SOME GLOBAL PROPERTIES OF COMPLETE MINIMAL SURFACES OF FINITE TOPOLOGY IN $\mathbb{R}^3$

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§1. INTRODUCTION

One of the outstanding questions in the classical theory of minimal surfaces asks whether or not the helicoid is the only properly embedded minimal surface in $\mathbb{R}^3$ with finite topology and infinite total curvature. Since the ends of a surface of finite topology are annuli, this question is essentially one of describing the asymptotic behavior of embedded minimal annuli in $\mathbb{R}^3$. When a minimal annulus has finite total curvature, it is asymptotic to either a plane or a half-catenoid [12]. Recently, Hoffman and Meeks [4] proved that a properly embedded minimal surface can have at most two annular ends of infinite total curvature.

In this paper we analyse further the geometry of properly embedded minimal surfaces of finite topology in $\mathbb{R}^3$ that have infinite total curvature. In order to carry out this analysis, we will need two technical results, the first of which we explain now. Consider the family of catenoids

$$C_t = \{(x, y, z) \in \mathbb{R}^3 | t^2 x^2 + t^2 y^2 = \cosh^2(tz)\},$$

for $t > 0$. The first technical result, just referred to, states that a properly immersed minimal annulus that lies above some $C_t$ must have finite total curvature. More precisely:

**Theorem 1.1.** Let

$$W_t = \{(x, y, z) \in \mathbb{R}^3 | t^2 x^2 + t^2 y^2 \leq \cosh^2(tz), z \geq 0\}.$$

Suppose $f: M \to \mathbb{R}^3$ is a proper minimal immersion of an annulus with smooth compact boundary such that the image is contained in $W_t$ for some $t > 0$. Then $M$ has finite total curvature.

One of our main applications of Theorem 1.1 is the following global result:

**Theorem 1.2.** Suppose $M$ is a properly embedded minimal surface in $\mathbb{R}^3$ that has two annular ends, each having infinite total curvature. Then these two ends have representatives $E_1, E_2$ satisfying the following:

1. There exist disjoint closed halfspaces $\mathbb{H}_1, \mathbb{H}_2$ such that $E_1 \subset \mathbb{H}_1$ and $E_2 \subset \mathbb{H}_2$.
2. All other annular ends of $M$ are asymptotic to flat planes parallel to $\partial \mathbb{H}_1$.
3. $M$ has only a finite number of normal vectors parallel to the normal vector of $\partial \mathbb{H}_1$.

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Since the proof of Theorem 1.2 holds when $M$ has compact boundary, the following corollary is a simple consequence of Theorem 1.2.

**Corollary 1.1.** Let $f: M \to \mathbb{R}^3$ be a properly embedded minimal surface with finite topology and smooth compact boundary. If $M$ has a catenoid end, then $M$ has at most one end with infinite total curvature.

Jorge and Xavier [6] proved that there exist many nonplanar complete minimal surfaces in the open slab $S = \{(x_1, x_2, x_3) | 0 < x_3 < 1\}$ in $\mathbb{R}^3$. The existence of these surfaces answered a question raised by Calabi as to whether a complete minimal surface in a halfspace must be a plane. Later Rosenberg and Toubiana [11], using a similar construction, proved that the slab $S$ contains a complete minimally immersed annulus $A$. The annulus $A$ intersects each plane $P_t = \{x_3 = t\}$ in a single closed immersed curve for every $t$ in the interval $(0, 1)$. In particular, the examples of Rosenberg and Toubiana are proper minimal immersions into $S$. These existence theorems contrast strongly with a recent theorem of Hoffman and Meeks [5]: A properly immersed minimal surface in $\mathbb{R}^3$ that is contained in a halfspace must be a plane. It follows that the examples of Jorge-Xavier and Rosenberg-Toubiana cannot be proper as maps into $\mathbb{R}^3$ even though they may be proper as maps into an open slab.

Suppose $X: M \to W$ is a proper minimal immersion of a surface with finite topology and compact boundary. By Theorem 1.1, $M$ has finite total curvature. In this case every normal vector to the surface is obtained finitely often. This result contrasts strongly with an example of Rosenberg and Toubiana [11]. Their example is a complete minimal immersion $X: A \to \mathbb{R}^3$ of the annulus $A = \{z \in \mathbb{C} | |z| > 1\}$ whose third coordinate function is $X_3(x) = \ln|z|$, and $A$ has infinite total curvature. Since $X_3$ is proper and nonnegative on $A$, $A$ is properly immersed in $\mathbb{R}^3$ and $X(A)$ lies in a halfspace. Results of Fujimoto [2], [3], Mo and Osserman [8], Osserman [9], Xavier [13] imply that the Gauss map of $X$ obtains every value infinitely often except for possibly 4 values.

We make one further remark that clarifies the reason for the finite topology hypotheses that will appear in the statement of our second technical result. Recently Callahan, Hoffman, and Meeks [1] constructed a periodic, properly embedded, minimal surface $M \subset \mathbb{R}^3$ that intersects the $x_1x_2$-plane in a simple closed curve. The portion $M^+ = \{(x_1, x_2, x_3) \in M | x_3 > 0\}$ is a proper minimal surface in $\mathbb{R}^3$ that is contained in a halfspace, has compact analytic boundary, and the Gauss map of $M^+$ obtains every value infinitely often. By the next theorem, $M^+$ must, as we already knew, have infinite topology. This technical theorem will be used in the proof of part 3 of Theorem 1.2.

**Theorem 1.3.** Suppose $X: M \to \mathbb{R}^3$ is a proper analytic minimal immersion of a non-flat analytic surface with compact boundary and with finite topology. If $X(M)$ is contained in a halfspace, then the collection of points of $M$ with normal vectors parallel to the normal vector of the boundary of the halfspace is a finite set.

§2. **The Proof of Theorem 1.1**

Let $C$ be a catenoid in $\mathbb{R}^3$ with the $z$-axis as symmetry axis. Let $X$ be the closure of the component of $\mathbb{R}^3 - C$ that contains the $z$-axis. Let $\mathbb{H} = \{(x, y, z) \in \mathbb{R}^3 | z > 0\}$ and $\overline{\mathbb{H}}$ its closure. Let $M$ be an annulus conformally diffeomorphic to $D_\varepsilon = \{z \in \mathbb{C} | |\varepsilon| \leq 1\}$ for some $\varepsilon$, $0 < r < 1$. Let $f: M \to \mathbb{R}^3$ be a proper conformal minimal immersion of $M$, $f(M) = A$ and $A \subset X \cap \overline{\mathbb{H}}$. 
After homothetically shrinking or expanding $C$ and $A$, we can assume $C$ is the standard catenoid, i.e., $C$ has the conformal structure of $C - \{0\}$ and is embedded in $\mathbb{R}^3$ as follows:

$$F : C - \{0\} \to \mathbb{R}^3$$

$$F(\zeta) = \text{Re} \left( \int_1^{\zeta} \omega_1, \int_1^{\zeta} \omega_2, \int_1^{\zeta} \omega_3 \right)$$

where

$$\omega_1 = \frac{1}{2} \left( 1 - \frac{\zeta^2}{\zeta^2} \right) d\zeta, \quad \omega_2 = \frac{i}{2} \left( \frac{\zeta^2}{\zeta^2} \right) d\zeta, \quad \omega_3 = \frac{d\zeta}{\zeta}.$$

The Gauss map of $C$ is

$$N_C(\zeta) = \frac{1}{1 + |\zeta|^2} (2 \text{Re} \zeta, 2 \text{Im} \zeta, |\zeta|^2 - 1).$$

**Lemma 2.1.** If $A \subset X \cap \mathbb{R}$, then $A$ contains a proper subannulus $A'$ that is conformally parametrized by $E = \{ \zeta \in C | |\zeta| \geq 1 \}$. Moreover, in this parametrization $G : E \to \mathbb{R}^3$ of $A'$, the third component of $G$ is

$$G_3(\zeta) = a \log |\zeta| + b$$

for some $a, b \in \mathbb{R}$, $a > 0$, $b > 0$.

**Proof.** Note that the third coordinate function of $X$ is proper. Since $f = (f_1, f_2, f_3) : M \to \mathbb{R}^3$ is a proper minimal immersion and $A = f(M) \subset X \cap \mathbb{R}$, $f_3 : M \to \mathbb{R}$ is a proper harmonic function. It is an elementary exercise in complex analysis that a proper harmonic function $h$ on an annulus $A$ with one boundary curve contains a subannulus $A'$, conformally parametrized by $E$, such that $h|E = a \log |\zeta| + b$.

Suppose $A'$ is the subannulus of $A$ described in Lemma 2.1. Since $A$ and $A'$ both have finite total curvature or both have infinite total curvature, we will assume, without loss of generality, that $A = A'$.

**Lemma 2.2.** Let $\Psi : E \to \mathbb{R}^3$ be a properly immersed minimal annulus, $S = \Psi(E) \subset \mathbb{R}^3$. Let $p \in \text{Int}(E)$. Suppose $P$ is the tangent plane of $S$ through $\Psi(p)$ and $P \cap \partial S = \emptyset$. Then the component of $P \cap S$ that contains $\Psi(p)$ is noncompact.

**Proof.** Since $S$ is noncompact, we may assume that $S$ is not part of a plane. If $\mathbf{n}$ is the normal vector of $P$, then $h = (\Psi - \Psi(p)) \cdot \mathbf{n}$ is a harmonic function on $E$ and $\Psi^{-1}(S \cap P) = h^{-1}(0)$. Since $h$ is harmonic and $h^{-1}(0) \subset \text{Int}(E)$, the maximum principle implies that every component of $h^{-1}(0)$ is a one-dimension analytic variety of $E$. Suppose that the component of $P \cap S$ containing $p$ is compact. Let $\Delta$ denote the preimage of this component on $E$. Note that $\Delta$ is compact since $\Psi$ is proper. Furthermore, since $p$ is a critical point of the harmonic function $h$, $\Delta$ is a singular compact analytic one-dimensional variety in $E$. But the complement of any such singular variety in the annulus $E$ disconnects $E$ into at least three components. One of the components of $E - \Delta$ is unbounded, another contains $\partial E$ and at least one, say $\Sigma$, has compact closure $\bar{\Sigma}$ and $h|\partial \bar{\Sigma} = 0$. By the maximum principle, $\Psi(\Sigma) = P$ which forces $S$ to be contained in the plane $P$. This contradiction proves the lemma.

Suppose now that $A$ has infinite total curvature. We will exhibit a family of tangent planes $P_n$ of $A$ at $G(p_n)$ such that the component of $P_n \cap A$ containing $G(p_n)$ is compact.
Furthermore, for \( n \) large enough, \( P_A \cap \partial A = \emptyset \). The existence of such tangent planes contradicts Lemma 2.2 and this contradiction will prove Theorem 1.1.

For the part of \( C \) in \( \mathfrak{H} \) we have following non-parametric expression: \( x^2 + y^2 = \cosh^2 z \), \( z \geq 0 \). Hence, at any point \( p = (x, y, z) \in C \cap \mathfrak{H} \), the normal vector is

\[
N^C(p) = \frac{1}{\sqrt{1 + z_x^2 + z_y^2}} (-z_x, -z_y, 1),
\]

where

\[
z_x = 2x / \sinh 2z, \quad z_y = 2y / \sinh 2z,
\]

and

\[
1 + z_x^2 + z_y^2 = (\sinh^2 2z + 4 \cosh^2 z) \sinh^2 2z = [4 \cosh^2 z(\sinh^2 z + 1)] / \sinh^2 2z = 4 \cosh^4 z / (4 \cosh^2 z \sinh^2 z) = \cosh^2 z / \sinh^2 z.
\]

Suppose \( p = (x, y, z) \in C \cap \mathfrak{H} \). Let \( \theta(p) \) be the angle such that

\[
\cos \theta(p) = N^C(p) \cdot (0, 0, 1) = -1.
\]

Then

\[
\sin \theta(p) = \sqrt{1 - \cos^2 \theta(p)} = \frac{1}{\cosh z}.
\]

Thus \( \sin \theta(p) \) is independent of \( x \) and \( y \). We denote it as \( \sin \theta(z) \). For \( p = (x_0, y_0, z_0) \in A \cap X \cap \mathfrak{H} \), consider the solid cylinders

\[
L^{z_0} = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 \leq \cosh^2 (z_0 + 1) \},
\]

\[
L_1^{z_0} = \{(x, y, z) \in L^{z_0} | -1 \leq z \leq z_0 + 1 \}.
\]

If \( P \) is a plane passing through \( p_0 = (x_0, y_0, z_0) \) and \( v_P \) is the normal vector of \( P \), define \( \Psi_P \) by the formula \( \cos \Psi_P = v_P \cdot (0, 0, 1) \).

**Lemma 2.3.** If \( z_0 \) is large and

\[
|\Psi_P| < \frac{1}{16 \cosh z_0} = \frac{\sin \theta(z_0)}{16},
\]

then the component of \( P \cap A \) that contains \( p_0 \) is compact.

**Proof.** Since \( p_0 = (x_0, y_0, z_0) \in L_1^{z_0} \), for any \( (x, y, z) \in P \cap \partial L^{z_0} \) we have

\[
|z - z_0| \leq 2 \cosh(z_0 + 1) \tan \Psi_P = 2 \cosh(z_0 + 1) \frac{\sin|\Psi_P|}{|\cos|\Psi_P|}.
\]

Choose \( z_0 \) large enough so that \( \cos |\Psi_P| > \frac{1}{2} \). Hence, \( |\Psi_P| \) satisfies \( \sin|\Psi_P| < |\Psi_P| < \frac{1}{16 \cosh z_0} \).

It follows that

\[
|z - z_0| < 4 \cdot \frac{\cosh(z_0 + 1)}{16 \cosh z_0}.
\]

Note that \( \cosh(z_0 + 1) = \cosh z_0 \cosh 1 + \sinh z_0 \sinh 1 \), \( \sin h 1 < \cosh 1 < 2 \), and \( \sinh z_0 < \cosh z_0 \). Hence, \( \cosh(z_0 + 1) < 4 \cosh z_0 \), and so \( |z - z_0| < 1 \). Hence, \( P \cap \partial L^{z_0} = P \cap \partial L_1^{z_0} \) and \( P \cap L^{z_0} = P \cap L_1^{z_0} \). This implies that the component \( \gamma \) of \( A \cap P \) that contains \( p_0 \) must be compact (since \( \gamma \subset P \cap L_1^{z_0} \) and \( L_1^{z_0} \) is compact).

Now we prove Theorem 1.1 that is stated in the Introduction.
Proof of Theorem 1.1. Assume $A$ has infinite total curvature. Let $g:E \to \mathbb{C} \cup \{\infty\}$ be the Gauss map of $A$ composed with stereographic projection. Similarly define $\tilde{g}:\mathbb{C} - \{0\} \to \mathbb{C} \cup \{\infty\}$ to be the Gauss map of $C$ composed with stereographic projection. Recall, in fact, that in our original parametrization $F$ of $C$, $\tilde{g}(\xi) = \xi$ for $\xi \in \mathbb{C} - \{0\}$. It is well-known that $g$ is holomorphic. In particular, if $A$ has infinite total curvature, then $g$ has an essential singularity at $\infty$ (see [10]). Recall that the Gauss map of $C$ is

$$N^C(\xi) = \frac{1}{1 + |\xi|^2} \left( 2 \text{Re } \xi, 2 \text{Im } \xi, |\xi|^2 - 1 \right)$$

for $\xi \in E$ and the Gauss map of $A$ is

$$N^A(\xi) = \frac{1}{1 + |g(\xi)|^2} \left( 2 \text{Re } g(\xi), 2 \text{Im } g(\xi), |g(\xi)|^2 - 1 \right).$$

Also, recall that for $p = (x, y, z) \in C \cap \mathbb{R}^3$, $\sin \theta(p) = \frac{1}{\cosh z}$. For any $(x, y, z) = F(\xi)$, $\cos \theta(z) = N^C \cdot (0, 0, 1) = |\xi|^2 - 1 / 1 + |\xi|^2$, so

$$\sin \theta(z) = \sqrt{1 - \cos^2 \theta(z)} = \frac{2|\xi|}{1 + |\xi|^2}. \quad (1)$$

Similarly define the angle $\Psi(\xi)$ such that $\cos \Psi(\xi) = N^A \cdot (0, 0, 1) = |g(\xi)|^2 - 1 / 1 + |g(\xi)|^2$. Then

$$\sin \Psi(\xi) = \sqrt{1 - \cos^2 \Psi(\xi)} = \frac{2|g(\xi)|}{1 + |g(\xi)|^2}. \quad (2)$$

Since for $p = (x, y, z) \in A$, $z = G_3(\xi) = F_3(\xi^a \cdot \exp b) = a \log |\xi| + b$, for some $a > 0$, $b \geq 0$, then

$$\sin \Psi(\xi) = \frac{|\xi^a \cdot \exp b|}{|g(\xi)|} \cdot \left( \frac{1 + 1/|\xi^a \cdot \exp b|^2}{1 + 1/|g(\xi)|^2} \right). \quad (3)$$

Choose a positive integer $m > a$. Since $(\xi^a \cdot \exp b)/g(\xi)$ has an essential singularity at $\infty$, there is a divergent sequence $(\xi^m_n)$ such that $|\xi^a_n \cdot \exp b|/|g(\xi^m_n)| \to 0$ as $n \to \infty$. Delete a ray in $C$ such that $l$ does not contain any $\xi^m_n$. Then on $C - l$, $\xi^m$ is well-defined and

$$\frac{|\xi^m \cdot \exp b|}{|g(\xi^m)|} \to 0 \quad (4)$$

as $n \to \infty$. In particular, $g(\xi^m_n) \to \infty$ as $n \to 0$. So $\theta(F_3(\xi^m_n \cdot \exp b)) \to 0$, $\Psi(\xi^m_n) \to 0$ as $n \to \infty$. We see by (3) and (4) that

$$\frac{\Psi(\xi^m_n)}{\sin \theta(F_3(\xi^m_n \cdot \exp b))} = \frac{\Psi(\xi^m_n)}{\sin \Psi(\xi^m_n)} \cdot \frac{\sin \Psi(\xi^m_n)}{\sin \theta(F_3(\xi^m_n \cdot \exp b))} \to 0, \quad (5)$$

as $n \to \infty$. Here $\sin \theta(F_3(\xi^m_n \cdot \exp b)) = \sin \theta(z_n) = 1 / \cosh z_n$, and $z_n = F_3(\xi^m_n \cdot \exp b) = G_3(\xi^m_n)$. The larger the $n$, the bigger the value of $G_3(\xi^m_n)$. By the argument in the proof of Lemma 2.3, we can choose $n$ so large that the tangent plane of $A$ at $G(\xi^m_n)$ does not intersect $\partial A$. By (5), we can also choose the $n$ so that

$$\Psi(\xi^m_n) < 1/16.$$  

It follows from Lemma 2.3 that the tangent plane of $A$ at $G(\xi^m_n)$ will have a compact
component that contains $G(\xi_i)$. The existence of such a tangent plane contradicts Lemma 2.2. This contradiction proves the theorem.

Remark 2.1. Rosenberg and Toubiana [11] have shown that there exist minimally immersed annuli in $\mathbb{R}^3$ with proper third coordinate function and that have infinite total curvature. Even though their examples have proper third coordinate function, Theorem 1.1 shows that they do not lie above any catenoid.

§3. SOME APPLICATIONS OF THEOREM 1.1

In this section we will apply Theorem 1.1 to derive some global results.

Theorem 3.1. Suppose $f: M \to \mathbb{R}^3$ is a smooth properly immersed minimal surface with smooth compact boundary and having finite topology. A sufficient condition for $M$ to have finite total curvature is that $f(M)$ intersects some catenoid in a compact set. If $M$ is embedded, this is also a necessary condition.

Proof. If $M$ is embedded, has finite total curvature, and compact boundary, then the ends of $M$ have a well-defined tangent plane parallel to a fixed plane $P$, say it is the $xy$-plane. Furthermore, annular end representatives of $M$ can be chosen to be graphs over $P$, each of some fixed logarithmic growth in terms of $\| (x, y) \|$ (see [12]). Any catenoid $C$ with waist circle $P \cap C$ and whose ends are graphs over $P$ with logarithmic growth greater than the logarithmic growth of all the ends of $M$, must intersect $M$ in a compact set. This proves the necessary part of the theorem.

Now suppose that $C$ is a catenoid such that $B = C \cap f(M)$ is compact. After removing a small regular neighborhood of $f^{-1}(B)$ from $M$, we may assume that each component of $f(M)$ is disjoint from $C$. Since $M$ has finite topology, we may assume that, without loss of generality, $M$ is connected and $f(M) \cap C = \emptyset$. Let $X$ and $Y$ be the closures of the components of $\mathbb{R}^3 - C$ and assume $X$ is the component that contains the symmetry axis of $C$. Thus either $f(M) \subset X$ or $f(M) \subset Y$. For the first case we apply Theorem 1.1 (in fact every annular end has a representative contained in the intersection of $X$ with a halfspace). For the second case we can use a theorem of Hoffman and Meeks (Lemma 4 (The Cone Lemma) in [4]) that states that any proper minimally immersed annulus in $Y$ has finite total curvature.

Theorem 3.2. Let $f: M \to \mathbb{R}^3$ be a smooth properly embedded minimal surface with smooth compact boundary and having finite topology. Suppose $M$ has two catenoid ends, each a graph over the $xy$-plane of opposite signed logarithmic growth. Then $M$ has finite total curvature.

Proof. In this case we may assume that $M$ has a catenoid end $E_+$ with positive $z$-coordinate and an end $E_-$ with negative $z$-coordinate. Since $M$ is proper, every end of $M$ eventually is contained in the region above $E_+$, below $E_-$ or in the region between $E_+$ and $E_-$. As in the proof of Theorem 3.1, all of the ends of $M$ must have finite total curvature.

Remark 3.1. We believe that a properly embedded minimal annulus $A \subset (\mathbb{R}^3 - X \cap \mathbb{H})$ must have finite total curvature. If that is true, then with the Annular End Theorem of Hoffman and Meeks, (for details see Theorem 1 in [4]), one can prove the following conjecture of Hoffman and Meeks:
Conjecture 3.1. Suppose $f: M \to \mathbb{R}^3$ is a properly embedded minimal surface of finite topology and more than one end. Then $M$ has finite total curvature.

Remark 3.2. It is well-known that a complete minimal surface with finite total curvature is conformally equivalent to a closed Riemann surface punctured in a finite number of points (see [10]). Hence, Conjecture 1 is equivalent to the following statement. A properly embedded minimal surface $M$ with more than one end has finite total curvature if and only if it has finite topology.

§4. THE PROOFS OF THE REMAINING THEOREMS

Let $A$ be the annulus $\{z = (x + iy) \in \mathbb{C} | 1 \leq |z| < 3\}$ and let $\Gamma = \partial A = \{z \in A | |z| = 1\}$ be its boundary. Let $X: A \to \mathbb{R}^3$ be a smooth proper minimal immersion of $A$ such that $X(A)$ is not contained in a plane. The annulus $A$ has a conformal structure that is induced by the pulled back metric of the immersion $X$ and in this conformal structure the coordinate functions are harmonic. Let $H^* = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_3 > 0\}$ be the upper halfspace in $\mathbb{R}^3$ and let $P_t = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_3 = t\}$. Further suppose $X(A) \subset H^*$. We will study the geometry of such minimal annuli. First we have

**Lemma 4.1.** Let $D$ be an open subset of $A$. Suppose that for some $t > 0$, $\partial D \subset X^{-1}(t)$ and $D \cap X^{-1}(t) = \emptyset$. Then for any $p \in D$, $X(p) > t$, i.e., if $X(\partial D) \subset P_t$, then $X(D)$ must lie above $P_t$.

**Proof.** Our proof of this lemma is taken from an argument given in the proof of the Halfspace Theorem in [5] that a proper immersed minimal surface in $\mathbb{R}^3$ that is contained in a halfspace must be a plane. Without loss of generality, we may assume that $D$ is connected. Suppose that there is a point $p \in D$ such that $X(p) < t$. Then $\inf_{p \in D}(X(p)) = t_0 \geq 0$ where $t_0 < t$. By the maximum principle, $X|_D > t_0$. Hence there exists a divergent sequence of points, $p_n \in D$, such that $X(p_n) \to t_0$. Since $X|D$ is proper, $X(D)$ is a closed subset of $\mathbb{R}^3$. It follows that we can choose a small ball $B$ of radius $\varepsilon > 0$ centered at $(0, 0, t_0)$ such that $B$ is disjoint from $X(D) \cup P_t$. Suppose $n$ is chosen large enough so that $X(p_n) < t_0 + \varepsilon/2$ and let $C$ be a small circle in $P_{t_0 + \varepsilon/2} \cap B$ centered at $(0, 0, t_0 + \varepsilon/2)$. Note that there exists a proper region $\mathcal{C}$ of a catenoid with $\partial \mathcal{C} = C$, $X(p_n)$ lies below $\mathcal{C}$, $\mathcal{C}$ is a graph over its projection onto $\partial H^*$ and the third coordinate of $\mathcal{C}$ is bounded from above by $t_0 + \varepsilon/2$. (See Fig. 1 for a picture. Note that $\partial \mathcal{C}$ is not necessarily the waist circle of the catenoid. In fact, there is one-parameter family of half-catenoids with boundary $C$ and we are choosing one that is flat enough to lie above $X(p_n)$.) Let $\mathcal{C}_s = \{p + (0, 0, s) | p \in \mathcal{C}\}$ denote the vertical translate of $\mathcal{C}$. Since $\mathcal{C} \cap \widetilde{H}^*$ is compact, $\mathcal{C} \cap X(D)$ is compact and therefore there is a smallest $s$, say $s_0$, such that $\mathcal{C}_{s_0} \cap X(D) \neq \emptyset$. Since the projection of $\mathcal{C} \cap X(D)$ onto $\partial H^*$ lies in the interior of the projection of $\mathcal{C} \cap \widetilde{H}^*$ onto $\partial H^*$, $\mathcal{C} \cap \widetilde{H}^*$ is a graph, and
$\mathcal{C}_\infty \cap X(D) \subset \text{Int}(\mathcal{C}_\infty \cap \text{Int}(X(D)))$, the maximum principle implies that $X(D) \subset \mathcal{C}_\infty$, a contradiction.\qed

We will fix some notation. We say $p \in A$ is a vertical point if $\frac{\partial X}{\partial x}(p) = 0$ and $\frac{\partial X}{\partial y}(p) = 0$, or equivalently, $X_*(T_pA)$ is a plane parallel to $\partial H^+$. Let $V$ be the set of all vertical points in $A$. Let $Y_t = X^{-1}(t) \subset A$ denote the level sets of $X_*$ for any $t > 0$ and let $V_t = V \cap Y_t$. An essential fact we need is that if $p \in V$ is a vertical point, then there is a cross or higher order singularity in $Y_t$ at $p$. In fact, if the Gauss curvature of $X(A)$ at $p$ is non-zero, then in $Y_t$ near $p$ there are two curves that cross orthogonally at $p$; if the Gauss curvature of $X(A)$ at $p$ is zero, then, near $p$, $Y_t$ consists of $k$ curves that cross at equal angles at $p$ and $k \geq 2$. $Y_t$ separates $A - Y_t$ into components, we will call them regions, on which $X_*$ is either greater than $t$ or less than $t$. For a fixed value of $t$, the corresponding regions will be called up regions or down regions depending on the value of $X_*$ on these regions. We say two regions in $A - Y_t$ are adjacent if they have an arc as part of their common boundary. By the maximum principle, for two adjacent regions, one is an up region and the other is a down region. Clearly every vertical point in $V_t$ is contained in the boundary of at least one down region. The following lemma further clarifies this property.

**Lemma 4.2.** If a vertical point $p \in V_t$ is contained in the boundary of only one down region, say $D$, then there is an embedded loop $\alpha$ in $D \cup \{p\}$ passing through $p$. Furthermore,

1. $\alpha$ can be chosen homotopic to $\Gamma$ in $A$;
2. $D$ is simply connected;
3. If $A_1$ and $A_2$ are the two subannuli of $A - \alpha$, labelled so that $\Gamma \subset A_1$, then $V_1 \cap A_2 = \emptyset$.

In particular, $V_t$ must be a finite set.

**Proof.** If $p \in Y_t$ is a vertical point, then there is a cross at $p$ in $Y_t$. Hence there is a small disk $K \subset \text{Int}(A)$ and centered at $p$, such that $Y_t \cap K$ divides $K - Y_t$ into at least 4 components and each of these components has $p$ in its boundary. Among these components there are at least two down components, say $E_1$ and $E_2$. If $p$ is contained in the boundary of only one down region in $A - Y_t$, then $E_1 \cup E_2$ is contained in the same down region, which we call $D$. Hence for any two points $p_1 \in E_1$, $p_2 \in E_2$, there is an embedded arc $\gamma_1$ in $D$ that joins these two points.

Since $p \in \partial E_1 \cap \partial E_2$, there exists an arc $\gamma_2$ in $K$ joining $p_1$ and $p_2$ such that $\gamma_2 \cap Y_t = \{p\}$ and $\alpha = \gamma_1 \cup \gamma_2$ is an embedded loop in $D \cup \{p\}$. Suppose $\alpha$ is homotopically trivial in $A$. Then $\alpha$ is the boundary of an open disk $F \subset A$ and the intersection $Y_t \cap F$ is a non-empty subvariety of $F$. Since $X_3 | F \leq t$, the maximum principle implies $X_3 | F < t$. Hence $F \cap Y_t = \emptyset$, a contradiction. Thus, $\alpha$ is not homotopically trivial in $A$. By elementary topology, $\alpha$ is homotopic to $\Gamma$.

Suppose $D$ is not simply connected. Let $\gamma$ be a simple closed curve in $D - \alpha$ that is not contractible in $D$. If $\gamma$ bounds a disk $K$ in $A$, then this disk must contain part of $\partial D$, an impossibility by the maximum principle since $X_3 | \partial K < t$. Hence, $\alpha \cup \gamma$ is the boundary of a subannulus $A'$ of $A$ with $Y_t \cap \text{Int}(A') \neq \emptyset$. Again, the maximum principle shows that this cannot happen. This contradiction proves that $D$ is simply connected.

The curve $\alpha$ separates $A$ into two annular components $A_1$ and $A_2$, where we let $A_1$ be the one whose boundary contains $\Gamma$. If $V_1 \cap A_2 \neq \emptyset$, then there is a $q \in V_1 \cap A_2$. The vertical point $q$ is contained in the boundary of a down region $E$. We claim $E = D$. By Lemma 4.1, $\partial E \cap \Gamma \neq \emptyset$, so we can connect $q$ to $\Gamma$ by a path $\beta$ in $E$ such that $(\beta \cap \partial E) - \Gamma = \{q\}$. Since $\alpha$
separates \( q \) and \( \Gamma, \beta \cap (z - \{ p \}) \neq \emptyset \) and so \( E \cap D \neq \emptyset \). We conclude that \( E = D \). This implies \( q \) is contained in the boundary of exactly one down region. The argument given for the construction of the loop \( z \) shows that there exists an embedded loop \( \gamma \subset D \cup \{ q \} \) that is homotopic to \( \Gamma \) and hence \( \gamma \) is homotopic to \( z \). Clearly we can choose \( \gamma \) so that \( \gamma \cap z = \emptyset \). The curves \( z \) and \( \gamma \) bound a subannulus \( A_0 \) of \( A \) with \( X_3|A_0| \leq t \). Since \( X_3 \) is a harmonic function, \( X_3|A_0| \leq t \). This contradicts the fact that there exists a \( w \in A_0 \) near \( q \) with \( X_3(w) > t \). This contradiction proves this lemma.

**Remark 4.1.** The proof of Lemma 4.2 also shows that if \( D \) is a down region in \( A - Y \) and \( \hat{D} \) is not simply connected, then \( V_i \) is finite. In particular, if \( V_i \) is infinite, then the boundary of every down region of \( A - Y_i \) is an embedded one-manifold. We will use this remark later.

In what follows, we will consider connected components of \( A - Y_i \) and of \( Y_i \). We will use "region" to refer to a component of \( A - Y_i \) and "component" to refer to a component of \( Y_i \).

**Lemma 4.3.** Let \( T = \max \{ X_3(p) \mid p \in \Gamma \} \). If \( X_3(q) = t \) and \( t > T \), then \( q \) is not a vertical point.

**Proof.** \( X_3 \) is a harmonic function (in the induced conformal structure) and the vertical points of \( A \) are critical points of \( X_3 \). If there exists a vertical point \( q \) such that \( X_3(q) > t \), then \( Y_i \) is a singular analytic 1-complex in \( A \) which is disjoint from \( \Gamma \). Any harmonic function \( f \) on \( A \) with \( f|\partial A| \leq T \) and with a critical point \( q' \) with \( t = f(q') > T \), has the property that there is a component \( D \) of \( f^{-1}((\infty, t)) \) with \( \partial D \subset f^{-1}(t) \). By letting \( f = X_3 \), the existence of \( D \) contradicts Lemma 4.1, and proves the lemma.

**Corollary 4.1.** If there is a piecewise smooth embedded loop \( \gamma \) in \( A \), homotopic to \( \Gamma \), and \( X_3|\gamma| \) is constant, then there are no vertical points outside \( \gamma \) in \( A \). In particular, if \( X_3|\Gamma| \) is constant, then there are no vertical points in \( A \).

**Proof.** Notice that the previous lemmas are true when \( X(A) \) has a piecewise smooth boundary. Let \( A \) be the component of \( A - \gamma \) which is disjoint from \( \Gamma \). Suppose \( T = X_3|\gamma| \). By Lemma 4.1 and the maximum principle, \( X_3|A| > T \) and by Lemma 4.3, \( A \) contains no vertical points. This proves the first statement in the corollary.

Suppose \( X_3|\Gamma| \) is constant. Since \( X(A) \) is smooth and \( X_3 \geq X_3|\Gamma| \), the Hopf boundary maximum principle implies that there are no vertical points on \( \Gamma \). This proves the last statement in the lemma.

If the boundary of \( X(A) \) is contained in a plane parallel to \( \partial H^* \), we have proved there are no vertical points, so we will only consider the case that \( X(\Gamma) \) is not contained in any \( P_r \). We will further assume that \( X_3|\partial A \) has only a finite number of critical points. This assumption on \( X_3|\partial A \) holds when \( X(\Gamma) \) is an analytic curve and \( X_3|\partial A \) is not constant.

**Lemma 4.4.** For every \( t > 0 \), \( V_t \) is a finite set.

**Proof.** Fix some \( t > 0 \). Notice that \( p \in V_t \) means that the tangent plane of \( X(p) \) is \( P_r \). We observe that \( Y_i \cap \Gamma \) must be a finite set, since \( X_3|\Gamma| \) has a finite number of critical points. By Lemma 4.1, the boundary of every down region of \( A - Y_i \) must contain an open arc of \( \Gamma \). Since \( \Gamma - Y_i \) consists of a finite number of components and each component of \( \Gamma - Y_i \), can be contained in the boundary of only one region of \( A - Y_i \), there are only a finite number of down regions. Let \( D_1, D_2, \ldots, D_n \) be the down regions of \( A - Y_i \). Since each vertical point is contained in the boundary of at least one down region, we have \( V_t \subset \bigcup_{i=1}^n \partial D_i \).
Suppose \( V_i \) is an infinite set. Since \( V_i \cup \bigcup_{i=1}^{\infty} \partial D_i \) there is at least one \( D_i \), say \( D_i \), such that \( \partial D_i \) contains an infinite number of points of \( V_i \). Choose an infinite sequence \( \{p_k\}_{k=1}^{\infty} \subset V_i \cap \partial D_i \). By Remark 4.1, \( D_i \) is simply connected and \( \partial D_i \) is an embedded one-manifold. Since \( D_i \) is simply connected, \( \partial D_i \) contains exactly one component, say \( \Delta_i \), that intersects \( \Gamma \) in a nonempty set. Let \( E \) be the component of \( A - \Delta_i \) that contains \( D_i \). Since \( E \) is disjoint from \( \Gamma \), Lemma 4.1 implies \( E \cap V_i = \emptyset \). Hence, \( V_i \cap \partial D_i \subset \Delta_i \).

By Lemma 4.2, each vertical point in \( V_i \) is contained in the boundary of at least two down regions, and so \( \{p_k\}_{k=1}^{\infty} \subset \bigcup_{i=2}^{\infty} \partial D_i \). Thus we can assume that there is a \( D_2 \) with an infinite subsequence \( \{p_k\}_{k=1}^{\infty} \subset \partial D_2 \). As before, there is a component \( \Delta_2 \) of \( \partial D_2 \) such that \( V_i \cap \partial D_2 \subset \Delta_2 \). Since \( \Delta_1 \cap \Delta_2 \cap \Gamma = \emptyset \) and \( \Delta_1 \cap \Delta_2 \) is a discrete set containing more than 2 points, \( A - \Delta \) must contain a component \( F \) whose closure is a compact disk and such that \( \partial F \cap \Gamma = \emptyset \). Since \( X_j \cap F = t \), the maximum principle implies \( X_j \) is constant. This contradiction proves this lemma.

**Lemma 4.4** tells us that each level set \( X_j^{-1}(t) \subset A \) contains only a finite number of vertical points. Our next step is to prove that \( V_i \) is nonempty for only a finite number of values of \( t \). We make a definition that is useful in proving this step.

**Definition 4.1** Let \( \mathcal{D}(t) \) denote the collection of all down regions in \( A - Y_t \) which have a vertical point on their boundary. Let \( \mathcal{D} \) be the union of the \( \mathcal{D}(t) \) for which \( t \) is not a critical point for \( X_j \). For \( D_1, D_2 \in \mathcal{D} \), we will write \( D_1 \leq D_2 \) if \( \bar{D}_1 \subset \bar{D}_2 \).

It follows immediately from the definition of the partial ordering \( \leq \) that if \( D_1, D_2 \in \mathcal{D} \) and \( D_1 \cap D_2 \neq \emptyset \), then \( D_1 \leq D_2 \) or \( D_2 \leq D_1 \). In particular, if \( D_1 \leq D_2 \) and \( D_1 \leq D_3 \), then \( D_2 \leq D_3 \) or \( D_3 \leq D_2 \). This ordering property implies that if \( \mathcal{D} \) has a finite number of minimal elements and each chain \( D_1 < D_2 < \ldots < D_n \) has bounded length, then \( \mathcal{D} \) is a finite set. Thus the next lemma implies \( \mathcal{D} \) is a finite set.

**Lemma 4.5.** Let \( N(\Gamma) \) denote the number of local minima of \( X_j \cap \Gamma \).

1. \( \mathcal{D} \) has at most \( N(\Gamma) \) minimal elements.
2. If \( D_1 < D_2 < \ldots < D_n \) is a chain in \( \mathcal{D} \), then \( n \leq N(\Gamma) + 1 \).

**Proof.** Since the minimal elements are disjoint from each other and a minimal element \( D \in \mathcal{D} \) has a local minimum in the interior of \( \partial D \cap \Gamma \), there can be no more than \( N(\Gamma) \) minimal elements. This proves the first statement in the lemma.

Consider a chain \( D_1 < D_2 < \ldots < D_n \) for some integer \( n \). Suppose for every \( i, 1 \leq i \leq n \), that \( D_i \in \mathcal{D}(t_i) \) and \( p_i \) is one of the vertical points in \( \partial D_i \). By Lemma 4.2, if \( p_i \) is contained in the boundary of only one down region, \( D_i \), in \( \mathcal{D}(t_i) \), then \( D_i \) is simply connected and furthermore any open subdomain containing \( \bar{D}_i \) is not simply connected. Since \( \bar{D}_i \subset \bar{D}_{i+1} \) for all \( j \leq n - 1 \), it follows from the previous statement that at most one \( p_i \) is contained in the boundary of exactly one down region.

Suppose for the moment that \( p_i \) is in the boundary of another down region \( \bar{D}_i \in \mathcal{D}(t_k) \). In this case there exists a local minima \( m_k+1 \in \partial \bar{D}_k \cap \Gamma \) for \( X_j \cap \Gamma \). Note that \( \bar{D}_k \subset B_{k+1} \) for \( k < n \), and hence \( m_{k+1} \in \partial D_{k+1} \cap \Gamma \) when \( k < n \). Let \( m_k \in \partial D_k \) be a local minimum of \( X_j \cap \Gamma \). Since \( m_k \) is defined for every \( i \) in \( \{1, \ldots, n\} \) except possibly one and the \( m_i \) are distinct local minimum of \( X_j \cap \Gamma \), then \( n \leq N(\Gamma) + 1 \). 

**Proof of Theorem 1.3.** After rotation of \( \mathbb{R}^3 \) assume that \( M \) is contained in \( H^+ \). Since \( X: M \rightarrow \mathbb{R}^3 \) is an analytic minimal immersion of analytic surface, the vertical points of \( M \) are isolated. Hence, on any compact subset of \( M \) there are only a finite number of vertical
normal vectors. Choose representative analytic annular ends $E_1, \ldots, E_n$ for $M$ and such the boundary curves of these ends are not contained in planes. We need to prove that each $E_i$ contains only a finite number of vertical points.

For any fixed integer $i, 1 \leq i \leq n$, let $A = E_i$. Then $A$ satisfies the conditions for the annulus $A$ discussed in Lemmas 4.1-4.5. By Lemma 4.4, $V_i$ is a finite set for every value $t$. On the other hand, Lemma 4.5 implies that $\mathcal{G}$ is a finite set, as we have already observed in the paragraph that follows Definition 4.1. Hence, for some $k, \mathcal{G} = \bigcup_{i=1}^{k} \mathcal{G}(t_i)$. If $t_{k+1}, \ldots, t_{k+n}$ are the critical values of $X_3|_{\Gamma}$, then $V \subseteq \bigcup_{i=1}^{k} V_i$. This proves $V$ is a finite set which completes the proof of Theorem 1.3.

Proof of Theorem 1.2. Given two properly embedded minimal annuli $A_1, A_2$ each with compact boundary curve, there exists a standard barrier between them (See Corollary 2 in [4]). This just means that there exists a half-catenoid or a plane $C$ such that outside of a sufficiently large ball $B$ the barrier $C$ is disjoint from $A_1 \cup A_2$ and also $C \cup B$ separates $A_1 - B$ from $A_2 - B$. Now consider the two annular ends $E_1$ and $E_2$ of $M$ with infinite total curvature and let $C$ be a standard barrier between them. Since $E_1$ and $E_2$ have infinite total curvature, Theorem 1.1 implies $C$ must be a plane. Since $C$ is disjoint from $E_1 \cup E_2$ outside of some ball, $C \cap (E_1 \cup E_2)$ is compact. Hence, after removing compact subannuli of $E_1$ and $E_2$, we may choose $E_1$ and $E_2$ to lie in the disjoint halfspaces determined by $C$. The weak maximum principle at infinity [7] states: If two properly immersed minimal surfaces with compact boundaries in a flat three-manifold are disjoint, they stay a bounded distance apart. Therefore, the distance from $C$ to $E_1 \cup E_2$ is greater than some $\varepsilon > 0$. It follows that we can choose closed disjoint halfspaces $H_1, H_2$ with $E_1 \subseteq H_1$ and $E_2 \subseteq H_2$. This proves the first statement in the theorem.

Suppose now that $E_3$ is another annular end of $M$ that is disjoint from $E_1$ and $E_2$. Corollary 3 in [4] states: Suppose $M_1, M_2, M_3$ are three pairwise-disjoint, properly embedded minimal surfaces in $\mathbb{R}^3$, each has compact boundary and one end. Then at least one of the surfaces lies between two standard barriers. On the other hand, Corollary 2 in [4] states that a minimal annulus between two standard barriers has finite total curvature. It is evident that $E_3$ has finite total curvature and lies between $E_1$ and $E_2$ (outside of some ball $B$). If $E_3$ asymptotic to a half-catenoid, then either $E_1$ or $E_2$ lies above a catenoid. By Theorem 1.1, $E_1$ or $E_2$ has finite total curvature which contradicts our hypotheses. Hence, $E_3$ is asymptotic to a flat plane $P$. By the weak maximum principle at infinity ([7]) the end of this plane $P$ stays a positive distance from both $E_1$ and $E_2$. This implies that $P$ intersects both $E_1$ and $E_2$ in a compact set and hence $E_1$ and $E_2$ have proper subends that are a positive distance from this plane. Theorem 3 in [5] states that a proper noncompact minimal surface with compact boundary in a halfspace of $\mathbb{R}^3$ has a convex hull that is a halfspace or a slab. Since $E_1$ and $E_2$ have proper subends that are disjoint from the plane $P$ and their convex hulls are halfspaces or slabs, we see that $P$ must be parallel to $\partial \mathbb{H}_1$. Since $E_3$ is an arbitrary annular end different from $E_1$ and $E_2$, the second part of Theorem 1.2 is proved.

Since $Y = M - E_1 \cup E_2$ has finite total curvature, $Y$ has only a finite number of tangent planes that are parallel to $\partial \mathbb{H}_1$. Since we can choose $E_1$, and also $E_2$, to satisfy the hypotheses of Theorem 1.3, $E_1 \cup E_2$ has only a finite number of tangent planes parallel to $\partial \mathbb{H}_1$. Thus, $M$ has only a finite number of such tangent planes which completes the proof of Theorem 1.2.

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