A}(\lambda)$ of $G_{n,\lfloor \lambda n \rfloor}$ with measure $v_{n,\lfloor \lambda n \rfloor}(\mathcal{A}(\lambda)) \geqslant 1 - c_1 \exp(-c_2\delta^2(1-\lambda)n)$, where $\delta = \frac{M^2-\lambda}{1-M^2}$ and $c_1, c_2 > 0$ are absolute constants.

Remark 2.5. The inequality $M > \sqrt{\lambda}$ is a necessary condition if we want to have such bounds for a random subspace $E \in G_{n,\lfloor \lambda n \rfloor}$. This can be checked by analyzing the example of the cylinder

$$C = \left\{ x \in \mathbb{R}^n : x_1^2 + \cdots + x_k^2 \leqslant 1 \right\},$$

where $k = \lfloor \lambda n \rfloor$. One should emphasize here the relation to Dvoretzky's theorem: for *some* $c \in (0,1)$ and for every symmetric convex body in $\mathbb{R}^n$ with $B_2^n \subseteq K$, a random $\lfloor cM^2 n \rfloor$-dimensional section $K \cap E$ of K satisfies $\frac{1}{2M} B_2^n \cap E \subseteq K \cap E \subseteq \frac{2}{M} B_2^n \cap E$. Theorem 2.3 shows that an isomorphic version of this fact is possible *"for all"* dimensions up to the natural bound $k_* =: M^2 n$.

Theorem 2.3 may be also stated in the following way.

Theorem 2.6. *Let $\alpha > 1$ and let K be a symmetric convex body in $\mathbb{R}^n$ with $B_2^n \subseteq K$. Assume that $M(K) = \sqrt{1-\varepsilon}$ for some $\varepsilon \in (0,1)$ with $\alpha\varepsilon < 1$. If $\varepsilon < 1/2$, then a random $E \in G_{n,\lfloor (1-\alpha\varepsilon)n \rfloor}$ satisfies*

$$D(K \cap E) \leqslant \frac{c\sqrt{\alpha}}{\alpha - 1} \frac{1}{\sqrt{\varepsilon}}, \tag{2.20}$$

where $c > 0$ is an absolute constant. If $\varepsilon \geqslant 1/2$, then a random $E \in G_{n,\lfloor (1-\alpha\varepsilon)n \rfloor}$ satisfies

$$D(K \cap E) \leqslant \frac{c\sqrt{1-\varepsilon}}{\alpha - 1}, \tag{2.21}$$

where $c > 0$ is an absolute constant.

Second approach (*Gaussian processes*). Vershynin's approach to the low M-estimate is based on Gordon's proof of the M^*-estimate. For the precise statement, we need to introduce the sequence

$$a_k = \mathbb{E}\left(\sum_{i=1}^{k} g_i^2\right)^{1/2} = \sqrt{2}\Gamma\left(\frac{k+1}{2}\right)\Big/\Gamma\left(\frac{k}{2}\right),$$

where $g_1, \ldots, g_k$ are independent standard Gaussian random variables on some probability space. It is not hard to check that $k/\sqrt{k+1} < a_k < \sqrt{k}$ (since k will be always assumed large, in what follows we can replace a_k by $\sqrt{k}$ for simplicity of the exposition; slight modifications would take care of the "error"). Theorem A is a consequence of the following very precise result of Gordon (see [2]).

Lemma 2.7 (*Gordon*). *Let S be a closed subset of S^{n-1}. If*

$$w(S) =: \int_{S^{n-1}} \max_{y\in S}\langle x, y\rangle \sigma(dx) < \frac{a_k}{a_n}, \tag{2.22}$$

then

$$\nu_{n,n-k}\left(E \in G_{n,n-k} : E \cap S = \emptyset\right) \geqslant 1 - \frac{7}{2}\exp\left(-\frac{(a_k - a_n w(S))^2}{18}\right). \tag{2.23}$$

We will use this criterion to prove a low M-estimate in the form of Theorem 2.3.

Proposition 2.8. *Let K be a symmetric convex body in $\mathbb{R}^n$ with $B_2^n \subseteq K$. Assume that $0 < \varepsilon < M(K)$ and set $N = M(K) - \varepsilon$. Let $0 < \alpha < N$ and define $S = \alpha K \cap S^{n-1}$. Then,*

$$w(S) =: \int_{S^{n-1}} \max_{y\in S}\langle x, y\rangle \sigma(dx) < \gamma(\alpha, N) + \exp(-c\varepsilon^2 n), \tag{2.24}$$

where $\gamma(\alpha, \beta) = \alpha\beta + \sqrt{(1-\alpha^2)(1-\beta^2)}$ and $c > 0$ is an absolute constant.

Proof. Since $\|\cdot\|$ is a 1-Lipschitz function on S^{n-1}, concentration of measure on the sphere (see [12]) shows that

$$\sigma(x \in S^{n-1} : \|x\| < N) \leqslant \exp(-c\varepsilon^2 n). \tag{2.25}$$

We will prove the following claim:

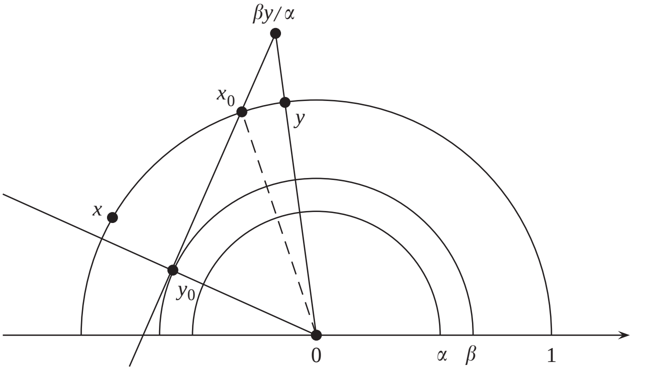
Claim. *If $0 < \alpha < \beta < 1$ and $S = \alpha K \cap S^{n-1}$, then for every $x \in S^{n-1}$ with $\|x\| \geqslant \beta$ we have*

$$\max_{y\in S}\langle x, y\rangle \leqslant \gamma(\alpha, \beta) =: \alpha\beta + \sqrt{(1-\alpha^2)(1-\beta^2)}. \tag{2.26}$$

After this is proved, we can write

$$\begin{aligned} w(S) &= \int_{S^{n-1}} \max_{y\in S}\langle x, y\rangle \sigma(dx) \\ &= \int_{\{x\in S^{n-1}:\|x\|\geqslant N\}} \gamma(\alpha, N)\sigma(dx) + \int_{\{x\in S^{n-1}:\|x\|<N\}} 1\sigma(dx) \\ &< \gamma(\alpha, N) + \exp(-c\varepsilon^2 n), \end{aligned}$$

which is the assertion of Proposition 2.8. $\square$



Proof of the Claim. To this end, assume that $x \in S^{n-1} \setminus \beta K$ and let $y \in S$. We may restrict ourselves to the two-dimensional plane E spanned by x and y. We know that $\beta K \cap E \supseteq \beta B_E$ and $\pm(\beta/\alpha)y \in \beta K \cap E$. Therefore, $x \notin \mathrm{co}\{\beta B_E, \pm(\beta/\alpha)y\}$. Consider the tangent from $(\beta/\alpha)y$ to βB_E. Let x_0 and y_0 be the points where this tangent meets S_E and βS_E, respectively (see the picture above).

Then, the angle $\phi =: \widehat{x0y}$ is greater than or equal to the angle $\phi_0 =: \widehat{x_0 0y}$. From the picture it is clear that $\phi_0 = \psi - \omega$, where $\psi = \widehat{y_0 0y}$ and $\omega = \widehat{y_0 0x_0}$. Since $\cos\psi = \alpha$ and $\cos\omega = \beta$, it follows that $\langle x, y\rangle = \cos\phi \leqslant \cos\phi_0 = \gamma(\alpha, \beta)$. $\square$

Proof of Theorem 2.3 (Continued). As in the first proof of the theorem, we define $\delta > 0$ by the equation $M^2 = \frac{\delta+\lambda}{\delta+1}$. We distinguish three cases.

(a) Assume first that $1 - 2\lambda \leqslant \delta < 1$ (this corresponds to the case $\frac{1}{2} \leqslant M^2 < \frac{1+\lambda}{2}$). Let $\varepsilon = s(1-\lambda)$ and $\eta = \alpha = s\sqrt{\frac{1-\lambda}{\delta+1}}$ where $s \in (0, 1)$ will be chosen. We define $S = \alpha K \cap S^{n-1}$. If $n \geqslant n_0(s, \lambda)$ then $\exp(-c\varepsilon^2 n) < \eta$, and Proposition 2.8 gives

$$w(S) =: \int_{S^{n-1}} \max_{y\in S}\langle x, y\rangle \sigma(dx) < \gamma(\alpha, N) + \eta < \alpha + \sqrt{1-N^2} + \eta, \tag{2.27}$$

where $N = M - \varepsilon$. Since

$$1 - N^2 = 1 - M^2 + \varepsilon(2M - \varepsilon) \leqslant 1 - \frac{\delta + \lambda}{\delta + 1} + 2s(1 - \lambda) = \frac{1 - \lambda}{\delta + 1} + 2s(1 - \lambda), \quad (2.28)$$

we get

$$1 - N^2 < \frac{1 - \lambda}{(\delta/2) + 1} \quad (2.29)$$

if we choose $s \simeq \delta$. Then,

$$w(S) + \eta < (1 + 3s)\sqrt{\frac{1 - \lambda}{(\delta/2) + 1}} < \sqrt{1 - \lambda} \quad (2.30)$$

provided (again) that $s \simeq \delta$. With this choice of s we have $\sqrt{1 - \lambda} - w(S) \geqslant \eta$, and Lemma 2.7 shows that (with probability greater than $1 - c_1 \exp(-c_2 \eta^2 n)$) a random $E \in G_{n, \lfloor \lambda n \rfloor}$ satisfies

$$E \cap \alpha K \cap S^{n-1} = \emptyset. \quad (2.31)$$

This implies easily that

$$D(K \cap E) \leqslant \frac{1}{\alpha} \simeq \frac{1}{\delta \sqrt{1 - \lambda}}. \quad (2.32)$$

(b) Next, assume that $\delta \geqslant 1$ (in this case we have $M^2 \geqslant \frac{1+\lambda}{2}$). We set $\varepsilon = s(1 - \lambda)$, $\eta = \alpha = s\sqrt{1 - \lambda}$ and define $S = \alpha K \cap S^{n-1}$. Then, we repeat the argument in (a). Observe that if s is small enough, we have

$$N^2 > M^2 - 2\varepsilon \geqslant \frac{1 + \lambda}{2} - 2s(1 - \lambda) > \frac{1 + 2\lambda}{3}. \quad (2.33)$$

Therefore,

$$w(S) + \eta < \alpha + \sqrt{1 - N^2} + 2\eta < 3s\sqrt{1 - \lambda} + \sqrt{\tfrac{2}{3}(1 - \lambda)} < \sqrt{1 - \lambda} \quad (2.34)$$

if s is small enough. This shows that

$$D(K \cap E) \leqslant \frac{1}{\alpha} \simeq \frac{1}{\sqrt{1 - \lambda}} \quad (2.35)$$

for a random $E \in G_{n, \lfloor \lambda n \rfloor}$.

By the definition of δ, the upper bounds in (2.32) and (2.35) are both of the order of $\sqrt{1-\lambda}/(M^2-\lambda)$. Thus, cases (a) and (b) prove Theorem 2.3(ii).

(c) Finally, assume that $\delta < 1-2\lambda$ (note that $\lambda < 1/2$ in this case). We now choose $\varepsilon = s(1-\lambda)$, $\alpha = s\sqrt{\frac{\delta+1}{\delta+\lambda}} = s/M$ and $\eta = s$. If $s \leqslant c\delta$ where $c > 0$ is an absolute constant, using (2.29) we get

$$w(S) + \eta < \alpha M + \sqrt{1-N^2} + 2\eta < \sqrt{\frac{1-\lambda}{(\delta/2)+1}} + 3\sqrt{2}s\sqrt{1-\lambda} < \sqrt{1-\lambda}. \quad (2.36)$$

It follows that

$$D(K \cap E) \leqslant \frac{1}{\alpha} \leqslant \frac{c_1\sqrt{\delta+\lambda}}{\delta}. \quad (2.37)$$

Taking into account the definition of δ we see that case (c) proves Theorem 2.3(i). $\square$

Remark 2.9. The second proof of Theorem 2.3 is based on Gordon's approach to Dvoretzky's theorem and to the M^*-estimate. In fact, after this paper was submitted, A. Litvak noted that the estimates of Theorem 2.3 may be also recovered from the methods developed in [2] for all $\lambda < M^2$. However, our first proof of Theorem 2.3 is based on purely geometric tools and could be useful in situations where one needs to consider $\lambda > M^2$. This can be done with a suitable choice of the parameters in Proposition 2.2. For example, assume that $B_2^n \subseteq K$ and $M(K)$ is small. Choose $\lambda = \delta = \alpha M^2 \in (0,1)$ where $\alpha \gg 1$. If $\rho > 0$ satisfies the equation

$$M(\mathrm{co}(K \cup \rho B_2^n)) = \frac{\sqrt{2\alpha}M}{\sqrt{\delta+1}}\frac{1}{\rho}, \quad (2.38)$$

then Proposition 2.2 implies that

$$D(K \cap E) \leqslant \frac{c\rho}{\sqrt{\delta}} \simeq \frac{\rho}{\sqrt{\alpha}M} \quad (2.39)$$

for a random $E \in G_{n,\lfloor \alpha M^2 n \rfloor}$. In cases where the solution ρ of (2.38) can be estimated, one has information on the diameter of proportional sections beyond $\lambda_0 =: M^2$.

3. Diameter of random sections and circumradius of random intersections

Let K be a symmetric convex body in $\mathbb{R}^n$ and let $t, k \geqslant 2$ be two integers. We define the minimal circumradius of an intersection of t rotations of K by

$$r_t(K) = \min\{\rho > 0 : \ u_1(K) \cap \cdots \cap u_t(K) \subseteq \rho B_2^n \text{ for some } u_1, \ldots, u_t \in O(n)\} \quad (3.1)$$

and the “upper radius” of a random $\lceil n/k\rceil$-dimensional central section of K by

$$R_k(K) = \min\left\{R>0 : \nu_{n,\lceil n/k\rceil}(E : K\cap E\subseteq R(B_2^n\cap E))\geqslant 1-\frac{1}{k+1}\right\} \tag{3.2}$$

(where $\lceil x\rceil$ denotes the least integer which is greater than or equal to x). In [11] it is proved that

$$r_{2k}(K)\leqslant\sqrt{k}R_k(K). \tag{3.3}$$

In [5] the following general reverse inequality was proved for fixed integer values of t (starting with $t=2$): For every symmetric convex body K in $\mathbb{R}^n$, where n is large enough depending on t, a random $c^t n$–dimensional section $K\cap E$ of K satisfies $D(K\cap E)\leqslant 20C^t r_t(K)$, where $0<c<1$ and $C>1$ are absolute constants.

Using Proposition 2.2 we are able to obtain sharper estimates in this direction.

Theorem 3.1. *Let $t\geqslant 2$ be an integer and let $n\geqslant 2(t+1)$. For every symmetric convex body K in $\mathbb{R}^n$, a random $\lfloor\frac{n}{c_1 t}\rfloor$-dimensional section $K\cap E$ of K satisfies*

$$D(K\cap E)\leqslant c_2\sqrt{t}r_t(K), \tag{3.4}$$

where $c_1, c_2>0$ are absolute constants.

Proof. Assume that for some body K in $\mathbb{R}^n$ and for some $\rho>0$ there exist rotations $u_1,\dots,u_t\in O(n)$ for which

$$u_1(K)\cap\cdots\cap u_t(K)\subseteq\rho B_2^n.$$

Let k be the least integer for which $\lambda=\frac{k}{n}>\frac{t}{t+1}$. There exists $r>0$ satisfying $M^*(u_j(K)\cap rB_2^n)=\sqrt{(3n+k)/4n}\,r$ for every $j=1,\dots,t$. We can then apply Proposition 2.2(ii) to find subsets $\mathcal{L}_j$ of $G_{n,k}$ with almost full measure (greater than $1-c_1\exp(-c_2(n-k))$) such that

$$[u_j(K)]^\circ\cap E\subseteq\frac{c_1}{r}\sqrt{\frac{n}{n-k}}(B_2^n\cap E) \tag{3.5}$$

for all $E\in\mathcal{L}_j$. Therefore, we can find $\mathcal{L}\subseteq G_{n,k}$ with $\nu_{n,k}(\mathcal{L})>0$ so that (3.5) holds for all $j\leqslant t$ and $E\in\mathcal{L}$. If $E\in\mathcal{L}$, passing to polar bodies we get

$$P_E(u_j(K))\supseteq\sqrt{\frac{n-k}{n}}c_2r(B_2^n\cap E)\quad,\quad j=1,\dots,t. \tag{3.6}$$

Without loss of generality we may assume that K is strictly convex. We then define a map $T : S(E) \to \mathbb{R}^{t(n-k)}$ as follows: Given $\theta \in S(E)$ we find $x_j = a_j\theta \in \mathrm{bd}(P_E(u_j(K)))$, $j = 1, \ldots, t$. Then, we have $x_j = P_E(y_j)$ for a unique point $y_j \in \mathrm{bd}(u_j(K))$. We define

$$T(\theta) = (y_1 - x_1, \ldots, y_t - x_t),$$

where we identify $(E^{\perp})^t$ with $\mathbb{R}^{t(n-k)}$. It is easy to check that T is an odd continuous function on $S(E)$. From the choice of k, we have $t(n-k) < k$. We can then apply Borsuk's antipodal theorem to find $\theta \in S(E)$ with $T(\theta) = 0$. Consider an index $j_0 \leqslant t$ for which $a_{j_0} = |x_{j_0}|$ is minimal. Since $x_{j_0} = y_{j_0}$, we have $x_{j_0} \in u_{j_0}(K) \cap E$, and since $a_{j_0} = \min_{j \leqslant t} a_j$ we see that $x_{j_0} \in u_1(K) \cap \cdots \cap u_t(K) \cap E$.

On the other hand, x_{j_0} is also on the boundary of $P_E(u_{j_0}(K))$, which gives

$$c_2 r\sqrt{\frac{n-k}{n}} \leqslant |x_{j_0}| \leqslant D(u_1(K) \cap \cdots \cap u_t(K) \cap E) \leqslant \rho. \tag{3.7}$$

This gives an upper bound for r in terms of ρ and t:

$$r \leqslant c_3\sqrt{\frac{n}{n-k}}\rho. \tag{3.8}$$

Let s be the least integer for which $(n-s)/n \leqslant \sqrt{(3n+k)/4n}$. We define $\varepsilon \in \mathbb{R}$ (which is easily checked to be in $(0,1)$) so that

$$M^*(K \cap rB_2^n) = (1-\varepsilon)\sqrt{(n-s)/n}\,r = \sqrt{(3n+k)/(4n)}\,r. \tag{3.9}$$

Theorem A implies that there is a subset $\mathcal{L}'$ of $G_{n,s}$ with almost full measure, such that

$$D(K \cap E) \leqslant r \leqslant c_3\sqrt{\frac{n}{n-k}}\rho \tag{3.10}$$

for all $E \in \mathcal{L}'$. It remains to estimate s and $n/(n-k)$ in terms of t. We had $k \leqslant nt/(t+1)+1$, which gives

$$\frac{n}{n-k} \leqslant 2(t+1) \tag{3.11}$$

if we assume $n \geqslant 2(t+1)$. Also, since $(n-s)/n \leqslant \sqrt{(3n+k)/4n}$, we have

$$s = n\frac{(n-k)/4n}{1+\sqrt{(3n+k)/4n}} \geqslant \frac{n}{16(t+1)}. \tag{3.12}$$

This completes the proof of the theorem. $\square$

By the definition of $r_t(K)$ and $R_k(K)$ we may rephrase Theorem 3.1 as follows.

Theorem 3.2. *There exist $c_1, c_2 > 0$ such that for every integer $t \geqslant 2$ and every $n \geqslant 2(t+1)$, the inequality*

$$R_{c_1 t}(K) \leqslant c_2 \sqrt{t}\, r_t(K) \tag{3.13}$$

holds true for every symmetric convex body K in $\mathbb{R}^n$. □

4. New tools

We consider S^{n-1} as a metric probability space, with the geodesic distance ρ and the probability measure σ_n. If $\theta > 0$ and A is a Borel subset of S^{n-1}, then the θ-extension of A is the set $A_\theta = \{x \in S^{n-1} : \rho(x, A) \leqslant \theta\}$. The following isoperimetric theorem of Gromov (see [3]) will be crucial for the results of Section 5.

Theorem 4.1 (*Gromov*). *Assume that $k < n$ are positive integers, n is even and $n - k = 2^m - 1$ for some positive integer m. For every odd continuous function $f : S^{k-1} \to S^{n-1}$ and every $\theta > 0$,*

$$\sigma_n\left(\left[f(S^{k-1})\right]_\theta\right) \geqslant \sigma_{n,k}(\theta), \tag{4.1}$$

where $\sigma_{n,k}(\theta)$ is the measure of the θ-extension of S^{k-1} in *S^{n-1}.*

Vershynin (see [19]) offers a relaxed version of Gromov's theorem for all k and n. This is done by embedding into a higher-dimensional sphere so that Theorem 4.1 can be applied. The embedding is possible because, as shown in [19], for every $\theta > 0$, for every symmetric Borel set $A \subseteq S^{n-1}$ and every $m \geqslant n$, one has $\sigma_n(A_\theta) \geqslant \sigma_m(A_\theta)$, where on the right hand side A is viewed as a subset of S^{m-1} via the natural embedding of S^{n-1} into S^{m-1}.

Proposition 4.2. *Assume that $k < n$ are positive integers. For every odd continuous function $f : S^{k-1} \to S^{n-1}$ and every $\theta > 0$,*

$$\sigma_n\left(\left[f(S^{k-1})\right]_\theta\right) \geqslant \sigma_{2n-k,k-2}(\theta), \tag{4.2}$$

where $\sigma_{m,k}(\theta)$ is the measure of the θ-extension of S^{k-1} in S^{m-1}.

The following lemma of Vershynin (see [18]) makes essential use of Proposition 4.2.

Lemma 4.3. *Let K be a symmetric convex body in $\mathbb{R}^n$ and assume that for some $a < 1 < b$ and some $E \in G_{n,k}$, $k > 2$ we have*

$$aB_2^n \subseteq K \quad \text{and} \quad b(B_2^n \cap E) \subseteq P_E(K). \tag{4.3}$$

Then,

$$\sigma_n(K \cap S^{n-1}) \geqslant \sigma_{2n-k,k-2}(\theta), \tag{4.4}$$

where $\theta = \arcsin(a) - \arcsin(a/b)$.

Proof. [Sketch; Vershynin]. Since $b(B_2^n \cap E) \subseteq P_E(K)$, there exists an odd continuous function $g : bS(E) \to K$. Consider the function $f : S(E) \to S^{n-1}$ defined by $f(x) = g(bx)/|g(bx)|$. We may clearly identify $S(E)$ with S^{k-1}, and hence, Proposition 4.2 shows that $\sigma_n\left(Y_\phi\right) \geqslant \sigma_{2n-k,k-2}(\phi)$ for every $\phi > 0$, where $Y = f(S(E))$. To complete the proof, we observe that

$$K \supseteq \mathrm{co}\{\pm g(bx), aB_2^n\} \supseteq B(f(x), \theta) \tag{4.5}$$

for every $x \in S(E)$, where $\theta = \arcsin(a) - \arcsin(a/b)$. Here, we only need the fact that $|g(bx)| \geqslant b > 1 > a$ and simple trigonometry. $\square$

Remark 4.4. Assume that Gromov's Theorem 4.1 holds true for every pair of positive integers $k < n$. Then, Lemma 4.3 takes a stronger form: under the same hypotheses we have

$$\sigma_n(K \cap S^{n-1}) \geqslant \sigma_{n,k}(\theta), \tag{4.6}$$

where $\theta = \arcsin(a) - \arcsin(a/b)$. In the end of the next section we discuss the consequences of this statement.

The asymptotic behaviour of $\sigma_{n,k}(\theta)$ has been determined by Artstein [1] (see also [19]): Let $\lambda \in (0, 1)$. Then, the following estimates hold as $n \to \infty$.

(1) If $\sin^2 \theta > 1 - \lambda$, then

$$\sigma_{n,k}(\theta) \simeq 1 - \frac{1}{\sqrt{n\pi}} \frac{\sqrt{\lambda(1-\lambda)}}{\sin^2 \theta - (1-\lambda)} e^{\frac{n}{2}u(\lambda,\theta)}.$$

(2) If $\sin^2 \theta < 1 - \lambda$, then

$$\sigma_{n,k}(\theta) \simeq \frac{1}{\sqrt{n\pi}} \frac{\sqrt{\lambda(1-\lambda)}}{(1-\lambda) - \sin^2 \theta} e^{\frac{n}{2}u(\lambda,\theta)},$$

where

$$u(\lambda, \theta) = (1-\lambda)\ln\frac{(1-\lambda)}{\sin^2\theta} + \lambda\ln\frac{\lambda}{\cos^2\theta}. \tag{4.7}$$

In particular, there exists a critical value $\theta(\lambda)$ such that: if $k \geqslant \lambda n$ and $\theta > \theta(\lambda)$ then $\sigma_{n,k}(\theta) \to 1$ as $n \to \infty$. What we really need is the fact that $\theta(\lambda) = \arcsin(\sqrt{1-\lambda})$. This already follows by a simple argument: in [1], it is observed that $\sigma_{n,k}(\theta) = \mathrm{Prob}(Y_n \leqslant \sin^2\theta)$, where Y_n is a random variable with distribution $\mathrm{Beta}\left(\frac{(1-\lambda)n}{2}, \frac{\lambda n}{2}\right)$. Since

$$\mathbb{E}(Y_n) = 1-\lambda \text{ and } \mathrm{Var}(Y_n) = \frac{2\lambda(1-\lambda)}{n+2}, \tag{4.8}$$

a simple application of Chebyshev's inequality shows that

$$\mathrm{Prob}(Y_n > (1-\lambda)+t) \leqslant \frac{\mathrm{Var}(Y_n)}{t^2} \leqslant \frac{2\lambda(1-\lambda)}{(n+2)t^2} \tag{4.9}$$

for every $t > 0$. Choosing $t = \delta(1-\lambda)$ we get the next lemma.

Lemma 4.5. *Let $\delta > 0$ and let $k = \lambda n$ for some positive integer $k < n$. If $n \geqslant \frac{4\lambda}{(1-\lambda)\delta^2}$ and*

$$\sin^2\theta > (1+\delta)(1-\lambda), \tag{4.10}$$

then $\sigma_{n,k}(\theta) > 1/2$.

5. Diameter of proportional sections

In this section, we obtain lower bounds for the diameter of proportional sections of a symmetric convex body K in $\mathbb{R}^n$. As a first step, we will use Lemma 4.3 to show the following: if K contains B_2^n, then a condition of the form $M(K) > g(\lambda)$ implies an upper bound for the inradius of *every* $\lceil \lambda n \rceil$-dimensional projection $P_E(K)$ of K.

Proposition 5.1. *Let $\lambda \in (0,1)$ and let K be a symmetric convex body in $\mathbb{R}^n$ such that $B_2^n \subseteq K$. If*

$$M > \beta(\lambda) =: \sqrt{\frac{2(1-\lambda)}{2-\lambda}} \tag{5.1}$$

and $n \geqslant C(M-\beta)^{-2}$, then

$$d(P_E(K)) \leqslant \frac{3}{M-\beta} \tag{5.2}$$

for every $E \in G_{n,\lfloor \lambda n \rfloor}$.

Proof. Let $k = \lfloor \lambda n \rfloor$ and let m be the Lévy mean of $\|\cdot\|$ on S^{n-1}. This is the unique $m > 0$ for which $\sigma_n(\|x\| \geqslant m) \geq 1/2$ and $\sigma_n(\|x\| \leqslant m) \geqslant 1/2$. Equivalently, $m = \max\{t > 0 : \sigma_n(tK \cap S^{n-1}) \leqslant 1/2\}$. Since $\|\cdot\|$ is a 1-Lipschitz function, one can check that $|M - m| \leqslant \delta_n$ where $\delta_n \leqslant c_1/\sqrt{n}$ for some absolute constant $c_1 > 0$ (see [12]).

Consider $E \in G_{n,k}$ for which $\rho =: d(P_E(K))$ is maximal. If $(M - \delta_n)\rho \leqslant 1$ then there is nothing to prove: observe that $\delta_n \leqslant (2M + \beta)/3$ if $n \geqslant C(M - \beta)^{-2}$. Otherwise, since $(M - \delta_n)K \supseteq (M - \delta_n)B_2^n$ we can apply Lemma 4.3 to the body $(M - \delta_n)K$. It follows that

$$\sigma_n((M - \delta_n)K \cap S^{n-1}) \geqslant \sigma_{2n-k,k-2}(\theta),$$

where $\theta = \arcsin(M - \delta_n) - \arcsin(1/\rho)$. On the other hand,

$$\sigma_n((M - \delta_n)K \cap S^{n-1}) \leqslant \sigma_n(mK \cap S^{n-1}) \leqslant 1/2. \tag{5.3}$$

We set $\lambda_0 = \frac{k-2}{2n-k}$ and $\delta_0 = \frac{M-\beta}{\beta}$. From Lemma 4.5 it follows that (for $n \geqslant n_0(\lambda_0, \delta_0) \simeq (M - \beta)^{-2}$) we must have

$$\sin\theta \leqslant \sqrt{(1 + \delta_0)\frac{2(n - k - 1)}{2n - k}} < \sqrt{1 + \delta_0}\,\beta. \tag{5.4}$$

Observe that

$$\begin{aligned}
\sin\theta &= \frac{(M - \delta_n)}{\rho}\sqrt{\rho^2 - 1} - \frac{1}{\rho}\sqrt{1 - (M - \delta_n)^2} \\
&= \frac{M - \delta_n}{\rho}\left(\sqrt{\rho^2 - 1} - \sqrt{(M - \delta_n)^{-2} - 1}\right) \\
&\geqslant \frac{M - \delta_n}{\rho}\,\frac{\rho^2 - (M - \delta_n)^{-2}}{\rho + (M - \delta_n)^{-1}} \\
&= (M - \delta_n) - \frac{1}{\rho}.
\end{aligned}$$

Then, (5.4) gives

$$\rho\left((M - \delta_n) - \sqrt{1 + \delta_0}\,\beta\right) \leqslant 1. \tag{5.5}$$

Finally, under the assumption $n \geqslant C(M - \beta)^{-2}$, it is easily checked that $\delta_n + \sqrt{1 + \delta_0}\,\beta \leqslant \left(1 + \frac{2\delta_0}{3}\right)\beta$. This proves the result. $\square$

The dual statement is now immediate.

Proposition 5.2. *Let $\lambda \in (0,1)$ and let K be a symmetric convex body in $\mathbb{R}^n$ such that $K \subseteq B_2^n$. If*

$$M^* > \beta(\lambda) =: \sqrt{\frac{2(1-\lambda)}{2-\lambda}} \tag{5.6}$$

and $n \geqslant C(M^* - \beta)^{-2}$, then

$$D(K \cap E) \geqslant \tfrac{1}{3}(M^* - \beta) \tag{5.7}$$

for every $E \in G_{n,\lfloor \lambda n \rfloor}$.

An equivalent formulation is the following.

Theorem 5.3. *Let $\lambda \in (0,1)$ and $\delta > 0$ satisfy $(1+\delta)\sqrt{\frac{2(1-\lambda)}{2-\lambda}} < 1$, and let $n \geqslant n_1(\lambda, \delta) \simeq \frac{1}{(1-\lambda)\delta^2}$. If K is a symmetric convex body in $\mathbb{R}^n$, and if r_2 is the solution of the equation*

$$M^*(K \cap rB_2^n) = (1+\delta)\sqrt{\frac{2(1-\lambda)}{2-\lambda}}\, r, \tag{5.8}$$

then

$$D(K \cap E) \geqslant \tfrac{1}{3}\delta\sqrt{1-\lambda}\, r_2 \tag{5.9}$$

for every $E \in G_{n,\lfloor \lambda n \rfloor}$.

Remark 5.4. We emphasize the fact that the lower bound for the diameter, in both Proposition 5.2 and Theorem 5.3, holds true for every $\lfloor \lambda n \rfloor$-dimensional section of *K*. Note also that Eq. (5.8) is "comparable" with Eq. (1.3) which implies an upper bound for the diameter of a random $\lfloor \lambda n \rfloor$-dimensional section of *K*. These observations lead to the next definition.

Definition 5.5. Let K be a symmetric convex body in $\mathbb{R}^n$. For every $\lambda \in (0,1)$ define

$$a(\lambda, K) = \min\left\{D(K \cap E) : \; E \in G_{n,\lfloor \lambda n \rfloor}\right\} \tag{5.10}$$

and

$$b(\lambda, K) = \min\left\{r > 0 : D(K \cap E) \leqslant r : \text{ with probability } \geqslant 1/2 \text{ in } G_{n,\lfloor \lambda n \rfloor}\right\}. \tag{5.11}$$

It is clear that $a(\lambda, K) \leqslant b(\lambda, K)$ for all λ and *K*. Combining Proposition 5.2 with Theorem A we see that $a(\lambda, K)$ and $b(\mu, K)$ are comparable in the following sense:

Theorem 5.6. *Let $0 < \mu < 1$ and $0 < s < 1/(2-\mu)$. There exists $n_0 = n_0(\mu, s)$ such that*

$$\left(\frac{c\mu(1-s(2-\mu))}{1-s\mu}\sqrt{1-\mu}\right) b(s\mu, K) \leqslant a(\mu, K) \tag{5.12}$$

for every $n \geqslant n_0$ and every symmetric convex body K in $\mathbb{R}^n$.

Proof. Let $\varepsilon \in (0, 1)$ be a constant (depending on μ and s) which will be suitably chosen. Let K be a symmetric convex body in $\mathbb{R}^n$ and let r_1 be the solution of the equation

$$M^*(K \cap rB_2^n) = (1-\varepsilon)\sqrt{1-s\mu}\,r. \tag{5.13}$$

If n is large enough, then from Theorem A we have

$$b(s\mu, K) \leqslant r_1. \tag{5.14}$$

We choose

$$\varepsilon = \frac{\mu(1-s(2-\mu))}{4(2-\mu)(1-s\mu)}. \tag{5.15}$$

Then, one can check that

$$(1-\varepsilon)\sqrt{1-s\mu} \geqslant (1+\varepsilon)\sqrt{\frac{2(1-\mu)}{2-\mu}}. \tag{5.16}$$

It follows that if r_2 is the solution of the equation

$$M^*(K \cap rB_2^n) = (1+\varepsilon)\sqrt{\frac{2(1-\mu)}{2-\mu}}\,r, \tag{5.17}$$

then $r_1 \leqslant r_2$. Now, Theorem 5.3 shows that

$$c\varepsilon\sqrt{1-\mu}\,r_2 \leqslant a(\mu, K). \tag{5.18}$$

Combining with (5.14) we complete the proof of (5.12). □

Remark 5.7. Assume that Gromov's Theorem 4.1 holds without any restrictions on n and k. Then, using Remark 4.4 and following the arguments of this Section one would be able to prove the next two statements:

Fact A (conditional). *Let $\lambda \in (0,1)$ and $\varepsilon > 0$ satisfy $(1+\varepsilon)\sqrt{1-\lambda} < 1$, and let $n \geqslant n_1(\lambda, \varepsilon) \simeq \frac{1}{(1-\lambda)\varepsilon^2}$. If K is a symmetric convex body in $\mathbb{R}^n$, and if r_2 is the solution of the equation*

$$M^*(K \cap rB_2^n) = (1+\varepsilon)\sqrt{1-\lambda}r, \tag{5.19}$$

then

$$D(K \cap E) \geqslant \tfrac{1}{2}\varepsilon\sqrt{1-\lambda}r_2 \tag{5.20}$$

for every $E \in G_{n,\lfloor \lambda n \rfloor}$.

Combined with Theorem A this would give a very precise "asymptotic formula" for the diameter of random $\lfloor \lambda n \rfloor$-dimensional sections of n-dimensional bodies. Solving the single "asymptotic equation" $M^*(K \cap rB_2^n) \asymp \sqrt{1-\lambda}r$ we would have an upper and a lower bound (up to a constant depending on λ) for the circumradius of a random $K \cap E$, $E \in G_{n,\lfloor \lambda n \rfloor}$. This would also lead to an improvement of Theorem 5.6.

Fact B (conditional). *Let $\mu, s \in (0,1)$. There exists $n_0 = n_0(\mu, s)$ such that*

$$b(s\mu, K) \leqslant \frac{c(1-s\mu)}{(1-s)\mu\sqrt{1-\mu}} a(\mu, K) \tag{5.21}$$

for every $n \geqslant n_0$ and every symmetric convex body K in $\mathbb{R}^n$.

This would show in a very exact way that (with a very small "loss in proportion") minimal and random diameter of μn-dimensional sections are comparable up to a constant depending on μ for every fixed proportion $\mu \in (0,1)$.

Remark 5.8. It is an interesting question to check whether *isometric* results complementing Theorem 2.3 are possible if we assume that $B_2^n \subseteq K$ and M is very close to 1. From Proposition 5.2 we can easily see that if $0 < \varepsilon < \varepsilon_0$ and if the symmetric convex body $K \subseteq B_2^n$ satisfies $M^* > 1-\varepsilon$, then $D(K \cap E) \geqslant 1 - c\varepsilon$ for every $E \in G_{n,k}$ where $n-k < \varepsilon n$.

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