



# A sufficient condition for the existence of an anti-directed 2-factor in a directed graph

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## ABSTRACT

Let  $D$  be a directed graph with vertex set  $V$ , arc set  $A$ , and order  $n$ . The *graph underlying*  $D$  is the graph obtained from  $D$  by replacing each arc  $(u, v) \in A$  by an undirected edge  $\{u, v\}$  and then replacing each double edge by a single edge. An *anti-directed (hamiltonian) cycle*  $H$  in  $D$  is a (hamiltonian) cycle in the graph underlying  $D$  such that no pair of consecutive arcs in  $H$  form a directed path in  $D$ . An *anti-directed 2-factor* in  $D$  is a vertex-disjoint collection of anti-directed cycles in  $D$  that span  $V$ . It was proved in Busch et al. (submitted for publication) [3] that if the indegree and the outdegree of each vertex of  $D$  is greater than  $\frac{9}{16}n$  then  $D$  contains an anti-directed Hamilton cycle. In this paper we prove that given a directed graph  $D$ , the problem of determining whether  $D$  has an anti-directed 2-factor is NP-complete, and we use a proof technique similar to the one used in Busch et al. (submitted for publication) [3] to prove that if the indegree and the outdegree of each vertex of  $D$  is greater than  $\frac{24}{46}n$  then  $D$  contains an anti-directed 2-factor.

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## 1. Introduction

Let  $G$  be a multigraph with vertex set  $V(G)$  and edge set  $E(G)$ . For a vertex  $v \in V(G)$ , the degree of  $v$  in  $G$ , denoted by  $\deg(v, G)$ , is the number of edges of  $G$  incident to  $v$ . Let  $\delta(G) = \min_{v \in V(G)} \{\deg(v, G)\}$ . The simple graph underlying  $G$ , denoted by  $\text{simp}(G)$ , is the graph obtained from  $G$  by replacing all multiple edges by single edges. A *2-factor* in  $G$  is a collection of vertex-disjoint cycles that span  $V(G)$ . Let  $D$  be a directed graph with vertex set  $V(D)$  and arc set  $A(D)$ . We recall that a directed graph  $D$  can contain arcs  $(u, v)$  and  $(v, u)$  for any two different vertices  $u, v$  but no parallel arcs. For a vertex  $v \in V(D)$ , the *outdegree* (respectively, *indegree*) of  $v$  in  $D$  denoted by  $d^+(v, D)$  (respectively,  $d^-(v, D)$ ) is the number of arcs of  $D$  directed out of  $v$  (respectively, directed into  $v$ ). Let  $\delta(D) = \min_{v \in V(D)} \{\min\{d^+(v, D), d^-(v, D)\}\}$ . The *multigraph underlying*  $D$  is the multigraph obtained from  $D$  by ignoring the directions of the arcs of  $D$ . A *directed (Hamilton) cycle*  $C$  in  $D$  is a (Hamilton) cycle in the multigraph underlying  $D$  such that all pairs of consecutive arcs in  $C$  form a directed path in  $D$ . An *anti-directed (Hamilton) cycle*  $C$  in  $D$  is a (Hamilton) cycle in the multigraph underlying  $D$  such that no pair of consecutive arcs in  $C$  form a directed path in  $D$ . A *directed 2-factor* in  $D$  is a collection of vertex-disjoint directed cycles in  $D$  that span  $V(D)$ . An *anti-directed 2-factor* in  $D$  is a collection of vertex-disjoint anti-directed cycles in  $D$  that span  $V(D)$ . Note that every anti-directed cycle in  $D$  must have an even number of vertices. We refer the reader to standard books on graph theory [1, 2, 9] for all terminology and notation that is not defined in this paper.

The following classical theorems by Dirac [6] and Ghouila-Houri [7] give sufficient conditions for the existence of a Hamilton cycle in a graph  $G$  and for the existence of a directed Hamilton cycle in a directed graph  $D$  respectively.

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**Theorem 1** ([6]). *If  $G$  is a graph of order  $n \geq 3$  and  $\delta(G) \geq \frac{n}{2}$ , then  $G$  contains a Hamilton cycle.*

**Theorem 2** ([7]). *If  $D$  is a directed graph of order  $n$  and  $\delta(D) \geq \frac{n}{2}$ , then  $D$  contains a directed Hamilton cycle.*

Note that if  $D$  is a directed graph of even order  $n$  and  $\delta(D) \geq \frac{3}{4}n$  then  $D$  contains an anti-directed Hamilton cycle. To see this, let  $G$  be the multigraph underlying  $D$  and let  $G'$  be the subgraph of  $G$  consisting of the parallel edges of  $G$ . Now,  $\delta(D) \geq \frac{3}{4}n$  implies that  $\delta(\text{simp}(G')) \geq \frac{n}{2}$  and hence **Theorem 1** implies that  $\text{simp}(G')$  contains a Hamilton cycle which in turn implies that  $D$  contains an anti-directed Hamilton cycle because for each edge  $\{u, v\}$  of  $\text{simp}(G')$  we have the directed arcs  $(u, v)$  and  $(v, u)$  in  $D$ .

The following theorem by Grant [8] gives a sufficient condition for the existence of an anti-directed Hamilton cycle in a directed graph  $D$ .

**Theorem 3** ([8]). *If  $D$  is a directed graph with even order  $n$  and if  $\delta(D) \geq \frac{2}{3}n + \sqrt{n \log(n)}$  then  $D$  contains an anti-directed Hamilton cycle.*

In his paper Grant [8] conjectured that the theorem above can be strengthened to assert that if  $D$  is a directed graph with even order  $n$  and if  $\delta(D) \geq \frac{1}{2}n$  then  $D$  contains an anti-directed Hamilton cycle. Mao-cheng Cai [4] gave a counter-example to this conjecture. In [3] the following sufficient condition for the existence of an anti-directed Hamilton cycle in a directed graph was proved.

**Theorem 4** ([3]). *Let  $D$  be a directed graph of even order  $n$  and suppose that  $\frac{1}{2} < p < \frac{3}{4}$ . If  $\delta(D) \geq pn$  and  $n > \frac{\ln(4)}{(p-\frac{1}{2}) \ln(\frac{p+\frac{1}{2}}{\frac{3}{2}-p})}$ ,*

*then  $D$  contains an anti-directed Hamilton cycle.*

It was shown in [3] that **Theorem 4** implies the following improvement on the result in **Theorem 3**.

**Corollary 1** ([3]). *If  $D$  is a directed graph of even order  $n$  and  $\delta(D) > \frac{9}{16}n$  then  $D$  contains an anti-directed Hamilton cycle.*

In this paper we seek to weaken the degree condition in **Corollary 1**, but still guarantee the existence of an anti-directed 2-factor. The following theorem (see [1]) gives a necessary and sufficient condition for the existence of a directed 2-factor in a digraph  $D$ .

**Theorem 5.** *A directed graph  $D = (V, A)$  has a directed 2-factor if and only if  $|\bigcup_{v \in X} N^+(v)| \geq |X|$  for all  $X \subseteq V$ .*

We note here that given a directed graph  $D$  the problem of determining whether  $D$  has a directed Hamilton cycle is known to be NP-complete, whereas, there exists an  $O(\sqrt{nm})$  algorithm (see [1]) to check if a directed graph  $D$  of order  $n$  and size  $m$  has a directed 2-factor. On the other hand, the following theorem proves that given a directed graph  $D$ , the problem of determining whether  $D$  has an anti-directed 2-factor is NP-complete. We are indebted to Sundar Vishwanathan [11] for pointing out the short proof of **Theorem 6** given below.

**Theorem 6.** *Given a directed graph  $D$ , the problem of determining whether  $D$  has an anti-directed 2-factor is NP-complete.*

**Proof.** Clearly the problem of determining whether  $D$  has an anti-directed 2-factor is in NP. A graph  $G$  is said to be  $k$ -edge colorable if the edges of  $G$  can be colored with  $k$  colors in such a way that no two adjacent edges receive the same color. It is well known that given a cubic graph  $G$ , it is NP-complete to determine if  $G$  is 3-edge colorable. Now, given a cubic graph  $G = (V, E)$ , construct a directed graph  $D = (V, A)$ , where for each  $\{u, v\} \in E$ , we have the oppositely directed arcs  $(u, v)$  and  $(v, u)$  in  $A$ . Now,  $G$  is 3-edge colorable if and only if  $E$  can be partitioned into 3 1-factors, or equivalently, a 1-factor and a 2-factor consisting of only even cycles. Thus it is clear that  $G$  is 3-edge colorable if and only if  $D$  contains an anti-directed 2-factor. This proves that the problem of determining whether a directed graph  $D$  has an anti-directed 2-factor is NP-complete.  $\square$

In Section 2 of this paper we prove the following theorem that gives a sufficient condition for the existence of an anti-directed 2-factor in a directed graph.

**Theorem 7.** *Let  $D$  be a directed graph of even order  $n$  and suppose that  $\frac{1}{2} < p < \frac{3}{4}$ . If  $\delta(D) \geq pn$  and  $n > \frac{\ln(4)}{(p-\frac{1}{2}) \ln(\frac{p+\frac{1}{2}}{\frac{3}{2}-p})} - \frac{1}{(p-\frac{1}{2})}$ , then  $D$  contains an anti-directed 2-factor.*

In Section 2 we will show that **Theorem 7** implies the following corollary.

**Corollary 2.** *If  $D$  is a directed graph of even order  $n$  and  $\delta(D) > \frac{24}{46}n$  then  $D$  contains an anti-directed 2-factor.*

The result in **Corollary 2** is almost certainly not the best possible. Let  $\vec{K}_k$  denote the complete directed graph on  $k$  vertices which has both oppositely directed arcs  $(u, v)$  and  $(v, u)$  for each pair of distinct vertices  $u$  and  $v$ . Let  $D(n)$  be the directed graph consisting of two disjoint copies of  $\vec{K}_{\frac{n}{2}}$  where  $n \equiv 2 \pmod{4}$ . Note that  $\delta(D(n)) = \frac{n}{2} - 1$  and that  $D(n)$  does not contain an anti-directed 2-factor. For each even integer  $n$ , Mao-cheng Cai [4] gave an example of a directed graph  $D'(n)$  on  $n$  vertices

with  $\delta(D'(n)) = \frac{n}{2}$ , and such that  $D'(n)$  does not contain an anti-directed Hamilton cycle. It is easy to see that the directed graph  $D'(6)$  given by Mao-cheng Cai does not contain an anti-directed 2-factor while  $D'(n)$  contains an anti-directed 2-factor for all  $n \geq 8$ . Based on these comments and our result in [Corollary 2](#) we offer the following conjecture.

**Conjecture 1.** *If  $D$  is a directed graph of even order  $n \geq 8$  and  $\delta(D) \geq \frac{1}{2}n$  then  $D$  contains an anti-directed 2-factor.*

## 2. Proof of Theorem 7 and its corollary

A partition of a set  $S$  with  $|S|$  being even into  $S = X \cup Y$  is an *equipartition* of  $S$  if  $|X| = |Y| = \frac{|S|}{2}$ . The proof of [Theorem 4](#) mentioned in the Introduction made extensive use of the following theorem by Chvátal [5].

**Theorem 8** ([5]). *Let  $G$  be a bipartite graph of even order  $n$  and with equipartition  $V(G) = X \cup Y$ . Let  $(d_1, d_2, \dots, d_n)$  be the degree sequence of  $G$  with  $d_1 \leq d_2 \leq \dots \leq d_n$ . If  $G$  does not contain a Hamilton cycle, then for some  $i \leq \frac{n}{4}$  we have  $d_i \leq i$  and  $d_{\frac{n}{2}} \leq \frac{n}{2} - i$ .*

We prepare for the proof of [Theorem 7](#) by proving [Theorems 10](#) and [11](#) which give necessary degree conditions (similar to those in [Theorem 8](#)) for the non-existence of a 2-factor in a bipartite graph  $G$  of even order  $n$  with equipartition  $V(G) = X \cup Y$ . Let  $G = (V, E)$  be a bipartite graph of even order  $n$  and with equipartition  $V(G) = X \cup Y$ . For  $U \subseteq X$  (respectively  $U \subseteq Y$ ) define  $N(U)$  as being the set of vertices  $v \in Y$  (respectively  $v \in X$ ) such that  $(u, v) \in E$  for some  $u \in U$ . For  $U \subseteq X$  (respectively  $U \subseteq Y$ ) define  $N^{(2)}(U)$  as being the multiset of vertices  $v \in Y$  (respectively  $v \in X$ ) such that  $(u, v) \in E$  for some  $u \in U$  and with  $v$  appearing twice in  $N^{(2)}(U)$  if there are two or more vertices  $u \in U$  with  $(u, v) \in E$  and  $v$  appearing once in  $N^{(2)}(U)$  if there is exactly one  $u \in U$  with  $(u, v) \in E$ . We will use the following theorem by Ore [10] that gives a necessary and sufficient condition for the non-existence of a 2-factor in a bipartite graph of even order  $n$  with equipartition  $V(G) = X \cup Y$ .

**Theorem 9** ([10]). *Let  $G = (V, E)$  be a bipartite graph of even order and with equipartition  $V(G) = X \cup Y$ .  $G$  contains no 2-factor if and only if there exists some  $U \subseteq X$  such that  $|N^{(2)}(U)| < 2|U|$ .*

For a bipartite graph  $G = (V, E)$  of even order  $n$  and with equipartition  $V(G) = X \cup Y$ , a set  $U \subseteq X$  or  $U \subseteq Y$  is defined to be a *deficient* set of vertices in  $G$  if  $|N^{(2)}(U)| < 2|U|$ . [Theorems 10](#) and [11](#) use [Theorem 9](#) to derive some degree conditions that are necessary for a bipartite graph to not have a 2-factor.

We first prove four lemmas that will be used in the proof of [Theorems 10](#) and [11](#).

**Lemma 1.** *Let  $G$  be a bipartite graph of even order  $n$  and with equipartition  $V(G) = X \cup Y$ . If  $U$  is a minimal deficient set of vertices in  $G$  then  $2|U| - 2 \leq |N^{(2)}(U)|$ .*

**Proof.** Clear by the minimality of  $U$ .  $\square$

**Lemma 2.** *Let  $G$  be a bipartite graph of even order  $n$  and with equipartition  $V(G) = X \cup Y$ , and let  $U$  be a minimal deficient set of vertices in  $G$ . Let  $M \subseteq N(U)$  be the set of vertices in  $N(U)$  that are adjacent to exactly one vertex in  $U$ . Then, no vertex of  $U$  is adjacent to more than one vertex of  $M$ .*

**Proof.** If a vertex  $u \in U$  is adjacent to two vertices of  $M$ , since  $U$  is a deficient set of vertices in  $G$ , we have  $|N^{(2)}(U - u)| \leq |N^{(2)}(U)| - 2 < 2|U| - 2 = 2|U - u|$ . This implies that  $U - u$  is a deficient set of vertices in  $G$ , which in turn contradicts the minimality of  $U$ .  $\square$

**Lemma 3.** *Let  $G$  be a bipartite graph of even order  $n$  and with equipartition  $V(G) = X \cup Y$ , and suppose that  $G$  does not contain a 2-factor. If  $U$  is a minimal deficient set in  $G$  with  $|U| = k$ , then  $\deg(u) \leq k$  for each  $u \in U$  and  $|\{u \in U : \deg(u) \leq k - 1\}| \geq k - 1$ .*

**Proof.** Suppose that  $\deg(u) \geq k + 1$  for some  $u \in U$  and let  $M \subseteq N(U)$  be the set of vertices in  $N(U)$  that are adjacent to exactly one vertex in  $U$ . Then [Lemma 2](#) implies that  $u$  is adjacent to at most one vertex in  $M$  which implies that  $u$  is adjacent to at least  $k$  vertices in  $N(U) - M$ . This implies that  $|N^{(2)}(U)| \geq 2k$ , which contradicts the assumption that  $U$  is a deficient set. This proves that  $\deg(u) \leq k$  for each  $u \in U$ . If two vertices in  $U$  have degree  $k$  then similarly [Lemma 2](#) implies that  $|N^{(2)}(U)| \geq 2k$ , which contradicts the assumption that  $U$  is a deficient set. This proves the second part of the lemma.  $\square$

**Lemma 4.** *Let  $G = (V, E)$  be a bipartite graph of even order  $n$  and with equipartition  $V(G) = X \cup Y$  and suppose that  $U \subseteq X$  is a minimal deficient set in  $G$ . Let  $Y_0 = \{v \in Y : v \notin N(U)\}$ ,  $Y_1 = \{v \in Y : |U \cap N(v)| = 1\}$ , and  $Y_2 = \{v \in Y : |U \cap N(v)| \geq 2\}$ . Let  $U^* = Y_0 \cup Y_1$ . Then  $U^*$  is a deficient set in  $G$ .*

**Proof.** Let  $X_0 = X - U$ ,  $X_1 = \{u \in U : (u, v) \in E \text{ for some } v \in Y_1\}$ , and  $X_2 = U - X_1$ . Note that  $|X| = |Y|$  implies that  $|X_0| + |X_1| + |X_2| = |Y_0| + |Y_1| + |Y_2|$ . Now, since by [Lemma 2](#) we have  $|X_1| = |Y_1|$ , this implies that  $|X_0| + |X_2| = |Y_0| + |Y_2|$ . Since  $U$  is a deficient set we have  $|N^{(2)}(U)| = |Y_1| + 2|Y_2| < 2|U| = 2(|X_1| + |X_2|)$ . Hence,  $|Y_1| + 2(|X_0| + |X_2| - |Y_0|) < 2(|X_1| + |X_2|)$ , which in turn implies that  $2|X_0| + |X_1| < 2(|Y_0| + |Y_1|)$ . This proves that  $U^*$  is a deficient set in  $G$ .  $\square$

We are now ready to prove two theorems which give necessary degree conditions (similar to those in [Theorem 8](#)) for the non-existence of a 2-factor in a bipartite graph  $G$  of even order  $n$  with equipartition  $V(G) = X \cup Y$ .

**Theorem 10.** Let  $G$  be a bipartite graph of even order  $n = 4s \geq 12$  and with equipartition  $V(G) = X \cup Y$ . Let  $(d_1, d_2, \dots, d_n)$  be the degree sequence of  $G$  with  $d_1 \leq d_2 \leq \dots \leq d_n$ . If  $G$  does not contain a 2-factor, then either

- (1) for some  $k \leq \frac{n}{4}$  we have  $d_k \leq k$  and  $d_{k-1} \leq k - 1$ , or,
- (2)  $d_{\frac{n}{4}-1} \leq \frac{n}{4} - 1$ .

**Proof.** We will prove that for some  $k \leq \frac{n}{4}$ ,  $G$  contains  $k$  vertices with degree at most  $k$ , and that of these  $k$  vertices,  $(k - 1)$  vertices have degree at most  $(k - 1)$ , or, that  $G$  contains at least  $\frac{n}{4} - 1$  vertices of degree at most  $\frac{n}{4} - 1$ .

Since  $G$  does not contain a 2-factor, **Theorem 9** implies that  $G$  contains a deficient set of vertices. Let  $U \subseteq X$  be a minimal deficient set of vertices in  $G$ . If  $|U| \leq \frac{n}{4}$ , then **Lemma 3** implies that statement (1) is verified and so the conclusion holds.

Now suppose that  $|U| > \frac{n}{4}$ . As in the statement of **Lemma 4**, let  $Y_0 = \{v \in Y : v \notin N(U)\}$ ,  $Y_1 = \{v \in Y : |U \cap N(v)| = 1\}$ , and  $Y_2 = \{v \in Y : |U \cap N(v)| \geq 2\}$ . Let  $U^* = Y_0 \cup Y_1$ . Then **Lemma 4** implies that  $U^*$  is a deficient set in  $G$ . If  $|U^*| \leq \frac{n}{4}$  then again statement (1) is verified and so the conclusion holds.

Now suppose that  $|U^*| > \frac{n}{4}$ , and as in the proof of **Lemma 4**, let  $X_0 = X - U$ ,  $X_1 = \{u \in U : (u, v) \in E \text{ for some } v \in Y_1\}$ , and  $X_2 = U - X_1$ . By **Lemma 2** we have  $\deg(u) \leq 1 + |Y_2|$  for each  $u \in U$ , and hence we may assume that  $|Y_2| \geq \frac{n}{4} - 1$ , else the conclusion holds. Similarly, since  $\deg(u) \leq 1 + |X_0|$  for each  $u \in U^*$ , we may assume that  $|X_0| \geq \frac{n}{4} - 1$ . Note that  $|U| > \frac{n}{4}$  and  $|X_0| \geq \frac{n}{4} - 1$  imply that  $|U| = \frac{n}{4} + 1$ , and that  $|U^*| > \frac{n}{4}$  and  $|Y_2| \geq \frac{n}{4} - 1$  implies that  $|U^*| = \frac{n}{4} + 1$ . Now, since  $U$  is a minimal deficient set of vertices in  $G$ , by **Lemma 1** we have  $2|U| - 2 \leq |N^{(2)}(U)| \leq 2|U| - 1$ . Substituting  $|U| = \frac{n}{4} + 1$ ,  $|N^{(2)}(U)| = 2|Y_2| + |Y_1| = 2|Y_2| + |X_1|$ , and  $|Y_2| = \frac{n}{4} - 1$  into the chain of inequalities  $2|U| - 2 \leq |N^{(2)}(U)| \leq 2|U| - 1$ , we have  $\frac{n}{2} \leq \frac{n}{2} - 2 + |X_1| \leq \frac{n}{2} + 1$ . Hence,  $|X_1| = 2$  or  $|X_1| = 3$ . If  $|X_1| = 2$  then at least  $\frac{n}{4} - 1$  of the vertices in  $U$  must have degree at most  $\frac{n}{4} - 1$ , and statement (2) of the theorem is true. Finally, if  $|X_1| = 3$  then at least  $\frac{n}{2} - 4$  (and hence at least  $\frac{n}{4} - 1$  because  $n \geq 12$ ) of the vertices in each of  $U$  and  $U^*$  must have degree at most  $\frac{n}{4} - 1$ , and statement (2) of the theorem is true.  $\square$

**Theorem 11.** Let  $G$  be a bipartite graph of even order  $n = 4s + 2 \geq 14$  and with equipartition  $V(G) = X \cup Y$ . Let  $(d_1, d_2, \dots, d_n)$  be the degree sequence of  $G$  with  $d_1 \leq d_2 \leq \dots \leq d_n$ . If  $G$  does not contain a 2-factor, then either

- (1) for some  $k \leq \frac{(n-2)}{4}$  we have  $d_k \leq k$  and  $d_{k-1} \leq k - 1$ , or,
- (2)  $d_{\frac{(n-2)}{2}} \leq \frac{(n-2)}{4}$ .

**Proof.** Since  $G$  does not contain a 2-factor, **Theorem 9** implies that  $G$  contains a deficient set of vertices. Without loss of generality let  $U \subseteq X$  be a minimum cardinality deficient set of vertices in  $G$ . If  $|U| \leq \frac{(n-2)}{4}$ , then **Lemma 3** implies that statement (1) is verified and so the conclusion holds.

Now suppose that  $|U| > \frac{(n-2)}{4}$ . As in the statement of **Lemma 4**, let  $Y_0 = \{v \in Y : v \notin N(U)\}$ ,  $Y_1 = \{v \in Y : |U \cap N(v)| = 1\}$ , and  $Y_2 = \{v \in Y : |U \cap N(v)| \geq 2\}$ . Let  $U^* = Y_0 \cup Y_1$ . Then **Lemma 4** implies that  $U^*$  is a deficient set in  $G$ . Since  $U$  is a minimum cardinality deficient set of vertices in  $G$ , we have  $|U^*| \geq |U| > \frac{(n-2)}{4}$ .

Now, as in the proof of **Lemma 4**, let  $X_0 = X - U$ ,  $X_1 = \{u \in U : (u, v) \in E \text{ for some } v \in Y_1\}$ , and  $X_2 = U - X_1$ . We have  $\deg(u) \leq 1 + |Y_2|$  for each  $u \in U$ , and hence we may assume that  $|Y_2| \geq \frac{(n-2)}{4} - 1$ , else the conclusion holds. Similarly, since  $\deg(u) \leq 1 + |X_0|$  for each  $u \in U^*$ , we may assume that  $|X_0| \geq \frac{(n-2)}{4} - 1$ . Note that  $|X| = \frac{n}{2}$ ,  $|U| > \frac{(n-2)}{4}$ , and  $|X_0| \geq \frac{(n-2)}{4} - 1$  imply that  $\frac{(n-2)}{4} + 1 \leq |U| \leq \frac{(n-2)}{4} + 2$ . We now examine the two cases:  $|U| = \frac{(n-2)}{4} + 1$  and  $|U| = \frac{(n-2)}{4} + 2$ .

- (1)  $|U| = \frac{(n-2)}{4} + 1$ . In this case we must have  $|X_0| = \frac{(n-2)}{4}$ . Note that  $|X_1| \leq 3$  because if  $|X_1| \geq 4$  then since  $U$  is a minimal deficient set of vertices, we would have  $|Y_2| \leq \frac{(n-2)}{4} - 2$ , a contradiction to the assumption at this point that  $|Y_2| \geq \frac{(n-2)}{4} - 1$ . We now examine the following four subcases separately.

- (1)a  $|X_1| = 0$ . In this case we have  $|Y_1| = 0$  and  $|X_2| = \frac{(n-2)}{4} + 1$ . Since  $U$  is a minimal deficient set of vertices, **Lemma 1** implies that  $|Y_2| = \frac{(n-2)}{4}$  and  $|Y_0| = \frac{(n-2)}{4} + 1$ . Thus,  $X_2 \cup Y_0$  is a set of  $\frac{n}{2} + 1$  vertices of degree at most  $\frac{(n-2)}{4}$  which shows that (2) is verified, and hence the conclusion holds.

- (1)b  $|X_1| = 1$ . In this case we have  $|Y_1| = 1$  and  $|X_2| = \frac{(n-2)}{4}$ . Since  $U$  is a minimal deficient set of vertices, **Lemma 1** implies that  $|Y_2| = \frac{(n-2)}{4}$  and  $|Y_0| = \frac{(n-2)}{4}$ . Thus,  $X_2 \cup Y_0$  is a set of  $\frac{n}{2} - 1$  vertices of degree at most  $\frac{(n-2)}{4}$  each as required by the theorem.

- (1)c  $|X_1| = 2$ . In this case we have  $|Y_1| = 2$  and  $|X_2| = \frac{(n-2)}{4} - 1$ . Since  $U$  is a minimal deficient set of vertices, **Lemma 1** implies that  $|Y_2| = \frac{(n-2)}{4} - 1$  and  $|Y_0| = \frac{(n-2)}{4}$ . Thus,  $X_2 \cup X_1 \cup Y_0$  is a set of  $\frac{n}{2}$  vertices of degree at most  $\frac{(n-2)}{4}$  which shows that (2) is verified, and hence the conclusion holds.

- (1)d  $|X_1| = 3$ . In this case we have  $|Y_1| = 3$  and  $|X_2| = \frac{(n-2)}{4} - 2$ . Since  $U$  is a minimal deficient set of vertices, **Lemma 1** implies that  $|Y_2| = \frac{(n-2)}{4} - 1$  and  $|Y_0| = \frac{(n-2)}{4} - 1$ . Thus,  $X_2 \cup X_1 \cup Y_0$  is a set of  $\frac{n}{2} - 1$  vertices of degree at most  $\frac{(n-2)}{4}$  as required by the theorem.

(2)  $|U| = \frac{(n-2)}{4} + 2$ . In this case we have  $|X_0| = \frac{(n-2)}{4} - 1$ . Recall that since  $\deg(u) \leq 1 + |Y_2|$  for each  $u \in U$  we have  $|Y_2| \geq \frac{(n-2)}{4} - 1$ . Hence we have  $|U^*| \leq \frac{n}{2} - \left(\frac{(n-2)}{4} - 1\right) = \frac{(n-2)}{4} + 2 = |U|$ . Thus,  $U^*$  is a minimum cardinality deficient set of vertices. Hence, we now have  $|Y_2| = |X_0| = \frac{(n-2)}{4} - 1$ . Thus,  $U \cup U^*$  is a set of  $\frac{n}{2} + 3$  vertices of degree at most  $\frac{(n-2)}{4}$  which shows that (2) is verified, and hence the conclusion holds.  $\square$

**Lemma 5.** Let  $x, y$ , and  $s$  be positive numbers such that  $x \geq y > \frac{s}{2}$ . Then,  $\frac{x(x+1)(x+2)\dots(x+s)}{y(y+1)(y+2)\dots(y+s)} \geq \left(\frac{x+\frac{s}{2}}{y+\frac{s}{2}}\right)^{(s+1)}$ .

**Proof.** Note that for positive numbers  $a, b, r$  such that  $a \geq b > r$ , since  $b^2(a^2 - r^2) \geq (b^2 - r^2)a^2$ , we have  $\frac{(a+r)(a-r)}{(b+r)(b-r)} \geq \left(\frac{a}{b}\right)^2$ . Applying this note with  $a = x + \frac{s}{2}, b = y + \frac{s}{2}$  and  $r$  ranging from 1 to  $\lceil \frac{s}{2} \rceil$  gives the result.  $\square$

We are now ready for a proof of Theorem 7.

**Proof.** For an equipartition of  $V(D)$  into  $V(D) = X \cup Y$ , let  $B(X \rightarrow Y)$  be the bipartite directed graph with vertex set  $V(D)$ , equipartition  $V(D) = X \cup Y$ , and with  $(x, y) \in A(B(X \rightarrow Y))$  if and only if  $x \in X, y \in Y$ , and  $(x, y) \in A(D)$ . Let  $B(X, Y)$  denote the bipartite graph underlying  $B(X \rightarrow Y)$ . It is clear that  $B(X, Y)$  contains a Hamilton cycle if and only if  $B(X \rightarrow Y)$  contains an anti-directed Hamilton cycle. We will prove that there exists an equipartition of  $V(D)$  into  $V(D) = X \cup Y$  such that  $B(X, Y)$  contains a Hamilton cycle. In this proof we abuse the notation and write  $d^+(v)$  (respectively  $d^-(v)$ ) in place of  $d^+(v, D)$  (respectively  $d^-(v, D)$ ).

In the argument below, we make the simplifying assumption that  $d^+(v) = d^-(v) = \delta(D)$  for each  $v \in V(D)$ . After presenting the proof of the theorem under this simplifying assumption it will be easy to see that the proof extends to the case in which some indegrees or outdegrees are greater than  $\delta(D)$ . We will supply a proof of the theorem only for the case in which  $n$  is a multiple of 4, and  $\delta$  is even; the other cases can be proved in a similar manner using Theorems 10 and 11.

So, let  $n = 4m$  and  $\delta = 2d$  for some positive integers  $m$  and  $d$ . Let  $v \in V(D)$  and let  $n_k$  denote the number of equipartitions of  $V(D)$  into  $V(D) = X \cup Y$  for which  $\deg(v, B(X, Y)) = k$ . Since  $v \in X$  or  $v \in Y$  and since  $d^+(v) = d^-(v) = \delta(D)$ , we have  $n_k = 2 \binom{\delta}{k} \binom{n-\delta-1}{\frac{n}{2}-k}$ . Note that if  $k > \frac{n}{2}$  or if  $k < \delta - \frac{n}{2} + 1$  then  $n_k = 0$ . Thus the total number of equipartitions of  $V(D)$  into  $V(D) = X \cup Y$  is

$$N = \sum_{k=\delta-\frac{n}{2}+1}^{\frac{n}{2}} n_k = \sum_{k=\delta-\frac{n}{2}+1}^{\frac{n}{2}} 2 \binom{\delta}{k} \binom{n-\delta-1}{\frac{n}{2}-k} = \binom{n}{\frac{n}{2}}. \tag{1}$$

For a particular equipartition of  $V(D)$  into  $V(D) = X_i \cup Y_i$ , let  $(d_1^{(i)}, d_2^{(i)}, \dots, d_n^{(i)})$  be the degree sequence of  $B(X_i, Y_i)$  with  $d_1^{(i)} \leq d_2^{(i)} \leq \dots \leq d_n^{(i)}, i = 1, 2, \dots, N$ . If  $B(X_i, Y_i)$  does not contain a 2-factor then Theorem 10 implies that there exists  $k \leq \frac{n}{4}$  such that  $d_k^{(i)} \leq k$  and  $d_k^{(i)} \leq (k-1)$ , or  $d_{\frac{n}{4}-1} \leq \frac{n}{4} - 1$ . Hence, the number of equipartitions of  $V(D)$  into  $V(D) = X \cup Y$  for which  $B(X, Y)$  does not contain a 2-factor is at most

$$S = n \left( \frac{n_2}{2} + \frac{n_3}{3} + \dots + \frac{n_{\lfloor \frac{n}{4} \rfloor - 1}}{\lfloor \frac{n}{4} \rfloor - 1} \right). \tag{2}$$

Thus, to show that there exists an equipartition of  $V(D)$  into  $V(D) = X \cup Y$  such that  $B(X, Y)$  contains a 2-factor, it suffices to show that  $N > S$ , i.e.,

$$\sum_{k=\delta-\frac{n}{2}+1}^{\frac{n}{2}} 2 \binom{\delta}{k} \binom{n-\delta-1}{\frac{n}{2}-k} > n \sum_{k=2}^{\lfloor \frac{n}{4} \rfloor - 1} \frac{2 \binom{\delta}{k} \binom{n-\delta-1}{\frac{n}{2}-k}}{k}. \tag{3}$$

For  $i = 0, 1, \dots, \frac{n}{4} - 3$ , let  $A_i = n_{(d+i)} = 2 \binom{\delta}{d+i} \binom{n-\delta-1}{2m-d-i}$ , and let  $B_i = n_{(\frac{n}{4}-i-1)} = 2 \binom{\delta}{m-i-1} \binom{n-\delta-1}{m+i+1}$ . Clearly, (3) is satisfied if we can show that

$$A_i > \frac{nB_i}{\frac{n}{4}-i-1}, \quad \text{for each } i = 0, 1, \dots, \frac{n}{4} - 3. \tag{4}$$

This is clear because the terms in  $\sum_{i=1}^{\frac{n}{4}-3} A_i$  form a subset of the terms in the sum on the left hand side of inequality (3), and the terms in  $\sum_{i=1}^{\frac{n}{4}-3} \frac{nB_i}{\frac{n}{4}-i-1}$  are precisely the terms in the sum on the right hand side of inequality (3). We prove (4) by recursion on  $i$ . We first show that  $A_0 > \frac{nB_0}{\frac{n}{4}-1}$ , i.e.  $\frac{(\frac{n}{4}-1)A_0}{B_0} > 1$ . Let  $s = \delta - \frac{n}{2}$ . We have

$$\begin{aligned}
 \frac{\binom{n}{\frac{n}{4}-1} A_0}{n B_0} &= \frac{\binom{n}{\frac{n}{4}-1} \binom{n-1}{\frac{n}{4}-1}! (\delta - \frac{n}{4} + 1)! \binom{n}{\frac{n}{4}+1}! (\frac{3n}{4} - \delta - 2)!}{n \frac{\delta! \frac{\delta}{2}! (\frac{n}{2} - \frac{\delta}{2})! (\frac{n}{2} - \frac{\delta}{2} - 1)!} \\
 &= \frac{\binom{n}{\frac{n}{4}-1} \binom{n}{\frac{n}{4}-1}! (\frac{n}{4} + s + 1)! \binom{n}{\frac{n}{4}+1}! (\frac{n}{4} - s - 2)!}{n \binom{n}{\frac{n}{4} + \frac{s}{2}}! \binom{n}{\frac{n}{4} + \frac{s}{2}}! \binom{n}{\frac{n}{4} - \frac{s}{2}}! (\frac{n}{4} - \frac{s}{2} - 1)!} \\
 &= \frac{\binom{n}{\frac{n}{4}-1} \binom{n}{\frac{n}{4}-1}! (\frac{n}{4} + s + 1)! \binom{n}{\frac{n}{4}+1}! (\frac{n}{4} - s - 2)!}{n \binom{n}{\frac{n}{4} + \frac{s}{2}}! (\frac{n}{4} + \frac{s}{2})! (\frac{n}{4} - \frac{s}{2})! (\frac{n}{4} - \frac{s}{2} - 1)!} \\
 &= \frac{\binom{n}{\frac{n}{4}-1} (\frac{n}{4} + s + 1) (\frac{n}{4} + s) \cdots (\frac{n}{4} + \frac{s}{2} + 1) \binom{n}{\frac{n}{4}+1} (\frac{n}{4}) \cdots (\frac{n}{4} - \frac{s}{2} + 1)}{n \binom{n}{\frac{n}{4}} (\frac{n}{4} + 1) \cdots (\frac{n}{4} + \frac{s}{2}) (\frac{n}{4} - \frac{s}{2} - 1) \cdots (\frac{n}{4} - s - 1)} \\
 &= \frac{\binom{n}{\frac{n}{4}-1} (\frac{n}{4} + 1) (\frac{n}{4} + s + 1) (\frac{n}{4} + s) \cdots (\frac{n}{4} + \frac{s}{2} + 1) (\frac{n}{4}) \cdots (\frac{n}{4} - \frac{s}{2} + 1)}{n \binom{n}{\frac{n}{4} - s - 1} \frac{n}{4} (\frac{n}{4} + \frac{s}{2}) \cdots (\frac{n}{4} + 1) (\frac{n}{4} - \frac{s}{2} - 1) \cdots (\frac{n}{4} - s)}.
 \end{aligned} \tag{5}$$

Since  $n \geq 4$  and  $s \geq 1$ , it is easy to check that  $\frac{\binom{\frac{n}{4}-1}{\frac{n}{4}-s-1} \binom{\frac{n}{4}+1}{\frac{n}{4}-s-1}}{n} \geq \frac{1}{4}$ . Now, applications of Lemma 5 give

$$\begin{aligned}
 \frac{\binom{n}{\frac{n}{4}-1} A_0}{n B_0} &\geq \frac{1}{4} \frac{\binom{n}{\frac{n}{4} + s + 1} \binom{n}{\frac{n}{4} + \frac{3s}{4} + \frac{1}{2}}^{\frac{s}{2}} (\frac{n}{4} - \frac{s}{4} + \frac{1}{2})^{\frac{s}{2}}}{\frac{n}{4} \binom{n}{\frac{n}{4} + \frac{s}{4} + \frac{1}{2}}^{\frac{s}{2}} (\frac{n}{4} - \frac{3s}{4} - \frac{1}{2})^{\frac{s}{2}}} \\
 &\geq \frac{1}{4} \frac{\binom{n}{\frac{n}{4} + s + 1} (\frac{n}{4} + \frac{s}{4} + \frac{1}{2})^s}{\frac{n}{4} (\frac{n}{4} - \frac{s}{4})^s} \\
 &\geq \frac{1}{4} \left( \frac{n+s}{n-s} \right)^{s+1}.
 \end{aligned} \tag{6}$$

Since  $\delta \geq pn$ , we have  $s = \delta - \frac{n}{2} \geq (p - \frac{1}{2})n$ . Thus, (6) gives

$$\frac{\binom{n}{\frac{n}{4}-1} A_0}{n B_0} \geq \frac{1}{4} \left( \frac{n + (p - \frac{1}{2})n}{n - (p - \frac{1}{2})n} \right)^{(p-\frac{1}{2})n+1} = \frac{1}{4} \left( \frac{p + \frac{1}{2}}{\frac{3}{2} - p} \right)^{(p-\frac{1}{2})n+1}. \tag{7}$$

Because  $n > \frac{\ln(4)}{(p-\frac{1}{2}) \ln(\frac{p+\frac{1}{2}}{\frac{3}{2}-p})} - \frac{1}{(p-\frac{1}{2})}$ , (7) implies that  $\frac{\binom{n}{\frac{n}{4}-1} A_0}{n B_0} > 1$  as desired.

We now turn to the recursive step in proving (4) and assume that  $A_k > \frac{nB_k}{\frac{n}{4}-k-1}$ , for  $0 < k < \frac{n}{4} - 3$ . We will show that

$$\frac{A_{k+1}}{A_k} \geq \left( \frac{\frac{n}{4} - k - 1}{\frac{n}{4} - k - 2} \right) \frac{B_{k+1}}{B_k}. \tag{8}$$

This will suffice because (8) together with the recursive hypothesis implies that  $A_{k+1} \geq \left( \frac{\frac{n}{4}-k-1}{\frac{n}{4}-k-2} \right) \frac{A_k B_{k+1}}{B_k} > \left( \frac{\frac{n}{4}-k-1}{\frac{n}{4}-k-2} \right) \frac{n}{\frac{n}{4}-k-1} B_{k+1} = \frac{n}{\frac{n}{4}-k-2} B_{k+1}$ . We have

$$\begin{aligned}
 \frac{A_{k+1}}{A_k} &= \frac{\binom{\delta}{\frac{\delta}{2}+k+1} \binom{n-\delta-1}{\frac{n}{2}-\frac{\delta}{2}-k-1}}{\binom{\delta}{\frac{\delta}{2}+k} \binom{n-\delta-1}{\frac{n}{2}-\frac{\delta}{2}-k}} = \frac{(\frac{\delta}{2} - k) (\frac{n}{2} - \frac{\delta}{2} - k)}{(\frac{\delta}{2} + k + 1) (\frac{n}{2} - \frac{\delta}{2} + k)}, \\
 \text{and, } \frac{B_{k+1}}{B_k} &= \frac{\binom{\delta}{\frac{n}{4}-k-2} \binom{n-\delta-1}{\frac{n}{4}+k+2}}{\binom{\delta}{\frac{n}{4}-k-1} \binom{n-\delta-1}{\frac{n}{4}+k+1}} = \frac{(\frac{n}{4} - k - 1) (\frac{3n}{4} - \delta - k - 2)}{(\delta - \frac{n}{4} + k + 2) (\frac{n}{4} + k + 2)}.
 \end{aligned}$$

Hence, letting  $\delta = \frac{n}{2} + s$ , we have

$$\begin{aligned}
 \frac{\left( \frac{A_{k+1}}{A_k} \right)}{\left( \frac{B_{k+1}}{B_k} \right)} &= \frac{(\frac{\delta}{2} - k) (\frac{n}{2} - \frac{\delta}{2} - k) (\delta - \frac{n}{4} + k + 2) (\frac{n}{4} + k + 2)}{(\frac{n}{4} - k - 1) (\frac{3n}{4} - \delta - k - 2) (\frac{\delta}{2} + k + 1) (\frac{n}{2} - \frac{\delta}{2} + k)} \\
 &= \frac{(\frac{n}{4} + \frac{s}{2} - k) (\frac{n}{4} - \frac{s}{2} - k) (\frac{n}{4} + s + k + 2) (\frac{n}{4} + k + 2)}{(\frac{n}{4} - k - 1) (\frac{n}{4} - s - k - 2) (\frac{n}{4} + \frac{s}{2} + k + 1) (\frac{n}{4} - \frac{s}{2} + k)}.
 \end{aligned} \tag{9}$$

Note that in Eq. (9) we have,  $\frac{(\frac{n}{4} + \frac{s}{2} - k)}{(\frac{n}{4} - k - 1)} > 1$ ,  $\frac{(\frac{n}{4} + s + k + 2)}{(\frac{n}{4} + \frac{s}{2} + k + 1)} > 1$ ,  $\frac{(\frac{n}{4} + k + 2)}{(\frac{n}{4} - \frac{s}{2} + k)} > 1$ , and in addition because  $k < \frac{n}{4}$ , it is easy to verify that  $\frac{(\frac{n}{4} - \frac{s}{2} - k)}{(\frac{n}{4} - s - k - 2)} \geq \frac{(\frac{n}{4} - k - 1)}{(\frac{n}{4} - k - 2)}$ . Now (9) implies (8) which in turn proves (4). This completes the proof.

**Remark.** We argue that there was no loss of generality in our assumption at the beginning of the proof of Theorem 7 that  $d^+(v) = d^-(v) = \delta(D)$  for each  $v \in V(D)$ . Let  $D^* = (V^*, A(D^*))$  be a directed graph with  $d^+(v) \geq \delta(D^*)$ , and  $d^-(v) \geq \delta(D^*)$  for each  $v \in V(D^*)$ . Let  $v \in V(D^*)$ , and let  $n_k^*$  denote the number of equipartitions of  $V(D^*)$  into  $V(D^*) = X \cup Y$  for which  $\deg(v, B(X, Y)) = k$ . We can delete some arcs pointed into  $v$  and some arcs pointed out of  $v$  to get a directed graph  $D = (V^*, A(D))$  in which  $d^+(v) = d^-(v) = \delta(D^*)$ . Now as before let  $n_k$  denote the number of equipartitions of  $V(D)$  into  $V(D) = X \cup Y$  for which  $\deg(v, B(X, Y)) = k$ . It is clear that  $\sum_{k=2}^q n_k \geq \sum_{k=2}^q n_k^*$  for each  $q$ , and that  $\sum_{k=\delta - \frac{n}{2} + 1}^{\frac{n}{2}} n_k = \sum_{k=\delta - \frac{n}{2} + 1}^{\frac{n}{2}} n_k^*$  is the total number of equipartitions of  $V(D^*)$ . Hence, the proof above that  $N > S$  holds with  $n_k$  replaced by  $n_k^*$ .  $\square$

We now prove Corollary 2 mentioned in the Introduction.

**Proof.** For  $p = \frac{24}{46}$ ,  $1420 < \frac{\ln(4)}{(p - \frac{1}{2}) \ln(\frac{p + \frac{1}{2}}{\frac{3}{2} - p})} - \frac{1}{(p - \frac{1}{2})} < 1421$ . Hence, Theorem 7 implies that the corollary is true for all

$n \geq 1420$ . If  $n < 1420$  and  $\delta > \frac{24}{46}n$  then we can verify by direct computation that inequality (3) in the proof of Theorem 7 is satisfied except for the case when  $n = 44$  and  $\delta = 23$ . In this case when  $n = 44$  and  $\delta = 23$ , using both conditions  $d_k \leq k$  and  $d_{k-1} \leq k - 1$  of condition (1) in Theorem 10 implies that  $D$  contains an anti-directed 2-factor.  $\square$

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