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Some Combinatorial Results for Complex Reflection Groups

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In this paper, we prove that a simple system for a subsystem Ψ of the complex root system Φ can always be chosen as a subset of the positive system Φ^+ of Φ. Furthermore, we show that a set of distinguished coset representatives can be found for every reflection subgroup of the complex reflection groups. The corresponding results for real crystallographic root systems and their reflection groups (i.e., Weyl groups) are well known (see [9]).

1. INTRODUCTION

The object of this paper is to determine some results relating to the complex reflection groups and their root systems which are more useful in an application to give a combinatorial construction of representations of complex reflection groups. The first example of these applications has been given in [3] which generalizes the $\lambda^{[m]}$-Young tableau method given in [2] for generalized symmetric groups. Since these results seem to be of independent interest, they are gathered together in the present paper.

We first establish the basic notation and state some results which are required later. We refer the reader to [4] and [8] for much of the undefined terminology. As a convention, throughout this paper, we assume that ξ is a primitive m-th root of unity.

1.1 Let V be a complex vector space of dimension n. A *reflection* in V is a linear transformation of V of finite order with exactly $(n-1)$ eigenvalues equal to 1. A *reflection group* G in V is a finite group generated by reflections in V. Define $o_G : V \to \mathbf{N}$ by $o_G(v) = |G_{\langle v\rangle^\perp}|$ $(v \in V)$. Then $o_G(v) > 1$ if and only if v is a root of G. In this case, $o_G(v)$ is the order of the cyclic group generated by the reflections in G with root v. If α is a root of G then the number $o_G(\alpha)$ is called the *order* of α (with respect to G).

1.2

(i) Let $\pi = (B, \theta)$ be a root graph, where $B = \{a_1, \ldots, a_n\}$. Denote by $W(\pi)$ the reflection group generated by the *simple reflections* $s_{a_i,\theta(a_i)}$ with $a_i \in B$, $i = 1, \ldots, n$. If $s \in W(\pi)$ then $s = r_{i(1)}r_{i(2)}\ldots r_{i(k)}$ where $r_{i(j)} \in \{r_i\ (i = 1, \ldots, n) \mid r_i = s_{a_i,\theta(a_i)},\ a_i \in B\}$ for $j = 1, \ldots, k$. The *length* of s, denoted by $l(s)$ is the smallest value of k for any such expression for s. By convention, $l(e) = 0$ [8]. Let $\pi' = (B', \theta')$ be another root graph. If $B \subset B'$ and $\theta'|_B = \theta$, we say that π' is an *extension* of π, or that π is a *sub-root graph* of π'. For any root graph $\pi = (B, \theta)$ and for any $w \in W(\pi)$, let $w\pi = (B_w, \theta_w)$, where $B_w = wB$ and $\theta_w(w(a)) = \theta(a)$ with $a \in B$, then by Cohen [4] $w\pi$ is again a root graph which is equivalent to π; in this case, $W(w\pi) = wW(\pi)w^{-1} = W(\pi)$ since $s_{w(a),\theta_w(w(a))} = ws_{a,\theta(a)}w^{-1}$ for all $a \in B$.

(ii) If $\pi = (B, \theta)$ is a root graph, then the pair $\Phi = (R, f)$ where $R = W(\pi)B$ and the map $f : R \to \mathbf{N}\backslash\{1\}$ is induced by the order function $o_{W(\pi)}$ defines a pre-root system with $W(\Phi) = W(\pi)$.

(iii) If $\Phi = (R, f)$ is a pre-root system, then there is a root system $\Sigma = (S, g)$ with $W(\Sigma) = W(\Phi)$, $S \subset R$ and $g = f|_S$.

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(iv) If $\Phi = (R, f)$ is a root system associated with $W(\Phi)$, where $W(\Phi)$ is one of the primitive reflection groups (in dimension greater than 2) $W(J_3(4))$, $W(L_3)$, $W(M_3)$, $W(J_3(5))$, $W(N_4)$, $EW(N_4)$, $W(L_4)$, $W(K_5)$, $W(K_6)$, then we say that Φ is a *primitive root system*.

1.3 Cohen [4] proves that all finite irreducible imprimitive reflection groups are of the form $G(m, p, n)$ for some $m, p \in \mathbf{N}$ with $p|m$ and $n \geq 2$. The reflection group $G(m, 1, n)$ has the following presentation (see [5]):

$$G(m,1,n) = \langle r_1, \ \ldots, \ r_{n-1}, \ w_1, \ \ldots, w_n \mid r_i^2 = (r_i r_{i+1})^3 = (r_i r_j)^2 = e, \ |i-j| \geq 2, w_i^m = e, \ w_i w_j = w_j w_i, \ r_i w_i = w_{i+1} r_i, \ r_i w_j = w_j r_i, \ j \neq i, \ i+1 \rangle.$$

The reflection group $G(m, p, n)$, where $p|m$, is the subgroup of $G(m, 1, n)$. Any element $w \in G(m, 1, n)$ can be decomposed as follows (see [2]): $w = \tau \prod_{i=1}^{n} w_i^{s_i}$, where $\tau \in W(A_{n-1})$ and $1 \leq s_i \leq m$.

If $\Phi(m, p, n)$ is a root system associated with an imprimitive reflection group $G(m, p, n)$, then we say that $\Phi(m, p, n)$ is an *imprimitive root system*.

1.4 Let π be a root graph and Φ be a root system. π is called a *simple system* in Φ if Φ is the pre-root system (in the manner of 1.2(ii)) corresponding to π with $W(\Phi) = W(\pi)$. If $w\pi = (B_w, \theta_w)$ is a root graph which is equivalent to π, where $B_w = wB$ and $\theta_w(w(a)) = \theta(a)$ with $a \in B$ for any $w \in W(\pi)$, then $w\pi$ gives rise to the same pre-root system Φ, and so $w\pi$ is another simple system in Φ. Hence the number of simple systems in Φ is equal to the order of $W(\pi)$. Let Φ be a root system with simple system π, then a graph associated with π is called a *Cohen (Dynkin) diagram* of Φ. Clearly if Φ is a root system then we may not have a simple system for Φ. For example, $G(m, p, n)$, for $p \neq 1, \ m$, is an n-dimensional reflection group which needs $n+1$ generating reflections, thus we do not have a root graph (see [4]) for $G(m, p, n)$, and so we do not have a simple system for the root system associated with $G(m, p, n)$.

1.5 Let $\Phi = (R, f)$ be a root system with $W(\Phi)$. Let S be a subset of R and $g = f|_S$. The pair $\Psi = (S, g)$ is called a *subsystem* of Φ if Ψ is a root system. A *reflection subgroup* $W(\Psi)$ of $W(\Phi)$ corresponding to the subsystem $\Psi = (S, g)$ of Φ is the subgroup generated by the $s_{a,g(a)}$ with $a \in S$. The subsystems $\Psi_1 = (S_1, g_1)$ and $\Psi_2 = (S_2, g_2)$ of Φ are *conjugate under* $W(\Phi)$ if $S_2 = wS_1$ and $g_2(w(a)) = g_1(a)$ for some $w \in W(\Phi)$ and for all $a \in S_1$; in this case, $W(w\Psi_1) = wW(\Psi_1)w^{-1}$ since $s_{w(a),g_2(w(a))} = ws_{a,g_1(a)}w^{-1}$ for all $a \in S_1$.

Let $\pi = (B, \theta)$ (resp. $\Phi = (R, f)$) be a root graph (resp. system), by abuse of notation we sometimes say $\pi = B$ (resp. $\Phi = R$).

1.6 Let Φ be a root system with simple system $\pi = (B, \theta)$, where $B = \{a_1, \ \ldots, \ a_n\}$. Hughes [8] defines a 'positive' system (which he calls a primary root system) in Φ by using the following algorithm:

(i) Let $B_1 = B$.

(ii) Let $B_2 = \{r_i(a_j) \mid i \neq j, \ i, \ j = 1, \ \ldots, \ n, \ a_j \in B_1, \ r_i(a_j) \notin B_1\}$, where $r_i = s_{a_i,\theta(a_i)}$ with $a_i \in B, \ i = 1, \ \ldots, \ n$.

(iii) For $k \geq 3$, put inductively

$$B_k = \{r_i(a)\ (i = 1,\ \dots,\ n) \mid a \in B_{k-1},\ r_i a \neq zb \text{ for all } b \in B_l\ (l < k)\},$$

where

$$z = \begin{cases} \pm 1, & \text{if } \pi = \pi(m,\ p,\ n)(p = 1,\ m), \\ \mu \in \mathbf{C}\ (|\mu| = 1), & \text{otherwise,} \end{cases}$$

(in fact, Hughes [8] takes the scalar z as $\mu \in \mathbf{C}$ with $|\mu| = 1$, but this is not a suitable choice for our later purposes when $\pi = \pi(m,\ p,\ n)\ (p = 1,\ m)$, where $\pi(m,\ p,\ n)$ is a simple system in $\Phi(m,\ p,\ n)$). For $\pi = \pi(m,\ p,\ n)\ (p = 1,\ m)$, if we have $r_i a = -r_j b$ for some $i,\ j,\ a,\ b$, which may occur when m is even, we choose either $r_i a$ or $r_j b$ for B_k. For $\pi \neq \pi(m,\ p,\ n)\ (p = 1,\ m)$, if we have $r_i a = \mu r_j b$ for some $i,\ j,\ a,\ b$ and $|\mu| = 1$ then we choose either $r_i a$ or $r_j b$ for B_k.

A *'positive' system* in Φ is defined to be the union of all B_k $(k \geq 1)$ and will be denoted by Φ^+. (The construction of a 'positive' system Φ^+ depends on the choice of a simple system π in Φ. Having fixed π and the corresponding 'positive' system Φ^+ in Φ, replacing π by another simple system $w\pi$, $w \in W(\pi)$, would just replace Φ^+ by its conjugate $w\Phi^+$. The proof of this fact will be given in Lemma 2.1.) By the construction of each B_k $(k \geq 1)$, it is clear that $\Phi^+ = \biguplus_{k \geq 1} B_k$ with $B_i \bigcap B_j = \emptyset$ whenever $i \neq j$. The roots in Φ^+ will be called *'positive' roots* and the remainder *'negative' roots*. The set of 'negative' roots in Φ is called a *'negative' system* in Φ and will be denoted by Φ^-.

The main results of this paper are summarized in the following theorems.

THEOREM 1. *Let $\Phi = (R,\ f)$ be a root system with a fixed simple system $\pi = (B, \theta)$ and Φ^+ be the 'positive' system determined by π. If Ψ is a subsystem of Φ, then a simple system $J = (B', \theta')$ of Ψ can be chosen such that $B' \subset \Phi^+$.*

The corresponding result for *real crystallographic* root systems Φ (i.e. Φ is one of the types $A_n\ (n \geq 1),\ B_n\ (= C_n)\ (n \geq 2),\ D_n\ (n \geq 4),\ E_6,\ E_7,\ E_8,\ F_4,\ G_2$) has been proved in Idowu and Morris [9]. As mentioned above, we shall show that a simple system for each subsystem Ψ of Φ can always be found as a subset of Φ^+. In fact, we shall show how such a subsystem may be constructed.

Now, let Φ be a root system with a fixed simple system $\pi = (B, \theta)$ and Φ^+ be the 'positive' system determined by π. Let Ψ be a subsystem of Φ with simple system $J \subset \Phi^+$. Let $D_\Psi = \{w \in W(\Phi) \mid w(\alpha) \in \Phi^+ \text{ for all } \alpha \in J\}$. We show that D_Ψ is a 'distinguished' set of coset representatives for $W(\Psi)$ in $W(\Phi)$. The corresponding result for real crystallographic root systems is well known (see, for example, [9]). We shall prove

THEOREM 2. *If Ψ is a subsystem of Φ then every element of $W(\Phi)$ can be uniquely expressed in the form $d_\Psi w_\Psi$ where $d_\Psi \in D_\Psi$ and $w_\Psi \in W(\Psi)$.*

2. PROOF OF THEOREM 1

We first show that differently chosen simple systems in Φ determine different positive systems.

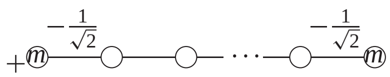
LEMMA 2.1. *Fix a simple system $\pi = (B, \theta)$ and the corresponding positive system Φ^+ in Φ. If $w\pi = (B_w, \theta_w)$ is another simple system in Φ, where $B_w = wB$ and $\theta_w(w(a)) = \theta(a)$ with $a \in B$ for any $w \in W(\pi)$, then $w\Phi^+$ is the positive system in Φ determined by $w\pi$.*

PROOF. Let $\Phi^+ = \biguplus_{k\geq 1} B_k$, then $w\Phi^+ = \biguplus_{k\geq 1} wB_k$ with $wB_i \bigcap wB_j = \emptyset$ whenever $i \neq j$. Now, let $B'_1 = B_w = wB$. By applying the above algorithm, suppose that $\Phi^+_{w\pi} = \biguplus_{k\geq 1} B'_k$ is the positive system determined by $w\pi$. The proof will be completed if we show that $\Phi^+_{w\pi} = w\Phi^+$. Since $B_1 = B$ then $B'_1 = wB_1$, and so it suffices to show that $B'_k = wB_k$ for all $k \geq 2$.

Let $w(a_i) \in B'_1$, $i = 1, \ldots, n$, then $r'_i = s_{w(a_i),\theta_w(w(a_i))} = ws_{a_i,\theta(a_i)}w^{-1} = wr_iw^{-1}$. If $\beta \in B'_k$ $(k \geq 2)$, then $\beta = r'_{i_1} \ldots r'_{i_k}(w(a))$, where $w(a) \in B'_1$ and $r'_{i_s} \in \{r'_1, \ldots, r'_n\}$ for $s = 1, \ldots, k$. Thus $\beta = wr_{i_1} \ldots r_{i_k}w^{-1}(w(a)) = wr_{i_1} \ldots r_{i_k}(a)$. But since $r_{i_1} \ldots r_{i_k}(a) \in B_k$ $(k \geq 2)$, it follows that $B'_k = wB_k$ for all $k \geq 2$. □

The previous lemma says that any two positive systems in Φ are conjugate under $W(\Phi)$. (Thus, Lemma 2.1 shows that it makes no great difference which Φ^+ we choose.)

Dynkin [6] gives an algorithm which gives all subsystems of a given root system relating to a Weyl group. Inspired by extended Dynkin diagrams Hughes [8] introduced extended Cohen diagrams in order to give an algorithm for obtaining subsystems of a given (real or complex) root system. For type $\pi(m, 1, n) = B^m_n$, he gives the following graph



$((n+1)$ points),

as an extended Cohen diagram, where the adjoined root is marked with the sign '+', but this is an error, since when m is odd there does not exist a root in $\Phi(m, 1, n)$ to obtain such a graph.

To prove Theorem 1, to use a similar argument to that in the proof of the corresponding theorem (Theorem 2.1 in [9] for the real case) is considerably more difficult as the idea of a subsystem is not as well developed in the context of complex root systems. Because of this, we now give an alternative way to obtain all subsystems of a given (real or complex) root system by using a new and independent approach without reference to the extended Dynkin (Cohen) diagram given in [6] and [8]. (This method is more useful from a computational point of view and, as an example, in [1] we interpreted it as a computer program written using the symbolic computation system Maple for the real crystallographic root systems. The outputs for these root systems were also given in [1] to illustrate how this method works. The computer program and outputs are available in [1] but are too long to be included in this paper.)

Theorem 1 comes as an immediate corollary of this method.

Let Φ be a root system with a fixed simple system $\pi = (B, \theta)$. The subsystems of Φ fall into two categories. Let Ψ be a subsystem of Φ with simple system $J = (B_\pi, \theta_\pi)$, where $B_\pi \subset B$ and $\theta_\pi = \theta|_{B_\pi}$. Replacing π by another simple system $w\pi$, $w \in W(\pi)$, would just replace Ψ by its conjugate $w\Psi$ by 1.2(ii). All subsystems of Φ obtainable in this way are called *parabolic subsystems*. A subsystem of Φ which is not the parabolic is called a *non-parabolic subsystem*. For example, in the type A_n, all subsystems are parabolic but in all the other root systems this is not the case.

The set of all parabolic subsystems of Φ is obtained by removing one or more nodes in all possible ways from the Cohen (Dynkin) diagram (and all equivalent diagrams) of Φ, that is,

LEMMA 2.2. *If $\Phi = (R, f)$ is a root system with a fixed simple system $\pi = (B, \theta)$ then the pair $J = (B_\pi, \theta_\pi)$, where $B_\pi \subset B$ and $\theta_\pi = \theta|_{B_\pi}$, is a sub-root graph of π. Furthermore, J yields a parabolic subsystem of Φ.*

PROOF. The result follows immediately from 1.2(ii) and (iii). □

If $\Psi = (S, g)$ is the parabolic subsystem of Φ corresponding to J, recall that its conjugates $w\Psi$, $w \in W(\pi)$, are also parabolic subsystems of Φ.

We shall now obtain non-parabolic subsystems of Φ by means of the parabolic subsystems of Φ, that is,

LEMMA 2.3. *Let $\Phi = (R, f)$ be a root system with a fixed simple system $\pi = (B, \theta)$ and Φ^+ be the 'positive' system determined by π. Let $\Psi = (S, g)$ be a parabolic subsystem of Φ with simple system $J = (B_\pi, \theta_\pi)$, where $B_\pi \subset B$ and $\theta_\pi = \theta|_{B_\pi}$, and let Ψ^+ be the 'positive' system determined by J. Define $\Phi^+_\Psi = \Phi^+ \backslash \Psi^+$, and let B_Ψ be a subset of Φ^+_Ψ such that*

$$B_\pi \cup B_\Psi \text{ is linearly independent over } \mathbf{C}. \tag{1}$$

Then the pair $J_0 = (B_0, \theta_0)$, where $B_0 = B_\pi \cup B_\Psi$ and $\theta_0 = f|_{B_0}$, is a root graph which is an extension of J. If $B_0 \not\subset wB$ for all $w \in W(\pi)$, then J_0 yields a non-parabolic subsystem of Φ. Furthermore, if $B_0 \subset wB$ for some $w \in W(\pi)$, then J_0 yields a parabolic subsystem of Φ.

PROOF. Since $B_0 \subset R$ and $\theta_0 = f|_{B_0}$, $J_0 = (B_0, \theta_0)$ is a vector graph. Denote by $W(J_0)$ the group generated by all reflections $s_{a,\theta_0(a)}$ with $a \in B_0$, then $W(J_0)$ is a subgroup of $W(\Phi)$ and so $W(J_0)$ is a finite reflection group. By Hypothesis (1), B_0 is linearly independent over $\mathbf{C}$. Thus $J_0 = (B_0, \theta_0)$ is a root graph which is an extension of J.

Let $S_0 = W(J_0)B_0$, and define a map $g_0 : S_0 \to \mathbf{N}\backslash\{1\}$ by $g_0(a) = o_{W(J_0)}(a)$ for all $a \in S_0$, then $S_0 \subset R$ and $g_0 = f|_{S_0}$, so the pair $\Psi_0 = (S_0, g_0)$ is the pre-root system corresponding to J_0 with $W(\Psi_0) = W(J_0)$ by 1.2(ii). By 1.2(iii), the pair $\Psi_0 = (S_0, g_0)$ is a root system and so is a subsystem of Φ. If $B_0 \not\subset wB$ for all $w \in W(\pi)$, then by definition of the non-parabolic subsystem Ψ_0 is a non-parabolic subsystem (note that its conjugates $w\Psi_0$, $w \in W(\pi)$, are also non-parabolic subsystems of Φ). On the other hand, if $B_0 \subset wB$ for some $w \in W(\pi)$, then by definition of the parabolic subsystem Ψ_0 is a parabolic subsystem. □

As we run through all the parabolic subsystems, we generate all the non-parabolic subsystems. Therefore, the above construction shows that all subsystems of Φ can be obtained up to conjugacy.

COROLLARY 2.4. *If Φ is a real root system, then we can replace Hypothesis (1) of Lemma 2.3 by*

$$(a, b) \le 0 \text{ for all pairs } a \ne b \text{ in } B_0. \tag{1$'$}$$

PROOF. We just need to show that B_0 is linearly independent over $\mathbf{R}$ under Hypothesis (1′). Let $B_0 = \{a_1, \ \dots, \ a_k\}$ and suppose that B_0 is linearly dependent over $\mathbf{R}$, i.e., let $\sum_{i=1}^k \gamma_i a_i = 0$ be a non-trivial relation.

Put $I = \{i \mid \gamma_i > 0\}$ and $K = \{i \mid \gamma_i < 0\}$, and write $\lambda_i = \gamma_i$, $i \in I$ and $\mu_i = -\gamma_i$, $i \in K$. Then

$$a = \sum_{i \in I} \lambda_i a_i = \sum_{j \in K} \mu_j a_j \ne 0$$

with λ_i, $\mu_j > 0$ for all $i \in I$ and $j \in K$. By Hypothesis (1′),

$$0 < (a, a) = \sum_{i,\ j} \lambda_i \mu_j (a_i, a_j) \le 0.$$

This forces $a = 0$ which is a contradiction, and so B_0 is linearly independent over $\mathbf{R}$. □

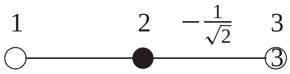
Let Φ be a root system with a fixed simple system π and Φ^+ be the 'positive' system of Φ determined by π. If Ψ is a subsystem of Φ obtained by means of the above construction, then a simple system J of Ψ can always be found such that $J \subset \Phi^{+}$. We recall that this is also true for its conjugates $w\Psi$, where $w \in W(\pi)$, because of the following reason. (Here, clearly, we need only consider an element w of $W(\pi) \setminus W(J)$, for if $w \in W(J)$ then $w\Psi = \Psi$ and we are done.)

Now, let $w\Psi$ be a subsystem of Φ which is conjugate to Ψ, where $w \in W(\pi) \setminus W(J)$. Since, at any stage of the above construction, we have a subsystem Υ of Φ such that $\Upsilon = w\Psi$ with simple system $\kappa \subset \Phi^{+}$, one does not need to worry about the conjugates of a subsystem of Φ obtained by means of the above construction.

We now give the following example to illustrate this fact.

EXAMPLE 2.5. Let Φ be the root system of type B_3^3 with simple system $\pi = \{e_1 - e_2,\ e_2 - e_3,\ e_3\}$. By applying the above algorithm, the 'positive' system in Φ with respect to π is obtained to be $\Phi^{+} = P \cup P' \cup Q$, where $P = \{e_i - \xi^{l} e_j \mid 1 \leq i < j \leq 3,\ 1 \leq l \leq 3\}$, $P' = -\{\xi,\ \xi^2\}P$ and $Q = \{e_k \mid 1 \leq k \leq 3\}$. (Here, ξ is a third root of unity.)

Now, consider $J = \{e_1 - e_2,\ e_3\}$ as a sub-root graph of π. Then by Lemma 2.2 $\Psi = W(J)J = A_1 + B_1^3 = \{e_1 - e_2,\ e_2 - e_1,\ e_3,\ \xi e_3,\ \xi^2 e_3\}$ is a parabolic subsystem of Φ with simple system $J \subset \Phi^{+}$. Thus, the corresponding Cohen diagram for Ψ is



where the nodes corresponding to $e_1 - e_2,\ e_2 - e_3$ and e_3 are denoted by 1, 2 and 3 respectively and the node 2 has been deleted.

By considering Lemma 2.3, let $J_0 = J \cup \{e_2\}$ where $e_2 \in \Phi_{\Psi}^{+}$, then J_0 is a root graph which is an extension of J. Since $J_0 \not\subset w\pi$ for all $w \in W(\pi)$, then $\Psi_0 = W(J_0)J_0 = B_2^3 + B_1^3 = \{1,\ \xi,\ \xi^2\}\{\pm(e_1 - \xi^{l} e_2),\ e_1,\ e_2,\ e_3 \mid 1 \leq l \leq 3\}$ is a non-parabolic subsystem of Φ with simple system $J_0 \subset \Phi^{+}$. Then the Cohen diagram for Ψ_0 is

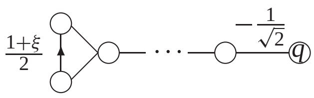
We first consider the conjugates of $\Psi = A_1 + B_1^3$. Let $w_1 w_2^2$ be an element of $W(\pi) \setminus W(J)$, then $w_1 w_2^2 \Psi = \{\xi e_1 - \xi^2 e_2,\ \xi^2 e_2 - \xi e_1,\ e_3,\ \xi e_3,\ \xi^2 e_3\}$ is a parabolic subsystem of Φ which is conjugate to Ψ. Now, consider a parabolic subsystem Γ of Φ which has a simple system $L = \{e_3\} \subset \Phi^{+}$. By following Lemma 2.3, if we put $L_0 = L \cup \{\xi^2 e_2 - \xi e_1\}$, where $\xi^2 e_2 - \xi e_1 \in \Phi_{\Gamma}^{+}$, then L_0 is a root graph which is an extention of L. Since $L_0 \subset w_1 w_2^2 r_1 \pi = \{\xi^2 e_2 - \xi e_1,\ \xi e_1 - e_3,\ e_3\}$ for $w_1 w_2^2 r_1 \in W(\pi)$, then $\Gamma_0 = W(L_0)L_0$ is a parabolic subsystem of Φ with simple system $L_0 \subset \Phi^{+}$. But we have $\Gamma_0 = w_1 w_2^2 \Psi$, so the L_0 can be chosen as a simple system of $w_1 w_2^2 \Psi$.

Secondly, we now consider the conjugates of $\Psi_0 = B_2^3 + B_1^3$. Let $r_2 w_1 w_2^2 w_3^2 \in W(\pi) \setminus W(J_0)$, then $r_2 w_1 w_2^2 w_3^2 \Psi_0 = \{1,\ \xi,\ \xi^2\}\{\pm(e_1 - \xi^{l} e_3),\ e_1,\ e_2,\ e_3 \mid 1 \leq l \leq 3\}$ is a non-parabolic subsystem of Φ. Now, put $L_0^* = L \cup \{e_1 - e_3,\ e_2\}$ where $e_1 - e_3,\ e_2 \in \Phi_{\Gamma}^{+}$, then L_0^* is a root graph which is an extention of L by Lemma 2.3. Since $L_0^* \not\subset w\pi$ for all $w \in W(\pi)$, then $\Gamma_0^* = W(L_0^*)L_0^*$ is a non-parabolic subsystem of Φ with simple system $L_0^* \subset \Phi^{+}$. On the other hand, since we have $\Gamma_0^* = r_2 w_1 w_2^2 w_3^2 \Psi_0$, then the L_0^* can be regarded as a simple system of $r_2 w_1 w_2^2 w_3^2 \Psi_0$.

Thus, as mentioned above, if $w\Psi$ is a subsystem of Φ which is conjugate to Ψ, then there exists a subystem Υ of Φ obtained by means of the above construction such that $\Upsilon = w\Psi$ with simple system $\kappa \subset \Phi^{+}$.

Let $\Phi = (R,\ f)$ be a root system which has a simple system. Having fixed a simple system $\pi = (B, \theta)$ and the corresponding 'positive' system Φ^+ in Φ, the above results enable us to construct all subsystems of Φ whose simple systems $J = (B', \theta')$ are such that $B' \subset \Phi^+$, so the proof of Theorem 1 is complete. If Φ is a real crystallographic root system, then we recover the result of Idowu and Morris [9].

REMARK 2.6. We shall now make a few remarks on the groups $G(m,\ p, n)$ for $p \neq 1,\ m$. The vector graph (see [10]) for $G(m,\ p, n)$ is



$(n+1 \text{ points},\ n \geq 2)$,

where $q = \frac{m}{p}$. If we denote this vector graph by $\pi(m,\ p, n)$ $(p \neq 1,\ m)$, then $W(\pi(m,\ p, n)) = G(m,\ p, n)$. But the elements of $\pi(m,\ p, n)$ are linearly dependent over $\mathbf{C}$, and so we do not have a simple system for the root system $\Phi(m,\ p, n)$ $(p \neq 1,\ m)$ associated with $G(m,\ p, n)$. If we delete a node from $\pi(m, p, n)$ then we obtain one of the vector graphs of the form $\pi(m, m, n),\ \pi(q, 1, n),\ \pi(m, m, r) + \pi(q, 1, n-r)$, which turn out to be root graphs. Let Σ be a root system such that its simple system is one of the root graphs obtained as above. Thus to obtain the subsystems of $\Phi(m,\ p, n)$ which have simple systems, we use Lemma 2.2 and Lemma 2.3 on the root system Σ.

3. PROOF OF THEOREM 2

Let $\Phi(m,\ p, n)$ $(p = 1,\ m)$ be an imprimitive root system with simple system $\pi(m,\ p, n) = (B, \theta)$, where

$$B = \begin{cases} \{\alpha_i = e_i - e_{i+1}\ (i = 1,\ \ldots,\ n-1),\ \alpha_n = e_n\}, & \text{if } p = 1, \\ \{\beta_i = e_i - e_{i+1}\ (i = 1,\ \ldots,\ n-1),\ \beta_n = e_{n-1} - \xi e_n\}, & \text{if } p = m. \end{cases}$$

By applying the above algorithm, a 'positive' system in $\Phi(m,\ p, n)$ is obtained to be

$$\Phi^+(m,\ p, n) = \begin{cases} P \cup P' \cup Q, & \text{if } p = 1, \\ P \cup P', & \text{if } p = m, \end{cases}$$

where

$$P = \{e_i - \xi^l e_j \mid 1 \leq i < j \leq n,\ 1 \leq l \leq m\},$$
$$Q = \{e_k \mid 1 \leq k \leq n\}$$

and

$$P' = -\lambda P \text{ with } \lambda = \begin{cases} \{\xi^l \mid 1 \leq l \leq m-1\}, & \text{if } m \text{ is odd} \\ \{\xi^l \mid 1 \leq l \leq \frac{m}{2} - 1\}, & \text{if } m \text{ is even.} \end{cases}$$

The 'positive' systems for the primitive root systems can be found in Hughes [7].

We are now in a position to give the proof of Theorem 2.

PROOF OF THEOREM 2. A simple system J of a subsystem Ψ of Φ is chosen as in Section 2 such that $J \subset \Phi^+$. We consider all the possible cases in terms of complex root systems Φ. Clearly, we do not need to consider the real crystallographic root systems, for these have been studied by Idowu and Morris [9].

Now, consider an arbitrary coset $wW(\Psi)$ where $w \notin W(\Psi)$. Suppose that all the elements $w' \in wW(\Psi)$ are such that $w' \notin D_\Psi$, that is, $w'(\alpha) \in \Phi^-$ for some $\alpha \in J$.

(i) If Ψ is a subsystem of $\Phi = \Phi(m, p, n)$ $(p = 1,\ m)$ then we have two possibilities on the α: the order of α, $o_{W(\Psi)}(\alpha)$ is either 2 or m. If $o_{W(\Psi)}(\alpha) = 2$ then $w'\tau_\alpha(\alpha) = -w'(\alpha) \in \Phi^+$ and $w'\tau_\alpha \in wW(\Psi)$. Since $\tau_\alpha^2 = e$ and $w'(\alpha) \in \Phi^-$ then $w'(\alpha) = \gamma$, where $\gamma \in \Phi^-$, and so $w' = \tau_\gamma w'\tau_\alpha$. Thus, $l(w'\tau_\alpha) = l(\tau_\gamma w') < l(\tau_\gamma w'\tau_\alpha) = l(w')$, where the length of an element $w \in W(\Phi)$, $l(w)$ is defined as in 1.2(i). (Here, it is clear that $\tau_\gamma w' \neq \tau_\alpha$, for if $\tau_\gamma w' = \tau_\alpha$ then $w' = e$, contradicting the choice of w'.) If $o_{W(\Psi)}(\alpha) = m$ then $\alpha = e_i$ $(1 \le i \le n)$. Since $w'(\alpha) \in \Phi^-$ then $w'(\alpha) = \xi^s e_j$, where $1 \le s < m$, $1 \le j \le n$, $w'\tau_\alpha^{m-s}(\alpha) \in \Phi^+$ and $w'\tau_\alpha^{m-s} \in wW(\Psi)$. This means that $w_i^s = \tau_\alpha^s$ is involved in w'. Furthermore, by 1.3, we can write $w' = \tau \prod_{i=1}^n w_i^{s_i}$, where $\tau \in W(A_{n-1})$, $s_i = s$ and $1 \le s_1, \ \ldots, \ s_{i-1}, \ s_{i+1}, \ \ldots, \ s_n \le m$. Thus,

$$l(w'\tau_\alpha^{m-s}) = l\left(\tau \prod_{i=1}^n w_i^{s_i} w_i^{m-s}\right) < l\left(\tau \prod_{i=1}^n w_i^{s_i}\right) = l(w').$$

Repeating this argument will eventually show that $e \in wW(\Psi)$, which is obviously not the case. Thus, there exists at least one element of $wW(\Psi)$ in D_Ψ. Denote this element by d_Ψ.

This element is unique, for if $\sigma \in d_\Psi W(\Psi)$ and $\sigma \in D_\Psi$ then $\sigma = d_\Psi \rho$ for some $\rho \in W(\Psi)$ and $\sigma(\alpha) \in \Phi^+$ for all $\alpha \in J$. Suppose that $\rho \neq e$, then for some $\alpha \in J$, $\rho(\alpha) \in \Phi^-$. If $o_{W(\Psi)}(\alpha) = 2$ then $-\rho(\alpha) \in \Phi^+$ and $-(d_\Psi\rho)\alpha \in \Phi^+$ and so $(d_\Psi\rho)\alpha \in \Phi^-$, that is, $d_\Psi\rho \notin D_\Psi$, which is a contradiction, Thus, $\rho = e$ and $\sigma = d_\Psi$. If $o_{W(\Psi)}(\alpha) = m$ then $\rho(\alpha) = \xi^k e_j$, where $1 \le k < m$, $1 \le j \le n$ and thus $\xi^{m-k}\rho(\alpha) \in \Phi^+$ and $\xi^{m-k}(d_\Psi\rho)\alpha \in \Phi^+$ and so $(d_\Psi\rho)\alpha \in \Phi^-$, that is, $d_\Psi\rho \notin D_\Psi$ which is a contradiction. Thus $\rho = e$ and $\sigma = d_\Psi$, and so the proof is complete for the imprimitive root systems $\Phi(m, p, n)$ $(p = 1,\ m)$.

(ii) If Ψ is a subsystem of the primitive root system Φ then we have again two possibilities on the α: the order of α, $o_{W(\Psi)}(\alpha)$ is either 2 or 3 (see [4]). By using a similar argument as above the result can be deduced for the primitive root systems.

Finally, let $\Phi = \Phi(m, p, n)$, where $p \neq 1,\ m$. Referring to Remark 2.6, let Σ be a root system such that its simple system is one of the root graphs $\pi(m, m, n)$, $\pi(q, 1, n)$, $\pi(m, m, r) + \pi(q, 1, n - r)$. If Ψ is a subsystem of Σ with simple system $J \subset \Sigma^+$, then $D_\Psi = \{w \in W(\Sigma) \mid w(\alpha) \in \Sigma^+ \text{ for all } \alpha \in J\}$ is a 'distinguished' set of coset representatives for $W(\Psi)$ in $W(\Sigma)$ since we have already dealt with these types in (i). This completes the proof of Theorem 2. $\square$

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