Discrete Mathematics 39 (1982) 105–109
North-Holland Publishing Company

NOTE

ON THE NON-EXISTENCE OF PERFECT AND NEARLY PERFECT CODES

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Received 2 July 1980
Revised 17 March 1981

The main result of the paper is the proof of the non-existence of a class of completely regular codes in certain distance-regular graphs. Corollaries of this result establish the non-existence of perfect and nearly perfect codes in the infinite families of distance-regular graphs $J(2b+1, b)$ and $J(2b+2, b)$.

1. Introduction

The setting for this paper is the class of distance-regular graphs. The reader should consult Biggs [2] for the various concepts of distance-regularity used. Throughout the paper Γ denotes a distance-regular graph with distance function ∂, diameter d, vertex set $V\Gamma$ and intersection array

$$\begin{Bmatrix} * & 1 & c_2 \cdots c_{d-1} & c_d \\ 0 & a_1 & a_2 \cdots a_{d-1} & a_d \\ k & b_1 & b_2 \cdots b_{d-1} & * \end{Bmatrix}$$

Let $[\boldsymbol{k}]_i = k_i$ denote the number of vertices in Γ at distance i from a particular vertex of Γ $(0 \le i \le d)$.

In the next section we state some preliminary results and in Section 3 we prove the main result of the paper. The objective is to establish a purely combinatorial proof for the non-existence of certain *completely regular* codes in the subclass of distance-regular graphs which satisfy both $a_d > 0$ and $k_d < k$. Corollaries of the main theorem establish the non-existence of perfect and nearly perfect codes in the infinite families of distance-regular graphs $J(2b+1, b)$ and $J(2b+2, b)$.

2. Definitions and preliminary results

Suppose that C is an e-code ($\partial(u, v) \ge 2e+1$ for all $u, v \in C$) in the graph Γ. C is non-trivial if $|C| \ge 2$ and we shall assume throughout that $d \ge 2e+1$. We say that C has *external distance* e' if the maximal distance of any vertex of Γ from C is e'. For each $j \in \{0, 1, \ldots, e'\}$ we choose z_j in $V\Gamma$ such that $\partial(z_j, C) = j$ and we call C *locally regular* if, for $0 \leq i \leq e'$ and $0 \leq j \leq e'$, the number

$$p_{ij}(C, z_j) = |\{c \in C \mid \partial(c, z_j) = i\}|$$

depends on the values of i and j and not on the choice of z_j. We say that C is *completely regular* if the same condition holds for $0 \leq i \leq d$. In either case we write $p_{ij}(C, z_j)$ as $p_{ij}(C)$ for the relevant ranges of i and j. It is shown in [3] that a locally regular code is completely regular. The following two Lemmas, also from [3], were originally stated for antipodal distance-regular graphs. However, it can be seen from their proofs that they are also valid for an arbitrary distance-regular graph.

Lemma. *If C is a locally regular e-code with external distance $e+m$, then there exist rational numbers $\gamma_{e+1}, \ldots, \gamma_{e+m}$ such that*

$$\sum_{j=0}^{e} k_j p_{ij}(C) + \sum_{q=1}^{m} \gamma_{e+q} p_{i,e+q}(C) = k_i \qquad (*)$$

for $e+1 \leq i \leq e+m$.

Lemma 2. *If also $m \leq e$, then* (*) *holds for $0 \leq i \leq d$.*

3. General results

We are now in a position to prove the main result of the paper.

Theorem. *If $m \leq e$, $k_d < k$ and $a_d > 0$, then Γ cannot contain a non-trivial completely regular e-code with external distance $e+m$ and with parameters which satisfy $\gamma_{e+s} > k_d$ $(1 \leq s \leq m)$.*

Proof. Suppose that Γ contains such an e-code. The dth component of the result of Lemma 2 gives

$$k_d = p_{d0}(C) + k p_{d1}(C) + \cdots + k_e p_{de}(C) + \sum_{s=1}^{m} \gamma_{e+s} p_{d,e+s}(C)$$

But $d \geq 2e+1$ and so $k_i \geq k > k_d$ $(1 \leq i \leq e)$ and if $\gamma_{e+s} > k_d$ $(1 \leq s \leq m)$, then $p_{d0}(C) = k_d$. However, $a_d > 0$ now implies that C contains adjacent vertices, an obvious contradiction. □

A *perfect e-code* is a completely regular e-code with external distance e and a *uniformly packed e-code* (first investigated in [5]) is a completely regular e-code with external distance $e+1$. Associated with a uniformly packed e-code are the parameters $\lambda = p_{e+1,e}(C)$, $\mu = p_{e+1,e+1}(C)$. If $\lambda = [b_e/c_{e+1}]$ and $\mu = [k/c_{e+1}]$, then C is called *nearly perfect* [4]. Before we investigate the existence of perfect and nearly perfect codes in particular families of distance-regular graphs, we prove two general results derived from the theorem above. Throughout the rest of the paper we shall assume that $a_d > 0$ and $k_d < k$.

Corollary 1. *Γ cannot contain a non-trivial perfect e-code for $e \geq 1$.*

Proof. The result follows immediately from the theorem with $m = 0$, since the conditions on the γ's are vacuous. □

Corollary 2. *If Γ contains a non-trivial uniformly packed e-code with parameters λ and μ, then*

$$\mu k_d + \lambda k_e \geq k_{e+1}.$$

Proof. Suppose that C is a non-trivial uniformly packed e-code with parameters λ and μ. Then $p_{e+1,j}(C) = 0$ for $j \leq e-1$ and Lemma 1 with $i = e+1$ gives

$$k_{e+1} = k_e\lambda + \mu\gamma_{e+1} \leq k_e\lambda + k_d\mu$$

by the theorem above. □

4. The graphs $J(a, b)$

The graph $J(a, b)$ has $\binom{a}{b}$ vertices indexed by the subsets of cardinality b of the set $\{1, \dots, a\}$. Two vertices of $J(a, b)$ are adjacent if and only if they have $b-1$ elements in common and the distance function ∂ is defined

$$\partial(u, v) = b - |u \cap v|.$$

$J(a, b)$ is distance-regular for $a \geq 2b$ and has intersection array

$$\begin{Bmatrix} * & \cdots & i^2 & \cdots & b^2 \\ 0 & \cdots & i(a-2i) & \cdots & b(a-2b) \\ b(a-b) & \cdots & (b-i)(a-b-i) & \cdots & * \end{Bmatrix}$$

4.1. *Perfect codes in $J(a, b)$*

In [1] Bannai proves the non-existence of perfect e-codes in $J(2b+1, b)$ for $e \geq 2$. The proof in [1] uses an analogue of Lloyd's theorem [3] and also some number-theoretic results. We illustrate a purely combinatorial proof of Bannai's result and at the same time prove the non-existence of perfect 1-codes in $J(2b+1, b)$ and perfect e-codes in $J(2b+2, b)$ for $e \geq 1$.

Corollary 3. *$J(a, b)$ cannot contain a non-trivial perfect e-code for $e \geq 1$ and $2b+1 \leq a \leq 2b+2$.*

Proof. For the graph $J(a, b)$ we have $k = b(a-b)$, $k_d = \binom{a-b}{b}$ and $a_d = b(a-2b)$. Obviously $a_d > 0$ for $a > 2b$ and it is easy to check that $k_d < k$ for $2b+1 \leqslant a \leqslant 2b+2$. The result follows from Corollary 1. □

4.2. *Nearly perfect codes in $J(a, b)$*

A nearly perfect e-code in Γ is perfect if and only if $b_e \equiv 0 \pmod{c_{e+1}}$. In [4] it is shown that $J(a, b)$ does not contain a non-trivial nearly perfect 1-code with $b_1 \not\equiv 0 \pmod{c_2}$. For the cases $2b+1 \leqslant a \leqslant 2b+2$ we can extend this to the following.

Corollary 4. *$J(a, b)$ cannot contain a non-trivial nearly perfect e-code for $e \geqslant 1$ when $a \in \{2b+1, 2b+2\}$.*

Proof. We can immediately assume that $e \geqslant 2$. Let $a = 2b + r$. If $b_e \not\equiv 0 \pmod{c_{e+1}}$, then

$$\gamma_{e+1} = \frac{k_e}{[k/c_{e+1}]}(b_e/c_{e+1} - [b_e/c_{e+1}]) \geqslant k_e/k.$$

Now, if $e \geqslant r+1$, then $b \geqslant 2e+1 \geqslant 2r+3$ and $k_e/k \geqslant k_{r+1}/k$. We need only prove that $k_{r+1}/k > k_d$ which is equivalent in this case to proving that

$$\binom{b}{r+1} > (r+1)(b+r).$$

In fact, it is not difficult to see that this is true for all $b \geqslant 7$ with $r = 2$ and for all $b > 5$ with $r = 1$. When $r = 1$ and $b = 5$, and hence $e = 2$, the non-integrability of $|C|$ ([4, p. 44]) rules out this case. When $e = 2, a = 2b+2$ and $b_2 = b(b-2) \equiv \alpha \pmod 9$, where $\alpha \in \{3, 6, 8\}$, we have

$$\gamma_3/k_d = \binom{b}{2}^{\alpha} \Big/ 9\left[\frac{b(b+2)}{9}\right].$$

It is a simple, but tedious, task to verify that $\gamma_3/k_d > 1$ for $b \equiv \beta \pmod 9$ with $\beta \in \{1, 3, 4, 5, 6, 7, 8\}$. □

5. Conclusion

We have shown that the proof of the non-existence of perfect and nearly perfect codes in the families $J(2b+1, b)$ and $J(2b+2, b)$ can be derived from the theorem in Section 3. We mention one further example in the infinite family of distance-regular graphs $\bar{Q}_m$. The graph Q_m is the generalised cube of dimension m and $\bar{Q}_m$ is obtained from Q_m by joining vertices at distance 2 in Q_m. $\bar{Q}_m$ is distance-regular with diameter $[\frac{1}{2}m]$ and Bannai has proved that $\bar{Q}_m$ does not contain a non-trivial perfect e-code for $e \geqslant 1$. Unfortunately our Corollary 1 guarantees the non-existence of perfect e-codes in $\bar{Q}_m$ only when m is odd, otherwise $a_d = 0$. However, we can derive the following for nearly perfect e-codes: $\bar{Q}_m$ (m *odd*) *cannot contain a non-trivial nearly perfect e-code with* $e \geqslant 2$ *and* $m \geqslant 17$.

It seems likely that other non-existence results can be obtained just as simply from the main theorem.

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