Two natural generalizations of locally symmetric spaces

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Abstract: One studies two classes of Riemannian manifolds which extend the class of locally symmetric spaces: manifolds all of whose Jacobi operators $R_\gamma$ have constant eigenvalues ($\mathcal{C}$-spaces) or parallel eigenspaces ($\Phi$-spaces) along geodesics $\gamma$. One gives several examples, derives equivalent characterizations and treats classifications for the two- and the three-dimensional case.

Keywords: Locally symmetric spaces, constant eigenvalues and parallel eigenspaces of the Jacobi operator; Gelfand, Riemannian g.o. and naturally reductive homogeneous spaces; geodesic spheres and tubes.


1. Introduction

Curvature is a fundamental notion of differential geometry. Therefore it is a natural problem to classify the Riemannian manifolds whose curvature tensor is "simple". The first and important candidates are symmetric spaces and their classification has been given by E. Cartan.

A useful technique to describe the curvature along a geodesic $\gamma$ in a Riemannian manifold $(M, g)$, with Riemannian curvature tensor $R$, is the use of the Jacobi operator $R_\gamma = R(\cdot, \gamma)\gamma$. $R_\gamma$ determines a self-adjoint tensor field along $\gamma$. As is well-known, this operator field has two remarkable properties when the manifold is locally symmetric:

(C) the eigenvalues of $R_\gamma$ are constant along $\gamma$;
(P) the eigenspaces of $R_\gamma$ are parallel along $\gamma$.

Here, the eigenspaces of $R_\gamma$ are said to be parallel along $\gamma$ if they are invariant with respect to parallel translation along $\gamma$.

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This paper is devoted to the study of these two properties. More precisely, let \((M, g)\) be a connected \(C^\infty\) Riemannian manifold. Then \((M, g)\) is said to be a \(c\)-space if \(R\) satisfies (C) for all geodesics \(\gamma\) and it is called a \(p\)-space if \(R\) satisfies (P) for all geodesics \(\gamma\). These two classes of manifolds will be denoted by \(c\) and \(p\), respectively. Since in case of non-constant eigenvalues the dimensions of the eigenspaces of \(R_\gamma\) are generically not constant along \(\gamma\), the meaning of parallel eigenspaces has to be changed slightly. We say that the eigenspaces of \(R_\gamma\) are parallel if \(R_\gamma\) is diagonalizable by a parallel orthonormal frame field along \(\gamma\).

We focus on the following natural question: what are the classes \(c\) and \(p\)?

Our first result is the following: \(c \cap p\) is the class formed by the locally symmetric spaces. (See Section 2.) Next, we prove that \(c \cap p\) is a strict subset of \(c\) and \(p\).

Although we are unable to classify the \(c\)- and \(p\)-spaces in full generality, we give the classification for dimensions two and three. This is done in Section 5 and Section 7.

The curvature theory, and hence the geometry, of \(c\)- and \(p\)-spaces shares some remarkable properties with that of the symmetric spaces. Although further research would be worthwhile, we obtain already some results in Section 4 and Section 6 where we concentrate on some alternative characterizations of the two classes. In particular, we derive some relations with the geometry of Jacobi vector fields and the geometry of geodesic spheres (Corollary 5) and we relate this to the geometry of tubes about the so-called “curvature-adapted” or “compatible” submanifolds (Theorem 6). Further we note in Section 8 that the classical product properties for locally symmetric spaces still hold for the classes \(c\) and \(p\).

Our whole study relies on fundamental properties of the self-adjoint Jacobi operator. Therefore, we collect in Section 3 some general results about perturbation theory for self-adjoint endomorphisms of a Euclidean vector space. They will prove to be useful to derive our results.

2. A characterization of locally symmetric spaces

At first, we set up some general notations. All manifolds, maps, vector fields, curves are assumed to be \(C^\infty\), if not otherwise stated. Let \(M\) be an \(n\)-dimensional connected Riemannian manifold. We shall denote by \(\langle \cdot, \cdot \rangle\), the Riemannian metric, by \(\nabla\) the Levi Civita connection, by \(TM\) the tangent bundle and by \(R\) the curvature tensor of \(M\), using the convention \(R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - [X,Y]Z\). We denote by Ric and ric the Ricci tensor of \(M\) in its \((1,1)\) and \((0,2)\) version, respectively. If \(\gamma\) is a
geodesic in $M$, the Jacobi operator $R_\gamma$ along $\gamma$ is defined by $R_\gamma := R(\cdot, \dot{\gamma})\dot{\gamma}$, where $\dot{\gamma}$ is the tangent vector field of $\gamma$. As a consequence of the well-known symmetries of $R$ one sees that $R_\gamma$, and therefore also its covariant derivative $R'_\gamma := (\nabla_\gamma R)(\cdot, \dot{\gamma})\dot{\gamma}$, are self-adjoint tensor fields along $\gamma$. For each $p \in M$ and each $v \in T_p M$ we define two self-adjoint endomorphisms $R_v$ and $R'_v$ of $T_p M$ by $R_v := R(\cdot, v)v$ and $R'_v := (\nabla_v R)(\cdot, v)v$. For any vector field $X$ along a curve in $M$ we define $X' := \nabla_\beta X$, where $\partial$ is the canonical unit vector field on $\mathbb{R}$. If $f$ is a map between manifolds, we shall denote by $f_*$ the differential of $f$.

Now, let us assume that $M$ is a locally symmetric space. Let $\gamma : I \to M$ be a geodesic in $M$, $v$ an eigenvector of $R_\gamma(t)$ for $t \in I$ with corresponding eigenvalue $\kappa$, and $E_v$ the parallel vector field along $\gamma$ with $E_v(t) = v$. Both $\kappa E_v$ and $R_\gamma E_v$ are parallel vector fields along $\gamma$, the latter one because the curvature tensor of a locally symmetric space is parallel. Since both vector fields coincide at $\gamma(t)$, we conclude $R_\gamma E_v = \kappa E_v$. This proves that

(C) the eigenvalues of $R_\gamma$ are constant along $\gamma$, and
(P) the eigenspaces of $R_\gamma$ are parallel along $\gamma$,

that is, they are invariant with respect to parallel translation along $\gamma$.

Conversely, let us assume that conditions (C) and (P) are valid for any geodesic in a $C^\infty$ Riemannian manifold $M$. Let $v \in T_p M$ be a tangent vector of $M$ at $p \in M$ and $\gamma$ be a geodesic in $M$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. By means of (P) we can diagonalize $R_\gamma$ by a parallel orthonormal frame field $E_1, \ldots, E_n$ along $\gamma$, say $R_\gamma E_i = \kappa_i E_i$ ($i = 1, \ldots, n = \dim M$). By differentiating these equations along $\gamma$ and using (C) we get $R'_\gamma = 0$, which implies $(\nabla_v R)(\cdot, v)v = 0$. Since the last equation holds for all $v \in TM$, we conclude $\nabla R = 0$ (see [12], [34, Lemma 5.1]). Hence, $M$ is a locally symmetric space. Thus we have proved

**Theorem 1.** Let $M$ be a $C^\infty$ Riemannian manifold. Then $M$ is a locally symmetric space if and only if for any geodesic $\gamma$ in $M$ the associated Jacobi operator $R_\gamma$ has the properties (C) and (P).

Due to this theorem it is natural to study conditions (C) and (P) separately. This leads to two different generalizations of locally symmetric spaces, namely $\mathcal{C}$-spaces and $\Phi$-spaces, as described in the Introduction.

### 3. Some perturbation theory of self-adjoint endomorphisms

Below we shall need some facts about the behavior of the eigenvalues and of the eigenvectors of the Jacobi operator. For the sake of convenience we summarize these facts here in a more general setting. Details for Lemmata 1 and 2 can be taken from [15, Chapter 2, Section 6].

Let $V$ be an $n$-dimensional ($n \geq 2$) Euclidean vector space and $A(t)$ be a $C^k$ family of self-adjoint endomorphisms of $V$ depending on a real parameter $t \in I$, where $k \in \mathbb{N} \cup \{\infty, \omega\}$, and $I$ is an open interval in $\mathbb{R}$. 
Lemma 4. The following statements are equivalent:

(a) the eigenvalues of $A$ are constant;
(b) there exists a $C^{k-1}$ family $T$ of skew-symmetric endomorphisms of $V$ such that $A' = A \circ T - T \circ A$;
(c) there exists a (not necessarily continuous) family $T$ of endomorphisms of $V$ such that $A' = A \circ T - T \circ A$.

Proof. We assume that the eigenvalues of $A$ are constant. Then $A$ is $C^k$ diagonalizable on $I$ and we readily get (b) from Lemma 3. Statement (c) follows obviously from (b). Finally, we assume (c). By means of Lemma 2 the family $A$ is $C^k$ diagonalizable on an open and dense subset $J$ of $I$, say $AE_i = \lambda_i E_i$. On $J$ we calculate

$$\lambda_i' E_i + \lambda_i E'_i = (AE_i)' = A' E_i + AE_i' = A \circ TE_i - T \circ AE_i + AE_i'.$$

Taking the inner product with $E_i$ yields $\lambda_i' = 0$ on $J$. Thus $\lambda_1, \ldots, \lambda_n$ are locally constant on $J$. By a continuity argument we conclude that $\lambda_1, \ldots, \lambda_n$ are constant on $I$.

Lemma 5. If $A$ is diagonalizable by an orthonormal basis (not depending on $t$) of $V$, then $A \circ A' = A' \circ A$. Moreover, if $A$ is real analytic, then the converse is also true.

Proof. Assume that $A$ is diagonalizable by an orthonormal basis $v_1, \ldots, v_n$ of $V$, say $Av_i = \lambda_i v_i$. Differentiating these equations yields $A'v_i = \lambda_i' v_i$. Thus $A$ and $A'$ are simultaneously diagonalizable, which proves $A \circ A' = A' \circ A$.

Conversely, we assume that $A$ is real analytic and $A \circ A' = A' \circ A$. By means of Lemma 2 both $A$ and $A'$ are $C^\omega$ diagonalizable on $I$. Since $A$ and $A'$ commute, it is not hard to see that they are diagonalizable simultaneously by a real analytic orthonormal frame field $E_1, \ldots, E_n$ of $V$, say $AE_i = \lambda_i E_i$ and $A'E_i = \mu_i E_i$ with some real analytic functions $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n$ (pointwise this is well-known from linear algebra). Let $W$ be an “eigenvector bundle” of $A$, that is, $W$ is the span of all those vector fields $E_1, \ldots, E_n$ which satisfy $AE_i = \lambda_i E_i$ for a fixed eigenvalue function $\lambda \in \{\lambda_1, \ldots, \lambda_n\}$. Since

$$\mu_i E_i = A'E_i = (AE_i)' - AE_i' = \lambda_i' E_i + \lambda_i E'_i - AE_i'',$$

we get $(\mu_i = \lambda_i')$ and $AE_i' = \lambda_i E'_i$ for all $i = 1, \ldots, n$. From this we infer that for each section $X$ in $W$ also $X'$ is a section in $W$, which implies that $W$ does not depend on $t$ and is therefore a vector subspace of $V$. Thus, any orthonormal basis of $W$ consists of eigenvectors of $A(t)$ for all $t \in I$. Since the eigenvalue function $\lambda$ has been chosen arbitrarily, the assertion is proved.

4. Characterizations and examples of $\mathcal{C}$-spaces

In this section we provide some equivalent characterizations of $\mathcal{C}$-spaces. From these characterizations we shall derive some classes of $\mathcal{C}$-spaces. Let $M$ be an $n$-dimensional ($n \geq 2$) connected $C^\infty$ Riemannian manifold.
Theorem 2. $M$ is a $\mathcal{C}$-space if and only if for each $p \in M$ and each $v \in T_p M$, there exists an endomorphism $T_v$ of $T_p M$ such that $R_v = R_v \circ T_v - T_v \circ R_v$.

Proof. Let $M$ be a $\mathcal{C}$-space, $p \in M$, $v \in T_p M$ and let $\gamma$ be a geodesic in $M$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Moreover, let $A(t)$ be the self-adjoint endomorphism of $T_p M$ which is obtained by parallel translation of $R_{\gamma(t)}$ along $\gamma$ from $\gamma(t)$ to $p$. We apply Lemma 4 to $A$, by which we obtain the endomorphism $T_v$ of $T_p M$.

In order to prove the converse, let $\gamma : I \to M$ be a geodesic in $M$, $t_0 \in I$, $p = \gamma(t_0)$. Let $A(t)$ be defined as above and $T(t)$ the endomorphism of $T_p M$ which is obtained by parallel translation of $T_{\gamma(t)}$ along $\gamma$ from $\gamma(t)$ to $p$. Then $A' = A \circ T - T \circ A$ in $T_p M$, and Lemma 4 implies that the eigenvalues of $A$, and therefore also of $R_v$, are constant. \hfill $\Box$

As an immediate consequence of Theorem 2 we get

Corollary 1. If $M$ carries a $(1,2)$ tensor field $T$ (written as $T(X,Y) = T(X,Y)$) such that
\[(\nabla_X R)(Y,X)X = T_X R(Y,X)X - R(T_X Y, X)X \]
for all vector fields $X$ and $Y$ on $M$, then $M$ is a $\mathcal{C}$-space.

(See [9] for more information concerning the preceding equation.)

Corollary 2. Any naturally reductive Riemannian homogeneous space is a $\mathcal{C}$-space.

Proof. Let $M$ be a naturally reductive Riemannian homogeneous space. Since $M$ is homogeneous, it carries a homogeneous structure, that is, there exists a $(1,2)$ tensor field $T$ on $M$ such that $\tilde{\nabla} := \nabla - T$ is a metric connection, $\tilde{\nabla} R = 0$ and $\tilde{\nabla} T = 0$ (see [31, p. 141]. Moreover, because $M$ is naturally reductive, we have $T_X X = 0$ for all vector fields $X$ on $M$ (see [31, p. 58]). Now, $\tilde{\nabla} R = 0$ gives $(\tilde{\nabla}_X R)(Y,X)X = 0$, or equivalently,
\[(\nabla_X R)(Y,X)X - T_X R(Y,X)X + R(T_X Y, X)X = 0 \]
for all vector fields $X$ and $Y$ on $M$. The assertion then follows from Corollary 1. \hfill $\Box$

For each $k \in \mathbb{N}$ we define
\[P_k : TM \to \mathbb{R}, \quad v \mapsto \text{trace}(R_v^k),\]
which is a homogeneous polynomial on $TM$ of degree $2k$, and put
\[P := P_1 + \ldots + P_{n-1}.
\]
Let $S_k$ be the symmetric $(0,2k)$ tensor field on $M$ which is obtained by polarization of $P_k$. Note that $S_1$ is just the Ricci tensor $\text{ric}$ of $M$. We recall that a first integral of the geodesic flow of $M$ is a function $F : TM \to \mathbb{R}$ such that $F \circ \gamma$ is constant for any geodesic $\gamma$ in $M$. A symmetric $(0,r)$ tensor field $S$ on $M$ is called a Killing tensor field if the cyclic sum of $(\nabla_X S)(X_2, \ldots, X_{r+1})$ is equal to zero for all vector fields $X_1, \ldots, X_{r+1}$ on $M$. 


Theorem 3. The following statements are equivalent:
(a) \( M \) is a \( C \)-space;
(b) \( P \) is a first integral of the geodesic flow of \( M \);
(c) \( P_1, \ldots, P_{n-1} \) are first integrals of the geodesic flow of \( M \);
(d) \( S_1, \ldots, S_{n-1} \) are Killing tensor fields.

Proof. The equivalence of (b), (c) and (d) is due to Levi Civita (for a proof see [30, p. 241]). If \( M \) is a \( C \)-space, then \( P_1, \ldots, P_{n-1} \) are evidently first integrals of the geodesic flow of \( M \). Conversely, assume that \( P_1, \ldots, P_{n-1} \) are first integrals of the geodesic flow of \( M \). Let \( \gamma \) be a geodesic in \( M \) and \( \kappa_1, \ldots, \kappa_{n-1} \) be the eigenvalue functions of \( R_{\gamma}(\mathbb{R} \dot{\gamma})^{-1} \) (see Lemma 1 and note that \( R_{\gamma} \dot{\gamma} = 0 \)). Then \( P_1 \circ \dot{\gamma}, \ldots, P_{n-1} \circ \dot{\gamma} \) are constant, or equivalently,
\[
\kappa_1 + \ldots + \kappa_{n-1} = \text{const},
\kappa_1^2 + \ldots + \kappa_{n-1}^2 = \text{const},
\ldots
\kappa_1^{n-1} + \ldots + \kappa_{n-1}^{n-1} = \text{const}.
\]
From this we can deduce that \( \kappa_1, \ldots, \kappa_{n-1} \) are constant. Hence \( M \) is a \( C \)-space. \( \square \)

Remarks. 1. It can easily be seen that, if \( P_1, \ldots, P_{n-1} \) are first integrals of the geodesic flow of \( M \), then \( P_k \) is a first integral of the geodesic flow of \( M \) for all \( k \in \mathbb{N} \). A corresponding statement is true for the tensor fields \( S_k \).

2. The proof shows that the number \( (n - 1) \) in Theorem 3 could be replaced by the maximal number \( s \leq n - 1 \) of distinct non-zero eigenvalues of \( R_v \) for all \( v \in TM \).

3. The conditions for \( P_1 \) and \( P_2 \) to be a first integral of the geodesic flow of \( M \) (or equivalently, for \( S_1 \) and \( S_2 \) to be a Killing tensor field) are also known as the Ledger conditions of order three and five, respectively (see [2, 27, 32, 33]). The fact that in any \( C \)-space \( S_1 (- \text{ric}) \) is a Killing tensor field yields

Corollary 3. Any \( C \)-space is real analytic and has constant scalar curvature.

(See for example [16, 29] and [8, Proposition 2.3].)

For another application we recall that a commutative space (or Gelfand space) is a Riemannian homogeneous space whose Lie algebra of all invariant (with respect to the connected component of the identity of the full isometry group) differential operators is commutative (see [17, 18, 19]). For simply connected manifolds in dimensions three, four and five the commutative spaces are precisely the naturally reductive Riemannian homogeneous spaces (see [5, 17] and [20]).

Corollary 4. Any commutative space is a \( C \)-space.

Proof. Let \( M \) be a commutative space. It can be seen readily that \( P_k(f \cdot v) = P_k(v) \) for any isometry \( f \) of \( M \). Thus each \( S_k \) is an invariant symmetric tensor field on \( M \). From a result of Sumitomo [28, Theorem 3.10] we conclude that each \( S_k \) is a Killing tensor field. Thus, by means of Theorem 3, \( M \) is a \( C \)-space. \( \square \)
We recall that a Riemannian g.o. space is a homogeneous Riemannian manifold $M$ such that all geodesics in $M$ are orbits of one-parameter groups of isometries of $M$ (see [22]). It is known that any naturally reductive Riemannian homogeneous space is a Riemannian g.o. space. The converse is proved to be true in dimensions three, four and five (see [22]). But in higher dimensions there are examples of Riemannian g.o. spaces which are in no way naturally reductive. In particular, any generalized Heisenberg group with two-dimensional center is a Riemannian g.o. space, which is not naturally reductive (see [25]). The classification of Riemannian g.o. spaces, which are not naturally reductive, in dimension six is given in [22]. The classification for naturally reductive Riemannian homogeneous spaces in dimension five is given in [21].

**Proposition 1.** Any Riemannian g.o. space is a $\mathfrak{C}$-space.

**Proof.** Let $M$ be a Riemannian g.o. space and $\gamma$ a geodesic in $M$. Since $M$ is complete, we may assume that $\gamma$ is defined on $\mathbb{R}$. By the assumption there exists a one-parameter group $\Phi_t$ of isometries of $M$ such that $\gamma(t) = \Phi_t(p)$ for all $t \in \mathbb{R}$, where $p := \gamma(0)$. Let $e_1, \ldots, e_n$ be an orthonormal basis of $T_pM$ consisting of eigenvectors of $R\gamma(0)$ with corresponding eigenvalues $\kappa_1, \ldots, \kappa_n$. Then, using the identity $\dot{\gamma}(t) = \Phi_t \dot{\gamma}(0)$, we get

$$\langle R_{\gamma}(t) \Phi_t e_i, \Phi_t e_j \rangle = \langle R_\gamma(0) e_i, e_j \rangle = \kappa_i \delta_{ij}.$$ 

Thus $\Phi_t e_1, \ldots, \Phi_t e_n$ is a global orthonormal frame field along $\gamma$ consisting of eigenvectors of $R_\gamma$ everywhere. The corresponding eigenvalue functions are constant, namely $\kappa_1, \ldots, \kappa_n$. \qed

**Remark.** Since any naturally reductive Riemannian homogeneous space is a Riemannian g.o. space, we obtain Corollary 2 also as a consequence of Proposition 1.

We conclude this section with a conjecture of Osserman, which originated from his studies in ergodic theory.

**Conjecture (Osserman [24]).** If $M$ is a $\mathfrak{C}$-space, and if the eigenvalues of the Jacobi operator $R_\gamma$ are the same for all geodesics $\gamma$ (parametrized by arc length) in $M$, then $M$ is locally isometric to a two-point homogeneous space.

This conjecture has already been proved to be true for odd dimensions and for dimensions two, four and $2(2k + 1)$ ($k \in \mathbb{N}$) (see [7]).

5. Classifications of $\mathfrak{C}$-spaces

Next, we treat the classification problem for $\mathfrak{C}$-spaces in dimensions two and three. Let $M$ be a two-dimensional Riemannian manifold and $\gamma$ a geodesic in $M$. The associated Jacobi operator $R_\gamma$ has two eigenvalue functions $0$ and $c\kappa$, where $c$ is a constant and $\kappa$ is the Riemannian sectional curvature of $M$ along $\gamma$. Thus we get

**Proposition 2.** A two-dimensional Riemannian manifold is a $\mathfrak{C}$-space if and only if it is of constant Riemannian sectional curvature.
For three-dimensional manifolds $M$, Theorem 3 and the preceding Remark 3 show that $M$ is a $\mathcal{C}$-space if and only if the Ledger conditions of order three and five hold on $M$. These three dimensional spaces are classified in [17] where it is proved that these conditions are valid precisely when the local geodesic symmetries are volume preserving (up to sign). Such spaces are locally homogeneous. More precisely, Kowalski's classification yields

**Theorem 4.** Let $M$ be a three-dimensional, connected, complete and simply connected $C^\infty$ Riemannian manifold. Then $M$ is a $\mathcal{C}$-space if and only if $M$ is a naturally reductive Riemannian homogeneous space; more explicitly, if $M$ is one of the following spaces:

1. a Riemannian symmetric space;
2. $SU(2)$ with a special left invariant Riemannian metric;
3. the universal covering of $SL(2, \mathbb{R})$ with a special left invariant Riemannian metric;
4. the three-dimensional Heisenberg group with any left invariant Riemannian metric.

If $M$ is not complete or not simply connected, it is locally isometric to one of these spaces.

(For a detailed description of these metrics see [17, Theorem 2].)

6. **Characterizations of $\mathfrak{P}$-spaces**

In this section we provide some equivalent characterizations of $\mathfrak{P}$-spaces. First of all we introduce some notions. Let $M$ be a connected $C^\infty$ Riemannian manifold and $\gamma : I \to M$ a geodesic in $M$, parametrized by arc length, and with $0 \in I$. We put $p := \gamma(0)$ and recall that $R_{\gamma} \dot{\gamma} = 0$. By means of Lemma 1 there exist $s$ distinct $C^\infty$ eigenvalue functions $\kappa_1, \ldots, \kappa_s : I \to \mathbb{R}$ of the normal Jacobi operator $R_\gamma((\mathbb{R} \dot{\gamma})^1)$; if $M$ is real analytic, then $\kappa_1, \ldots, \kappa_s$ are also real analytic. From now on we suppose that $R_\gamma$ is $C^\infty$ diagonalizable on the whole of $I$ (see Lemma 2). The assumption of the existence of a global diagonalizing orthonormal frame field along $\gamma$ will be important in Theorem 5. Then there exist $s$ distinct $C^\infty$ eigenvector subbundles $V_1, \ldots, V_s$ of $TM$ along $\gamma$ satisfying

$$R_\gamma | V_j = \kappa_j \, \text{id}_{V_j} \quad (j = 1, \ldots, s) \quad \text{and} \quad T_\gamma M = V_1 \oplus \ldots \oplus V_s \oplus \mathbb{R} \dot{\gamma}.$$

If $M$ is real analytic, then $V_1, \ldots, V_s$ always exist and are also real analytic (see Lemma 2).

One of the major tools in Riemannian geometry are Jacobi fields, by which the curvature of a Riemannian manifold along a geodesic is controlled infinitesimally. Any Jacobi field along $\gamma$ with initial values tangent to $\gamma$ is of the form $(a + bt)\dot{\gamma}(t)$ with some $a, b \in \mathbb{R}$. Of greater interest are the Jacobi fields along $\gamma$ whose initial values are orthogonal to $\gamma$; such Jacobi fields are called normal. For $v \in T_pM$ we denote by $Z_v$ (resp. $\dot{Z}_v$) the Jacobi field along $\gamma$ with initial values $Z_v(0) = 0$ and $\dot{Z}_v(0) = v$ (resp.
\( \dot{Z}_v(0) = v \) and \( \dot{Z}'_v(0) = 0 \). If \( v \in V_j(0) \), then we call \( Z_v \) (and \( \dot{Z}_v \)) a basic Jacobi field along \( \gamma \). Each normal Jacobi field along \( \gamma \) can be expressed as a linear combination of basic Jacobi fields along \( \gamma \). In a locally symmetric space the basic Jacobi fields are of a well-known form, namely

\[
Z_v = d_j E'_v \quad \text{resp.} \quad \dot{Z}_v = \dot{d}_j E'_v,
\]

where \( v \in V_j(0) \) and \( d_j \) (resp. \( \dot{d}_j \)) is the solution of the scalar Jacobi equation \( y'' + \kappa_j y = 0 \) with initial values \( y(0) = 0 \) and \( y'(0) = 1 \) (resp. \( y(0) = 1 \) and \( y'(0) = 0 \)). Here, and henceforth, \( E_v \) denotes the parallel vector field along \( \gamma \) with initial value \( E_v(0) = v \in T_pM \). We shall see that \( \mathfrak{q} \)-spaces are characterized by this specific feature of their Jacobi fields.

Any solution \( Y \) of the endomorphism-valued Jacobi equation

\[
Y'' + R_{\gamma} \circ Y = 0
\]

along \( \gamma \) is called a Jacobi tensor along \( \gamma \). A tensor field \( Y \) of type \((1,1)\) along \( \gamma \) is a Jacobi tensor if and only if, for each parallel vector field \( E \) along \( \gamma \), the vector field \( YE \) is a Jacobi field. We denote by \( D \) (resp. \( \dot{D} \)) the Jacobi tensor along \( \gamma \) with initial values \( D(0) = 0 \) and \( D'(0) = \text{id}_{T_pM} \) (resp. \( \dot{D}(0) = \text{id}_{T_pM} \) and \( \dot{D}'(0) = 0 \)). Then \( DE_v = Z_v \) and \( \dot{D}E'_v = \dot{Z}_v \) for all \( v \in T_pM \).

Let \( N \) be a submanifold of \( M \), \( \xi \) a unit normal vector of \( N \) at \( p \in N \) and \( \gamma : I \to M \) a geodesic in \( M \) with \( \gamma(0) = p \) and \( \dot{\gamma}(0) = \xi \). Moreover, let \( Y \) be the Jacobi tensor along \( \gamma \) with initial values

\[
Y(0) = \begin{pmatrix}
\text{id}_{T_pN} & 0 \\
0 & 0
\end{pmatrix}
\quad \text{and} \quad
Y'(0) = \begin{pmatrix}
-B_\xi & 0 \\
0 & \text{id}_{\perp_pN}
\end{pmatrix},
\]

where \( B_\xi \) is the shape operator of \( N \) at \( p \) with respect to \( \xi \) and the matrix decomposition is with respect to the orthogonal decomposition \( T_pM = T_pN \oplus \perp_pN \). At the regular points of \( Y \) we put \( X := Y' \circ Y^{-1} \), which is a solution of the endomorphism-valued Riccati equation

\[
X' + X^2 + R_{\gamma} = 0
\]

along \( \gamma \). \( X(t) \) is well-defined for sufficiently small \( t \in \mathbb{R}_+ \). The geometric significance of \( X \) is the following: At least locally and for sufficiently small \( r \in \mathbb{R}_+ \) we can define the tube \( G_N(r) \) of radius \( r \) about \( N \). Then \( \dot{\gamma}(r) \) is a unit normal vector of \( G_N(r) \) and \( X(r)|(\mathbb{R}\dot{\gamma}(r))\perp \) is the shape operator of \( G_N(r) \) with respect to \(-\dot{\gamma}(r)\). In the special situation when \( N \) is just a single point \( p \in M \), then \( Y \) is exactly the Jacobi tensor \( D \) (see above) and \( G_p(r) \) is the distance sphere of radius \( r \) with center \( p \). In this situation we shall write \( A := D' \circ D^{-1} \), which gives the shape operator of the distance spheres about \( p \) and along \( \gamma \).

We shall characterize \( \mathfrak{q} \)-spaces by properties of their distance spheres. For this we introduce one more notion. \( N \) is said to be curvature-adapted to \( M \) at \( p \) with respect to \( \xi \), if \( R_\xi(T_pN) \subset T_pN \) and \( R_\xi \circ B_\xi = B_\xi \circ R_\xi \). The last equation means that \( R_\xi|T_pN \)
and $B_{ij}$ are simultaneously diagonalizable. $N$ is called \textit{curvature-adapted to $M$ at $p$}, if $N$ is curvature-adapted to $M$ at $p$ for every (unit) normal vector at $p$; and $N$ is called \textit{curvature-adapted (to $M$)} if it is curvature-adapted at each point of $N$. Obviously, every submanifold of a space of constant curvature is curvature-adapted. But in other spaces the definition is restrictive. Examples of curvature adapted submanifolds are provided by complex submanifolds in spaces of constant holomorphic sectional curvature and by totally umbilical hypersurfaces in arbitrary Riemannian manifolds. A classification of curvature adapted hypersurfaces in quaternionic projective spaces is given in [1]. Note that the notion of "curvature-adapted" tallies with the one of "compatible" of Gray [13, p. 104]. We shall see that a characteristic feature of $\mathfrak{P}$-spaces is that their distance spheres are curvature-adapted.

**Theorem 5.** (i) If $M$ is real analytic, then the following statements are equivalent:

(a) $R_\gamma$ is diagonalizable by a real analytic parallel orthonormal frame field;

(b) $V_1, \ldots, V_s$ are parallel;

(c) $R_\gamma \circ R'_\gamma = R'_\gamma \circ R_\gamma$;

(d) $D$ (or equivalently, $\hat{D}$) is self-adjoint;

(e) if $v \in V_j(0)$, then $Z_v = d_j E_v$ and $\hat{Z}_v = \hat{d}_j E_v$;

(f) if $v \in V_j(0)$, then $E_v(r)$ is a principal curvature vector of $G_p(r)$ for all sufficiently small $r \in \mathbb{R}_+$;

(g) if $v \in V_j(r)$, then $v$ is a principal curvature vector of $G_p(r)$ for all sufficiently small $r \in \mathbb{R}_+$;

(h) $G_p(r)$ is curvature-adapted to $M$ at $\gamma(r)$ for all sufficiently small $r \in \mathbb{R}_+$.

(ii) If $M$ is $C^\infty$, then (a) (with $C^\infty$ instead of real analytic), (b) and (c) are equivalent and each of the statements (d)-(h) is a consequence of (a).

**Remark.** In (f) and (g) the corresponding principal curvature is $d'_j(r)/d_j(r)$.

**Proof.** ad (i): The equivalence of (a) and (b) is obvious, and the one of (a) and (c) is a consequence of Lemma 5.

"(b) $\Rightarrow$ (e)" : If $v \in V_j(0)$, then $R_\gamma E = \kappa_i E_v$ by the assumption. Hence,

$$ (d_i E_v)'' = d''_i E_v = -\kappa_i d_i E_v = -d_i \kappa_i E_v = -d_i R_\gamma E_v = -R_\gamma (d_i E_v), $$

that is, $d_i E_v$ is a Jacobi field along $\gamma$. The initial values of $d_i E_v$ at 0 coincide with those of $\dot{Z}_v$ at 0. The uniqueness of Jacobi fields then provides $Z_v = d_j E_v$. The equation $\dot{Z}_v = \hat{d}_j E_v$ can be proved analogously.

"(e) $\Rightarrow$ (d)" : We choose an orthonormal basis $v_1, \ldots, v_{n-1}$ of $(\mathbb{R} \gamma(0))^\perp$ consisting of vectors belonging to $V_1(0), \ldots, V_s(0)$. Then, by means of the assumption, $D$ is diagonalized by $E_{v_1}, \ldots, E_{v_{n-1}}$, $\gamma$ (note that $D \gamma(t) = t \dot{\gamma}(t)$) and hence self-adjoint. $\hat{D}$ can be treated analogously.

"(d) $\Rightarrow$ (a)" : We assume that $D$ is self-adjoint (and $\hat{D}$ can be treated in the same manner). Then the adjoint equation of $D'' + R_\gamma \circ D = 0$ is $D'' + D \circ R_\gamma = 0$, from which we infer
\[ R_\gamma \circ D = D \circ R_\gamma \text{ and hence} \]
\[ 0 = (R_\gamma \circ D - D \circ R_\gamma) \circ D = R_\gamma \circ D \circ D - D \circ R_\gamma \circ D \]
\[ = D \circ D'' - D'' \circ D = (D \circ D' - D' \circ D'). \]

Since \((D \circ D' - D' \circ D)(0) = 0\), we get \(D \circ D' = D' \circ D\). As a solution of a Jacobi equation with real analytic coefficients \(D\) is also real analytic. Regarding to Lemma 5 there exists a real analytic parallel orthonormal frame field \(E_1, \ldots, E_n\) of \(TM\) along \(\gamma\) such that \(DE_i = \delta_i E_i\); the eigenvalue functions \(\delta_1, \ldots, \delta_n\) of \(D\) are also real analytic (see Lemma 1). The Jacobi equation yields
\[ 0 = \langle (D'' + R_\gamma \circ D)E_i, E_j \rangle = \delta_i \langle R_\gamma E_i, E_j \rangle \]
for \(i \neq j\). If we assume \(\delta_i = 0\), then
\[ 0 = \delta'_i(0)E_i(0) = D'(0)E_i(0) = E_i(0), \]
which is a contradiction. Since \(\delta_i\) is analytic and \(\delta_i \neq 0\), the zeros of \(\delta_i\) are isolated, by which we get \(\langle R_\gamma E_i, E_j \rangle = 0\) for \(i \neq j\). Thus \(R_\gamma\) is diagonalized by \(E_1, \ldots, E_n\).

\((e) \Rightarrow (f)\): If \(v \in V_j(0)\), then
\[ AE_v(r) = D' \circ D^{-1} E_v(r) = \frac{d'_i(r)}{d_j(r)} E_v(r) \]
for all \(r \in I\) where \(D(r)\) is regular. The assertion is now a consequence of the geometric interpretation of \(A\) described above.

\((f) \Rightarrow (a)\): By the assumption \(A || 0, \epsilon\) is diagonalizable by a parallel orthonormal frame field \(E_1, \ldots, E_n\) along \(\gamma|| 0, \epsilon\) for a sufficiently small \(\epsilon \in \mathbb{R}_+\) (note that \(A\gamma(t) = (1/t)^n \gamma(t)\)). Using the Riccati equation \(A' + A^2 + R_\gamma = 0\) one can readily see that \(R_\gamma|| 0, \epsilon\) is diagonalized by \(E_1, \ldots, E_n\). The analyticity of \(R_\gamma\) and \(E_1, \ldots, E_n\) then imply that \(R_\gamma\) is diagonalized by \(F_1, \ldots, F_n\), where \(F_i\) is the parallel vector field along \(\gamma\) with \(F_i|| 0, \epsilon\) = \(E_i\).

\((b) \Rightarrow (g)\): Let \(v \in V_j(0)\). By means of the assumption there is a \(w \in V_j(0)\) such that \(v = E_w(0)\). Since we already may use the implication from (b) to (f), we get
\[ Av = AE_w(r) = \frac{d'_i(r)}{d_j(r)} E_w(r) = \frac{d'_i(r)}{d_j(r)} v, \]
by which \((g)\) is proved.

\((g) \Rightarrow (h)\): The assumption implies that \(A\) and \(R\) are simultaneously diagonalizable on \(|0, \epsilon|\) for sufficiently small \(\epsilon \in \mathbb{R}_+\), that is, \(A \circ R_\gamma = R_\gamma \circ A\) on \(|0, \epsilon|\). Since the tangent space \((\mathbb{R}\gamma(r))_t^1\) of \(G_p(r)\) at \(\gamma(r)\) is invariant with respect to \(R_\gamma(r)\), the assertion follows.

\((h) \Rightarrow (f)\): By means of the Riccati equation for \(A\) we get
\[ R_\gamma \circ A - A \circ R_\gamma = A \circ A' - A' \circ A. \]

By utilizing Lemma 5 we easily get the assertion.

\(\text{ad (ii)}\): The equivalence of (a) and (b) in the \(C^\infty\) case is obvious. The proof of (i) shows, that the implications (a) \(\Rightarrow (c)\), (b) \(\Rightarrow (e) \Rightarrow (d)\) and (f), (b) \(\Rightarrow (g) \Rightarrow (h)\) are
also valid in the $C^\infty$ case. Moreover, since the converse in Lemma 5 is also valid under the “global” $C^\infty$ diagonalizable condition, (c) $\Rightarrow$ (a) also follows. $\square$

**Corollary 5.** (i) If $M$ is real analytic, then the following statements are equivalent:

(a) $M$ is a $\mathfrak{P}$-space;

(b) $R \circ R_v = R'_v \circ R_v$ for all $v \in TM$;

(c) the basic Jacobi fields in $M$ are of the form as in locally symmetric spaces, that is, they arise from multiplying appropriate parallel vector fields with particular solutions of scalar Jacobi equations (as described explicitly above);

(d) the principal curvature spaces of any family of (sufficiently small) distance spheres in $M$ are invariant with respect to parallel translation along the radial geodesics emanating from the center of this family (as described explicitly above);

(e) all (sufficiently small) distance spheres in $M$ are curvature-adapted.

(ii) If $M$ is a $C^\infty$ $\mathfrak{P}$-space, then each of the statements (b)-(e) is valid.

**Remark.** Corollary 5 provides also a contribution to a question posed by Chen and the second author in [6], namely “To what extent do the properties of sufficiently small geodesic spheres determine the Riemannian geometry of the ambient space?”

The particular form of the Jacobi fields in $\mathfrak{P}$-spaces has consequences on the geometry of such spaces. We give here an example from submanifold theory. Let $N$ be a submanifold of a $\mathfrak{P}$-space $M$, $\xi$ a unit normal vector of $N$ at $p \in N$, and $\gamma : I \to M$ a geodesic in $M$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = \xi$. We denote by $\lambda_1, \ldots, \lambda_k$ the distinct principal curvatures of $N$ at $p$ with respect to $\xi$ (that is, the distinct eigenvalues of $B_\xi$) and by $T_1, \ldots, T_k$ the spaces of corresponding principal curvature vectors. Let $Y$ be the Jacobi tensor along $\gamma$ for which $X = Y' \circ Y^{-1}$ describes the shape operators of the tubes about $N$ (for the exact description of $Y$ see the beginning of this section). From the particular form of the Jacobi fields in $\mathfrak{P}$-spaces we deduce

$$YE_v = (\tilde{d}_j - \lambda_i d_j)E_v, \quad \text{if } v \in V_j(0) \cap T_i,$$

$$YE_v = d_j E_v, \quad \text{if } v \in V_j(0) \cap \perp_p N,$$

and therefore

$$XE_v = \frac{\tilde{d}_j - \lambda_i d_j}{d_j - \lambda_i d_j} E_v = (\ln(\tilde{d}_j - \lambda_i d_j))' E_v, \quad \text{if } v \in V_j(0) \cap T_i,$$

$$XE_v = \frac{d_j'}{d_j} E_v = (\ln(d_j))' E_v, \quad \text{if } v \in V_j(0) \cap \perp_p N.$$
(i) the shape operators of the tubes about $N$ are diagonalizable by a parallel orthonormal frame field of $(\mathbb{R} \dot{\gamma})^\perp$ along $\gamma$;
(ii) the tubes about $N$ are also curvature-adapted.

7. Classifications of $\mathfrak{X}$-spaces

In the final section we treat the classification problem for $\mathfrak{X}$-spaces in dimensions two and three. Let $\gamma$ be a geodesic in a two-dimensional Riemannian manifold $M$. We can decompose $T\gamma M$ orthogonally into the two parallel vector subbundles $\mathbb{R} \dot{\gamma}$ and $(\mathbb{R} \dot{\gamma})^\perp$, which are both invariant with respect to the Jacobi operator $R_\gamma$. Thus $R_\gamma$ is diagonalizable by a parallel orthonormal frame field along $\gamma$, by which we have proved

**Proposition 3.** Any connected two-dimensional Riemannian manifold is a $\mathfrak{X}$-space.

**Remark.** Since $\mathcal{C} \cap \mathfrak{X}$ consists exactly of the locally symmetric spaces, the combination of Proposition 2 and Proposition 3 reaffirms the well-known fact that the two-dimensional locally symmetric spaces are precisely the two-dimensional spaces of constant curvature.

We recall that a **Liouville surface** is a two-dimensional Riemannian manifold, for which locally there exist isothermal coordinates $x$ and $y$, such that the Riemannian metric of $M$ is of the form

$$(\varphi(x) + \psi(y))(dx^2 + dy^2)$$

for some functions $\varphi$ and $\psi$ (see for instance [4, p. 170]). The remaining part is to prove the following classification of three-dimensional $\mathfrak{X}$-spaces.

**Theorem 7.** (i) Let $M$ be a three-dimensional $\mathfrak{X}$-space of class $C^\infty$. Then $M$ is almost everywhere (that is, on an open and dense subset of $M$) locally isometric to one of the following spaces:

(I) a space of constant Riemannian sectional curvature;
(II) a warped product of the form $M_1 \times_f M_2$, where $M_1$ is a one-dimensional Riemannian manifold, $M_2$ is a two-dimensional Riemannian manifold, and $f$ is a positive function on $M_1$;
(III) a warped product of the form $M_2 \times_f n/r$, where $M_1$ is a one-dimensional Riemannian manifold, $M_2$ is a Liouville surface, and $f$ is given by

$$f^2(x, y) = |\varphi(x)\psi(y)|,$$

where the functions $\varphi$ and $\psi$ come from a (local) Liouville form

$$(\varphi(x) + \psi(y))(dx^2 + dy^2)$$

of the Riemannian metric of $M_2$;
(IV) a three-dimensional Riemannian manifold with Riemannian metric of the form

$$\mathcal{S} \int_{1,2,3} F_1(x_1)|x_1 - x_2||x_1 - x_3|dx_1,$$

where $\mathcal{S}$ denotes the cyclic sum and $F_1, F_2, F_3$ are positive functions.

(ii) Any connected real analytic Riemannian manifold of type (I), (II), (III) or (IV) is a $\mathfrak{P}$-space.

**Proof.** Regarding to Corollary 5 it is sufficient to study Riemannian manifolds satisfying

$$L_v := R'_v \circ R_v - R_v \circ R'_v = 0 \quad \text{for all } v \in T M. \quad (1)$$

Since $L_v$ is a skew-symmetric endomorphism of $T_p M$ (where $p \in M$ such that $v \in T_p M$), $L_v v = 0$ and $L_{tv} = t^5 L_v$ for all $t \in \mathbb{R}_+$, condition (1) is equivalent to

$$\langle L_{v_1} v_2, v_3 \rangle = 0 \quad \text{for each orthonormal basis } v_1, v_2, v_3 \text{ of } T_p M, \ p \in M. \quad (2)$$

As is well-known, the curvature tensor $R$ of a three-dimensional Riemannian manifold $M$ can be expressed in terms of the Ricci tensors $\text{Ric}$ and $\text{ric}$ and the scalar curvature $s$ of $M$, namely

$$R(X, Y)Z = \text{ric}(Y, Z)X - \text{ric}(X, Z)Y + \langle Y, Z \rangle \text{Ric}X - \langle X, Z \rangle \text{Ric}Y - \frac{1}{2} s(\langle Y, Z \rangle X - \langle X, Z \rangle Y). \quad (3)$$

By a straightforward computation we get for $v_1, v_2, v_3$ as in (2)

$$\langle L_{v_1} v_2, v_3 \rangle = (\nabla_{v_1} \text{ric})(\text{Ric} v_2, v_3) - (\nabla_{v_1} \text{ric})(v_2, \text{Ric} v_3)$$

$$- \text{ric}(v_1, v_2)(\nabla_{v_1} \text{ric})(v_1, v_3) + \text{ric}(v_1, v_3)(\nabla_{v_1} \text{ric})(v_1, v_2). \quad (4)$$

Let $W$ be the subset of $M$ on which the number of distinct eigenvalues of $\text{Ric}$ is locally constant. This set is open and dense in $M$. On $W$ we can choose $C^\infty$ eigenvalue functions of $\text{Ric}$, say $\lambda_1, \lambda_2, \lambda_3$, such that they form at each point of $W$ the spectrum of $\text{Ric}$. (This is the set where our classification holds.) We fix a point $p \in W$. Then there exists a local orthonormal frame field $E_1, E_2, E_3$ of $TM$ on an open connected neighborhood $U$ of $p$ in $W$ such that $\text{Ric} E_i = \lambda_i E_i$ ($i = 1, 2, 3$). Henceforth, the index $i$ has to be taken modulo three. We define

$$s_i := \lambda_{i+1} - \lambda_{i+2} \quad (i = 1, 2, 3),$$

$$\Lambda := s_1 d\lambda_1 + s_2 d\lambda_2 + s_3 d\lambda_3,$$

$$\omega_{ij}^k := \langle \nabla_{E_i} E_j, E_k \rangle \quad (i, j, k = 1, 2, 3).$$

**Lemma 6.** Condition (1) holds on $U$ if and only if

$$0 = a_1 a_2 a_3 \Lambda(V_a) + \sum_{i=1}^{3} a_i [(s_i + a_{i+1}^2 s_{i+2} + a_{i+2}^2 s_{i+1})a_i s_i \omega_{i,i+1}^{i+2}$$

$$+ (s_{i+2} + a_{i+1}^2 s_{i+1} + a_{i+1}^2 s_i) a_{i+2} s_{i+2} \omega_{i+1}^{i+2}$$

$$- (s_{i+1} + a_{i+2}^2 s_i + a_{i+2}^2 s_{i+2}) a_{i+1} s_{i+1} \omega_{i+1}^{i+2}] \quad (5)$$
for all \( a = (a_1, a_2, a_3) \in S^2 \), where \( S^2 \) is the unit sphere in \( \mathbb{R}^3 \) and

\[
V_a := a_1 E_1 + a_2 E_2 + a_3 E_3.
\]

Proof. Let \( a, b, c \in \mathbb{R}^3 \) be orthonormal. Then \( V_a, V_b, V_c \) is a local orthonormal frame field of \( TM \) over \( U \). Evaluating (4) for \( V_a, V_b, V_c \) proves that \( (LV_a V_b, V_c) \) coincides with the right-hand term of (5) up to sign (the sign depends on whether \( a \) equals the vector product \( b \times c \) or \( -b \times c \)). This implies Lemma 6. \( \square \)

We shall investigate three cases, namely the cases that the number \( d \) of distinct eigenvalues of the Ricci tensor in \( U \) is one, two or three.

Case A: \( d = 1 \). In this case \( M \) is an Einstein manifold over \( U \). The three-dimensional Einstein manifolds are precisely the spaces of constant curvature (see [3, Proposition 1.120]), which are, of course, \( \mathfrak{g} \)-spaces.

Case B: \( d = 2 \). We may assume \( \lambda_1 \neq \lambda_2 = \lambda_3 \). Then \( s_1 = 0, s_2 = -s_3 \neq 0 \) and \( \Lambda = 0 \).

Let \( a = (a_1, a_2, a_3) \in S^2 \). From Lemma 6 we infer successively for

\[
\begin{align*}
& a_1 = 0, \quad a_2 = 0, \quad a_3 = 1: \quad \omega_{31}^1 = 0, \quad \text{and thus } \omega_{32}^1 = 0; \quad (6) \\
& a_1 = 0, \quad a_2 = 1, \quad a_3 = 0: \quad \omega_{23}^1 = 0; \quad (7) \\
& a_1 \neq 0, \quad a_2 = 0, \quad a_3 \neq 0: \quad \omega_{11}^2 = 0; \quad (8) \\
& a_1 \neq 0, \quad a_2 \neq 0, \quad a_3 = 0: \quad \omega_{11}^3 = 0; \quad (9) \\
& a_1 \neq 0, \quad a_2 \neq 0, \quad a_3 \neq 0: \quad \omega_{22}^1 = \omega_{33}^1. \quad (10)
\end{align*}
\]

Utilizing (6), (7) and (10) we calculate

\[
\begin{align*}
0 &= \text{ric}(E_2, E_1) = \langle R(E_2, E_3)E_3, E_1 \rangle = d\omega_{33}^1(E_2); \quad (11) \\
0 &= \text{ric}(E_3, E_1) = \langle R(E_3, E_2)E_2, E_1 \rangle = d\omega_{22}^1(E_3). \quad (12)
\end{align*}
\]

We define \( V_1 := \mathbb{R}E_1 \) and \( V_2 := V_1^\perp = \mathbb{R}E_2 \oplus \mathbb{R}E_3 \). Then \( V_1 \) is an autoparallel (apply (8) and (9)) subbundle and \( V_2 \) is an integrable (apply (6) and (7)) and spherical (apply (10)–(12)) subbundle of \( TU \). In this situation we can apply a result of Hiepko [14, p. 211] and obtain that \( U \) is locally a warped product of type (II). Conversely, it can be calculated easily, that any warped product of type (II) satisfies equation (5) and hence (1) (for instance, one might use the formulae for the Ricci tensor of warped products in [23, p. 211]).

Case C: \( d = 3 \). Then \( s_1, s_2, s_3 \neq 0 \).

Lemma 7. Condition (1) is valid on \( U \) if and only if the following equations are valid for all distinct \( i, j, k \in \{1, 2, 3\} \):

\[
\begin{align*}
& \omega_{ij}^k = 0; \quad (13) \\
& s_i \omega_{jj}^k + s_j \omega_{ii}^k = 0; \quad (14) \\
& \Lambda(E_k) = 2s_is_k \omega_{jj}^k \quad \text{for } k = (j + 1) \mod 3. \quad (15)
\end{align*}
\]
**Proof.** Assume that (1) is valid on $U$. We apply Lemma 6. By choosing $a = (a_1, a_2, a_3) \in S^2$ with $a_i = 1$ for $i = 1, 2, 3$ successively, equation (5) yields (13) (note that $\omega_i^k + \omega_j^k = 0$). In the next step we choose $a \in S^2$ with $a_k = 0$ for just one $k \in \{1, 2, 3\}$. Then (5) implies

$$0 = (s_i + a_j^2 s_k) s_j \omega_i^k - (s_j + a_j^2 s_k) s_i \omega_j^k.$$  

Since

$$s_j + a_j^2 s_k = s_j + (1 - a_j^2) s_k = -s_i - a_j^2 s_k$$

we get

$$0 = (s_i + a_j^2 s_k)(s_j \omega_i^k + s_i \omega_j^k).$$

Pointwise we may choose $a_j$ in such a way that $s_i + a_j^2 s_k \neq 0$, whence (14) follows. Under consideration of (13) and (14), as well as $s_i + s_{i+2} = -s_{i+1}$, equation (5) becomes

$$a_1 a_2 a_3 \Lambda(V_\alpha) = - \sum_{i=1}^3 a_{i+1} a_{i+2} [(s_{i+1} + a_{i+2}^2 s_i + a_i^2 s_{i+2}) s_{i+1} \omega_{i+2,i+2}^i - (s_{i+2} + a_i^2 s_{i+1} + a_{i+1}^2 s_i) s_{i+2} \omega_{i+1,i+1}^i]$$

$$= - \sum_{i=1}^3 [s_{i+1} + s_{i+2} + (a^2_{i+1} + a^2_{i+2}) s_i + a^2_i (s_{i+1} + s_{i+2})] a_{i+1} a_{i+2} s_{i+1} \omega_{i+2,i+2}^i$$

$$= 2a_1 a_2 a_3 \sum_{i=1}^3 a_i s_i s_{i+1} \omega_{i+2,i+2}^i.$$  

Since this is true for all $a \in S^2$, we get by a continuity argument, that

$$0 = \sum_{i=1}^3 a_i [\Lambda(E_i) - 2 s_i s_{i+1} \omega_{i+2,i+2}^i]$$

for all $a \in S^2$. From this we infer easily (15). Conversely, if (13)–(15) are valid, one can check quite easily that (5), and hence (1), is valid over $U$. \(\square\)

**Remark.** Equations (13) and (14) are equivalent to

$$(\nabla_{E_i} \text{ric})(E_j, E_k) = 0 \quad \text{and} \quad (\nabla_{E_i} \text{ric})(E_i, E_k) = (\nabla_{E_j} \text{ric})(E_j, E_k),$$

respectively.

The geometric significance of (13) is the following (see also [10, p. 117]):

**Lemma 8.** The equations $\omega_{ij}^k = 0$ for all distinct $i, j, k \in \{1, 2, 3\}$ are equivalent to the integrability of $W_i := \mathbb{R} E_{i+1} \oplus \mathbb{R} E_{i+2}$ for all $i \in \{1, 2, 3\}$.  

Proof. The “only if” part of the statement is obvious. Conversely, assume that \( W_1, W_2 \) and \( W_3 \) are integrable. We apply repeatedly the Ricci identity and use the fact that \( \nabla \) is of zero torsion to get

\[
\omega_{ij}^k = \omega_{ji}^k = -\omega_{jk}^i = -\omega_{kj}^i = \omega_{ki}^j = -\omega_{ik}^j,
\]

by which the lemma is proved.  

Thus the equations (13) are equivalent to the existence of a triply orthogonal system of surfaces in \( U \) all of whose orthogonal trajectories are precisely the Ricci curvature lines. Therefore (see [10, p. 43]) the Riemannian metric in \( U \) is locally of the form

\[
\sum_{i=1}^{3} \mu_i^2(x_1, x_2, x_3) dx_i^2
\]

with some positive functions \( \mu_i \) and with the property, that the level surfaces \( x_i \equiv \text{const} \) correspond to the integral manifolds of \( W_i \). The coordinate vector fields

\[
X_i := \frac{\partial}{\partial x_i}
\]

are eigenvectors of the Ricci tensor everywhere. We may assume that \( X_i = +\mu_i E_i \) (otherwise we replace \( x_i \) by \(-x_i\)). In the following we shall need some formulae for Riemannian metrics of the form (16). We define \( \Gamma_{ij}^k, R_{klj}^i \) and \( S_{jk} \) by

\[
\nabla X_i X_j = \sum_k \Gamma_{ij}^k X_k,
\]

\[
R(X_i, X_j)X_k = \sum_l R_{klj}^i X_l,
\]

\[
\text{Ric} X_j = \sum_k S_{jk}^i X_k,
\]

and put

\[
\nu_i := \ln(\mu_i), \quad \nu_{i,j} := \frac{\partial \nu_i}{\partial x_j} \quad \text{and} \quad \nu_{i,j,k} := \frac{\partial^2 \nu_i}{\partial x_j \partial x_k}
\]

for \( i, j, k \in \{1, 2, 3\} \). By the well-known formulae (see for instance [10, pp. 17-22]) we get for distinct \( i, j, k \):

\[
\Gamma_{ij}^k = 0,
\]

\[
\Gamma_{ij}^i = \Gamma_{ji}^i = \nu_{i,j},
\]

\[
\Gamma_{ii}^j = -\frac{\mu_j^2}{\mu_i^2} \nu_{i,j},
\]

\[
\Gamma_{ii}^i = \nu_{i,i},
\]

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\[ R^k_{kij} = 0, \quad \text{for all distinct } i, j, k \in \{1, 2, 3\}. \]

\[ R^i_{kij} = \nu_{i,j} \nu_{j,k} + \nu_{i,k} \nu_{k,j} - \nu_{i,j} \nu_{i,k} - \nu_{i,jk}, \]

\[ R^i_{jij} = \nu_{i,j} \nu_{j,j} - \nu_{i,j} \nu_{j,j} + \frac{\mu^2}{\mu_i^2} (\nu_{j,i} \nu_{i,i} - \nu_{j,j} - \nu_{j,ii}) - \frac{\mu^2}{\mu_k^2} \nu_{i,k} \nu_{j,k}, \]

\[ S^k_j = \frac{1}{\mu_k^2} R^i_{kij}, \]

\[ S^i_j = \frac{1}{\mu_j^2} R^i_{jij} + \frac{1}{\mu_k^2} R^i_{kik}. \]

**Lemma 9.** If (13)-(15) are valid, then the functions \( \nu_1, \nu_2, \nu_3 \) satisfy the following system of nonlinear partial differential equations:

\[ 0 = \nu_{i,jk}, \quad \text{for all distinct } i, j, k \in \{1, 2, 3\}. \]

\[ 0 = \nu_{i,j} \nu_{i,k} - \nu_{i,j} \nu_{k,j} - \nu_{i,k} \nu_{k,j}, \]

\[ 0 = \nu_{i,ij} + 2 \nu_{i,j} \nu_{j,i}, \]

**Proof.** With regard to Lemma 8 condition (13) induces that locally around \( p \) the Riemannian metric of \( M \) is of the form (16) and the coordinate vector fields \( X_i \) are tangent to Ricci curvature lines. According to (22) and (24) the latter condition can be expressed by the equations

\[ \nu_{i,jk} \quad \text{for all distinct } i, j, k \in \{1, 2, 3\}. \]

By means of (19) we get

\[ \nu_{i,k} + \mu_k \omega^k_i = 0 \quad \text{for all distinct } i, k \in \{1, 2, 3\}. \]

which implies that the equations (14) are equivalent to

\[ s_i \nu_{j,k} + s_j \nu_{i,k} = 0 \quad \text{for all distinct } i, j, k \in \{1, 2, 3\}. \]

Now, if \( \nu_{i,j} = 0 \), then \( \nu_{k,j} = 0 \) by means of (31) and hence, (26) and (27) are valid. Thus we assume \( \nu_{i,j} \neq 0 \). Then \( \nu_{k,j} \neq 0 \) and \( s_i/s_k = -\nu_{i,j}/\nu_{k,j} \) by (31). Since \( s_j = -s_i - s_k \), we get

\[ 0 = s_i \nu_{j,k} + s_j \nu_{i,k} = s_i (\nu_{j,k} - \nu_{i,k}) - s_k \nu_{i,k}. \]

Dividing by \( s_k \), multiplying with \( \nu_{k,j} \), and replacing \( s_i/s_k \) by \( -\nu_{i,j}/\nu_{k,j} \) finally yields

\[ 0 = -\nu_{i,j} \nu_{j,k} + \nu_{i,j} \nu_{i,k} - \nu_{i,k} \nu_{k,j}. \]

From this and (29) we readily obtain (26) and (27). We now turn our attention to equation (15). It can easily be checked that

\[ \Lambda = s_i^2 d \left( \frac{s_i+2}{s_i} \right) = -s_i^2 d \left( \frac{s_i+1}{s_i} \right). \]
Combining (15), (30), (32) and replacing $F_i$ by $\frac{1}{\mu_i}X_i$, we obtain that (15) is equivalent to
\[
2\nu_{i+1} = d\left(\ln\frac{s_{i+2}}{s_{i+1}}\right)(X_{i+1}).
\] (33)

Next, we calculate from (31) and (33)
\[
2\nu_{i+1} = -2\frac{s_{i+1}}{s_{i+2}}\nu_{i+2,i} = -\frac{s_i}{s_{i+2}}d\left(\ln\frac{s_{i+1}}{s_i}\right)(X_i) = \frac{s_i}{s_{i+2}}d\left(\frac{s_{i+2}}{s_i}\right)(X_i) = d\left(\ln\frac{s_{i+2}}{s_i}\right)(X_i).
\] (34)

Summing up we see that condition (15) is equivalent to
\[
2\nu_{i,j} = d\left(\ln\frac{s_k}{s_j}\right)(X_j) \quad (i, j, k \text{ distinct}).
\] (35)

We now assume $\nu_{i,j} \neq 0$ (otherwise (28) is obvious). Then, by (31), also $\nu_{k,j} \neq 0$ and we calculate, using (35), (31) and (26),
\[
2\nu_{j,i} = d\left(\ln\frac{s_k}{s_j}\right)(X_i) = \frac{s_i}{s_k}d\left(\frac{s_k}{s_j}\right)(X_i) - \frac{\nu_{i,j}}{\nu_{k,j}}\nu_{i,j} = \frac{\nu_{i,j}}{\nu_{k,j}},
\]
which is (28). So the whole lemma is proved. $\square$

**Remark.** From the proof we keep in mind that
\[
(i) \iff s_i\nu_{j,k} + s_j\nu_{i,k} = 0 \quad \text{for all distinct } i, j, k \in \{1, 2, 3\}; \quad (36)
\]
\[
(ii) \iff 2\nu_{i,j} = d\left(\ln\frac{s_k}{s_j}\right)(X_j) \quad \text{for all distinct } i, j, k \in \{1, 2, 3\}. \quad (37)
\]

The system (26)-(28) of nonlinear partial differential equations is well-known; it arises from a quantum mechanical problem. One could regard the three-dimensional Riemannian manifold $M$ as a conservative system with zero potential energy and kinetic energy equal to $\frac{1}{2}\langle v, v \rangle$ for all $v \in TM$. As is known, the state of the system can be described by a Schrödinger equation. Robertson [26] studied the problem (also in higher dimensions) whether there exists a solution of this Schrödinger equation arising from an ansatz of simple separation of variables in orthogonal coordinates (that is, for
Riemannian metrics of the form (16)), and derived necessary and sufficient conditions for the existence of such solutions. Later on Eisenhart [11] proved that Robertson’s conditions on the metric coincide with the PDE-system (26)–(28). Moreover, in [11] Eisenhart solved this system completely. He obtained four kinds of solutions, namely

\begin{align*}
\text{(E1)} \quad & \mu^2_1 = 1, \mu^2_2 = G(x_2)\eta(x_1)(\varphi(x_2) + \psi(x_3)), \mu^2_3 = H(x_3)\eta(x_1)(\varphi(x_2) + \psi(x_3)), \\
& \text{where } \eta \text{ is a function of } x_1, \varphi \text{ and } \psi \text{ are functions of } x_2, \text{ and } H \text{ and } \psi \text{ are functions of } x_3; G, H, \eta > 0; \\
\text{(E2)} \quad & \mu^2_1 = 1, \mu^2_2 = \varphi^2(x_1), \mu^2_3 = \psi^2(x_1) \text{ where } \varphi \text{ and } \psi \text{ are functions of } x_1; \\
\text{(E3)} \quad & \mu^2_1 = \mu^2_2 = \varphi(x_1) + \psi(x_2), \mu^2_3 = |\varphi(x_1)\psi(x_2)|, \text{ where } \varphi \text{ is a function of } x_1 \text{ and } \\
& \psi \text{ is a function of } x_2; \\
\text{(E4)} \quad & \mu^2_i = F_i(x_i)|x_i - x_j||x_i - x_k| \text{ for } i, j, k \in \{1, 2, 3\} \text{ distinct, where } F_i \text{ is a function of } x_i.
\end{align*}

In order to complete the proof of Theorem 7 we have to pick out those of the spaces (E1)–(E4) which have three distinct eigenvalues of Ric and, in view of Lemmata 7 and 9 and the preceding Remark, satisfy the equations (31) and (35).

(a) \textit{Spaces of type (E1).} The spaces of type (E1) are precisely the warped products of the form $M_1 \times_f M_2$ (see Theorem 6, type (II)), where $M_2$ is a Liouville surface. (This can be seen easily by defining new coordinates $y_2$ and $y_3$ by $dy_2 = \sqrt{G(x_2)}\, dx_2$ and $dy_3 = \sqrt{H(x_3)}\, dx_3$.) Hence the number of distinct eigenvalues of Ric is at most two (see the case $d = 2$).

(b) \textit{Spaces of type (E2).} With (23) and (25) we calculate

\[ s_2 = \nu_2^2 + \nu_{2,11} - \nu_{2,1}\nu_{3,1} \quad \text{and} \quad s_3 = -\nu_{3,1}^2 - \nu_{3,11} + \nu_{2,1}\nu_{3,1}, \]

and thus

\[ s_2\nu_{3,1} + s_3\nu_{2,1} = 2\nu_{2,1}\nu_{3,1} - 2\nu_{3,1}\nu_{2,1} + \nu_{2,11}\nu_{3,1} - \nu_{3,11}\nu_{2,1} = \nu_{2,1}\nu_{3,1}(2\nu_{2,1} - 2\nu_{3,1} + (\ln |\nu_{2,1}|)_1 - (\ln |\nu_{3,1}|)_1). \]

If $\nu_{2,1} = 0$, then also $\nu_{3,1} = 0$ and hence $\varphi$ and $\psi$ are constant, which implies that $M$ is flat. Thus $\nu_{2,1} \neq 0$ and $\nu_{3,1} \neq 0$, and the necessary condition (31) yields

\[ 2\nu_3 + \ln |\nu_{3,1}| = 2\nu_2 + \ln |\nu_{2,1}| + \alpha \]

with some $\alpha \in \mathbb{R}$. Standard calculations then imply

\[ \mu_3^2 = \gamma \mu_2^2 + \delta, \quad \text{that is,} \quad \psi^2 = \gamma \varphi^2 + \delta \]

with some constants $\gamma, \delta \in \mathbb{R}$. From this we can deduce, with some slight coordinate transformations, that the manifold is of type (E3) with either $\varphi$ or $\psi$ constant.

\textbf{Remark.} At this point we have finished the proof of Theorem 7 (i).
(c) Spaces of type \((E3)\). For such spaces we calculate

\[
\begin{align*}
\nu_{1,1} &= \nu_{2,1} = \frac{\varphi'}{2(\varphi + \psi)}, & \nu_{3,1} &= \frac{\varphi'}{2\varphi}, \\
\nu_{1,2} &= \nu_{2,2} = \frac{\psi'}{2(\varphi + \psi)}, & \nu_{3,2} &= \frac{\psi'}{2\psi}, \\
\nu_{1,3} &= \nu_{2,3} = \nu_{3,3} = 0, \\
S_1 &= \lambda_2 - \lambda_3 = \psi T, & S_2 &= \lambda_3, & S_3 &= \lambda_1, & S_4 &= (\varphi \mid \psi)T,
\end{align*}
\]

where

\[
T := \frac{1}{\varphi} \left( 2\varphi''(\varphi + \psi) - \frac{\varphi'^2}{\varphi'}(3\varphi + \psi) \right) - \frac{1}{\psi} \left( 2\psi''(\varphi + \psi) - \frac{\psi'^2}{\psi'}(3\varphi + \varphi) \right).
\]

Thus the number of distinct eigenvalues of \(\text{Ric} \) at each point is either one or three. Moreover, equations (31) and (35) can easily be verified with the above formulae. Hence, in the real analytic case, the space is either of constant curvature, or the number of distinct eigenvalues of \(\text{Ric} \) is three on an open and dense subset. On this subset condition (1) is valid. By a continuity argument (1) is valid everywhere.

(d) Spaces of type \((E4)\). For \(i \neq j \) we calculate

\[
\nu_{i,j} = \frac{1}{x_j - x_i} \quad \text{and} \quad \lambda_i - \lambda_j = (x_i \ x_j)T,
\]

where

\[
T := \frac{S_{2,3}}{\frac{3}{F_1} \left( \frac{1}{x_1 - x_3} + \frac{1}{x_1 - x_2} \right) - \left( \frac{1}{F_1} \right)'} \left( x_2 - x_3 \right)^2.
\]

An analogous argumentation as in (c) completes the proof of Theorem 7. \( \Box \)

8. Locally reducible \(\mathcal{C}\)-spaces and \(\mathfrak{P}\)-spaces.

Let \(M \) be a locally reducible Riemannian manifold \(M_1 \times \ldots \times M_r \). Then it follows at once from Theorem 2 that \(M \) is a \(\mathcal{C}\)-space if and only if each factor \(M_i \) is a \(\mathcal{C}\)-space. Corollary 5 implies that the analogous statement holds for \(\mathfrak{P}\)-spaces when each \(M_i \) and \(M \) are real analytic.

Remark. As a consequence of this, using our examples in Section 7, we can get examples of \(\mathfrak{P}\)-spaces in arbitrary dimensions by taking, for example, the product with symmetric spaces.
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References


Lemma 1. There exist $C^k$ functions $\lambda_1, \ldots, \lambda_n : I \to \mathbb{R}$ representing at each point the eigenvalues of $A$ (counted with its multiplicities).

Lemma 2. $A$ is $C^k$ diagonalizable on an open and dense subset of $\mathbb{R}$, that is, there exist an open and dense subset $J$ of $I$ and $C^k$ maps $E_1, \ldots, E_n : J \to V$ such that $E_1(t), \ldots, E_n(t)$ is an orthonormal basis of $V$ consisting of eigenvectors of $A(t)$ for all $t \in J$. In addition to this one might always assume $I = J$ in case $k = \omega$. For the case $k = \infty$, however, there are examples where $I = J$ under no circumstances.

Lemma 3. Let $A$ be $C^k$ diagonalizable on $I$, say $AE_i = \lambda_i E_i$ $(i = 1, \ldots, n)$. Then there exist a $C^k$ family $Q$ of orthogonal endomorphisms of $V$, a $C^k$ family $D$ of self-adjoint endomorphisms of $V$, and a $C^{k-1}$ family $T$ of skew-symmetric endomorphisms of $V$, such that

(a) $Q E_1, \ldots, Q E_n$ is an orthonormal basis of $V$ not depending on $t$;
(b) $D \circ Q E_i = \lambda_i Q E_i$ $(i = 1, \ldots, n)$;
(c) $A = Q^{-1} \circ D \circ Q$;
(d) $Q' = Q \circ T$;
(e) $A' = A \circ T - T \circ A + Q^{-1} \circ D' \circ Q$.

Proof. Let $T$ be the $C^{k-1}$ family of skew-symmetric endomorphisms of $V$ defined by $TE_i := -E_i'(i = 1, \ldots, n)$. Moreover, let $Q$ be the solution of the endomorphism-valued linear equation

$$Y' = Y \circ T, \quad Y(t_0) = \text{id}_V \quad (t_0 \in I).$$

Then $Q$ is $C^k$ and

$$(Q \circ Q^*)' = Q' \circ Q^* + Q \circ (Q^*)' = Q' \circ Q^* + Q \circ (Q')^* = Q \circ T \circ Q^* + Q \circ T^* \circ Q^* = 0,$$

since $T^* = -T$. Therefore

$$Q \circ Q^* = Q \circ Q^*(t_0) = \text{id}_V$$

and hence $Q^* = Q^{-1}$, that is, $Q$ is a family of orthogonal endomorphisms of $V$. Next, we calculate

$$(QE_i)' = Q'E_i + Q E_i' = Q(TE_i + E_i') = 0.$$

Thus $QE_1, \ldots, QE_n$ do not depend on $t$ and, because $Q$ is orthogonal, is an orthonormal basis of $V$. Now we define the $C^k$ family $D$ of self-adjoint endomorphisms of $V$ by

$$D := Q \circ A \circ Q^{-1}.$$ Then $D \circ Q E_i = Q \circ A E_i = \lambda_i Q E_i$. Finally, using

$$(Q^{-1})' = (Q'^*) = (Q')^* = (Q \circ T)^* = T^* \circ Q^* = -T \circ Q^* = -T \circ Q^{-1},$$

we calculate the derivative $A'$ of $A$:

$$A' = (Q^{-1} \circ D \circ Q)' = (Q^{-1})' \circ D \circ Q + Q^{-1} \circ D' \circ Q + Q^{-1} \circ D \circ Q' = -T \circ Q^{-1} \circ D \circ Q + Q^{-1} \circ D' \circ Q + Q^{-1} \circ D \circ Q \circ T = A \circ T - T \circ A + Q^{-1} \circ D' \circ Q. \quad \Box$$