An iterative algorithm of NW TLS-EC for three dimensional-datum transformation with large rotation angle

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Abstract: The Gauss-Markov (GM) model and the Errors-in-Variables (EIV) model are frequently used to perform 3D coordinate transformations in geodesy and engineering surveys. In these applications, because the observation errors in original coordinates system are also taken into account, the latter is more accurate and reasonable than the former. Although the Weighted Total Least Squares (WTLS) technique has been introduced into coordinate transformations as the measured points are heteroscedastic and correlated, the Variance-Covariance Matrix (VCM) of observations is restricted by a particular structure, namely, only the correlations of each points are taken into account. Because the 3D datum transformation with large rotation angle is a nonlinear problem, the WTLS is no longer suitable in this case. In this contribution, we suggested the nonlinear WTLS adjustments with equality constraints (NWTLS-EC) for 3D datum transformation with large rotation angle, which removed the particular structure restriction on the VCM. The Least Squares adjustment with Equality (LSE) constraints is employed to solve NWTLS-EC as the nonlinear model has been linearized, and an iterative algorithm is proposed with the LSE solution. A simulation study of 3D datum transformation with large rotation angle is given to insight into the feasibility of our algorithm at last.

Key words: nonlinear weighted total least squares; equality constraints; 3D datum transformation; heteroscedastic and correlated; orthogonal transformation

1 Introduction

The Least Squares (LS) adjustment has been frequently used for parameters estimate in geodetic under the assumption that only the variables in observation are affected by random errors which is normally distributed with zero mean, but the coefficient matrix A is considered as a fixed matrix. This is not the case in geodetic applications, however, such as in 3D datum transformations¹, where the errors existed both in the target coordinates system and the original coordinates system (which is used to organize the coefficient matrix). Such an adjustment model that considered all kinds of random errors involved in an over-determined system is called EIV². Total Least Squares (TLS), the terminology originally introduced by Golub³, had been developed to solve this model about 30 years ago. For the solution of TLS can be readily obtained from Singular Value Decomposition (SVD), it has been widely applied in system identification⁴, signal processing⁵, astronomy⁶ and so on. As reported in previous works⁷,⁸, the TLS is a relatively new approach in estimating parameters compared with the LS adjustment. Naturally, TLS adjustment is introduced into geodetic
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As we know, the datum transformations, employing a set of common points, which measured respectively in original coordinates system (henceforth, all coordinates are in the Cartesian coordinates system) and target coordinates system, to compute the transformation parameters, have been frequently encountered in geodesy and engineering surveys. Although there are numerous TLS algorithms have been developed to compute the transformation parameters in 2D coordinate system\cite{9-12}, only a few literatures represented those approaches for 3D coordinate transformations currently.

Acari\cite{11} introduced the generalized TLS (GTLS), also called Mixed Least Squares-Total Least Squares (MIS-TLS), into deformation analysis, intrinsically, which is a 3D coordinate transformations problem. Unfortunately, only the homoscedastic case is discussed in this study. Felus\cite{13} provided a good review of 3D affine transformations within GM model and EIV model, and proposed a new algorithm for computing the transformation parameters respect to multivariate EIV model under the heteroscedastic assumption. However, this algorithm cannot be employed to deal with the case that the errors of coordinates are correlated. Although Schaffrin\cite{14} and Mahboub\cite{12} have discussed WTLS dealing with the structured TLS problem in 2D coordinate transformations, they still did not take into account the correlation of coordinate components.

On the other hand, in many applications, in order to simplify the computation, a reasonable assumption always plus on the rotation angles is that sine and cosine of these angles are equal to their radian value and one, respectively. However, this is not the case in 3D datum transformation with large rotation angle. For all of these reasons, the NWTLs-EC approach will be discussed for 3D datum transformation with large rotation angle, and an iterative algorithm will also be designed to solve it in this paper.

2 Model of nonlinear weighted total least squares adjustment with equality constraints

As we know, the seven-parameter model of 3D datum transformations can be mathematically described as follows\cite{15},

\[
\begin{bmatrix}
X_i \\
Y_i \\
Z_i
\end{bmatrix}
= s
\begin{bmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{bmatrix}
\begin{bmatrix}
X_o \\
Y_o \\
Z_o
\end{bmatrix}
+ \begin{bmatrix}
\Delta X \\
\Delta Y \\
\Delta Z
\end{bmatrix}
\]

(1)

where \([X,Y,Z]^T\) denotes the vector of target coordinates. \([X_o,Y_o,Z_o]^T\) denotes the vector of original coordinates. \(s\) denotes the scale factor. \([\Delta X,\Delta Y,\Delta Z]^T\) denotes the vector of translation parameters. \(i\) is the amount of common points. The elements of rotation matrix can be calculated by

\[
\begin{align*}
a_1 &= \cos(e_x)\cos(e_y) \\
a_2 &= \cos(e_x)\sin(e_y) + \sin(e_x)\sin(e_y)\cos(e_z) \\
a_3 &= \sin(e_x)\sin(e_y) - \cos(e_x)\sin(e_y)\cos(e_z) \\
b_1 &= -\cos(e_y)\sin(e_x) \\
b_2 &= \cos(e_y)\cos(e_x) - \sin(e_y)\sin(e_x)\sin(e_z) \\
b_3 &= \sin(e_y)\cos(e_x) + \cos(e_y)\sin(e_x)\sin(e_z) \\
c_1 &= \sin(e_y) \\
c_2 &= -\sin(e_x)\cos(e_y) \\
c_3 &= \cos(e_x)\cos(e_y)
\end{align*}
\]

(2)

where \(e_x, e_y, e_z\) are the rotation angles with respect to \(X, Y, Z\) axis, respectively. We employ \(R\) to represent the rotation matrix as

\[
R = \begin{bmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{bmatrix}
\]

(3)

where \(R\) is an orthogonal matrix, namely \(R^TR = I\). Chen\cite{15} pointed out that the identity equation \(R^TR = I\) can be represented as

\[
\mathbf{g}(x_i) = \begin{bmatrix}
a_1a_1 + a_2a_2 + a_3a_3 - 1 \\
b_1b_1 + b_2b_2 + b_3b_3 - 1 \\
c_1c_1 + c_2c_2 + c_3c_3 - 1 \\
a_1a_2 + b_1b_2 + c_1c_2 \\
a_1a_3 + b_1b_3 + c_1c_3 \\
a_2a_3 + b_2b_3 + c_2c_3
\end{bmatrix} = 0
\]

(4)

In order to compute the transformation parameters by NWTLs-EC method, the transformation model (1)
should be re-expressed as

\[ y - e_1 = A_1 x_1 + (A_2 - E_2) f(x_2) \]  

(5)

where \( y = [X_{zd} \ Y_{zd} \ \ldots \ X_{zd} \ Y_{zd} \ Z_{zd}]^T \) corresponding to the random error vector \( e_1 (m \times 1) \), \( m = 3 \times i \).

\[ x_i = [\Delta X \ \Delta Y \ \Delta Z]^T. \]

\[ A_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 1 & 0 & 0 \end{bmatrix}^T \]

(6)

is an \( m \times 3 \) fixed matrix.

\[ A_2 = \begin{bmatrix} X_{zd} & Y_{zd} & Z_{zd} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_{zd} & Y_{zd} & Z_{zd} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ X_{zd} & Y_{zd} & Z_{zd} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_{zd} & Y_{zd} & Z_{zd} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & X_{zd} & Y_{zd} & Z_{zd} \end{bmatrix} \]

(7)

is an \( m \times 9 \) variable matrix affected by random matrix \( E_2 (m \times 9) \).

\[ f(x_2) = s [a_1 \ a_2 \ a_3 \ b_1 \ b_2 \ b_3 \ \cdots \]  

(8)

with \( x_2 = [a_1 \ a_2 \ a_3 \ b_1 \ b_2 \ b_3 \ \cdots \ s]^T. \)

If we don’t consider the constraints (4), the NWTLS-EC model becomes the MLS-STLS adjustment. Here we can easily discover that not only the coefficient matrix \( A \), but also the function of parameters \( f(x_2) \) is nonlinear. That is why the model is named as nonlinear weighted TLS adjustment with equality constraints, and used to estimate the parameters of 3D affine transformation under the assumption that the VCM is heterogeneous and correlated without structure constraints.

The stochastic properties of errors are characterized as follows,

\[ \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} e_1 \\ \text{vec}(E_2) \end{bmatrix} - \begin{bmatrix} 0 \\ \sigma^2 \begin{bmatrix} Q \ & 0 \\ 0 & O \end{bmatrix} \end{bmatrix} \]

(9)

where \( e_1 \) and \( e_2 \) are the \( m \times 1 \) error vector of measured coordinates in target system and original system, respectively. \( \sigma^2 \) denotes the (unknown) variance component. \( Q_1 \) and \( Q_2 \) are symmetric and non-negative definite matrices with the size of \( m \times m \).

Compared with other methods to define the covariance matrix such as described by Felus[15], this definition not only take into account the correlation of points but also the correlation of coordinate components that constantly appeared among the local coordinates system construction by GPS technology.

With above discussion, the objective function to solve NWTLS-EC model can be described as

\[ e_1^T P_1 e_1 + e_2^T P_2 e_2 = \min \]

subject to

\[ y - e_1 = A_1 x_1 + (A_2 - E(e_2)) f(x_2) \]  

(11)

\[ g(x_2) = 0 \]  

(12)

where \( P_1 = Q_1^{-1} \), and \( P_2 = Q_2^{-1} \).

3 Solution of NWTLS-EC

To a Structured Total Least Squares (STLS) problem, there are numerous algorithms have been proposed over the past twenty years[16-19]. As reported previously[17], \( e_1 \) can be expressed as

\[ e_1 = y - A_1 x_1 - (A_2 - E(e_2)) f(x_2) \]

(13)

This means if \( [x_1^T, x_2^T]^T \) and \( e_2 \) are estimated, \( e_1 \) can also be yielded accordingly. Assume

\[ x_1 = x_1^0 + \delta x_1 \]  

(14)

\[ x_2 = x_2^0 + \delta x_2 \]  

(15)

\[ e_2 = e_2^0 + \delta e_2 \]  

(16)

and insert these equations into, the expansion of observation function is expressed as
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\[
e_z = y - A_1 x_1^0 - (A_2 - E(e_2^0)) f(x_2^0) - A_1 \delta x_1 - (A_2 - E(e_2^0)) J(x_2^0) \delta x_2 + G(x_2^0) \delta e_2
\]  
(17)

where \(E(e_2^0)\) has the identical structure with the coefficient matrix \(A_2\) described as

\[
\begin{bmatrix}
\Delta X_2 & \Delta Y_2 & \Delta Z_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Delta X_2 & \Delta Y_2 & \Delta Z_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Delta X_2 & \Delta Y_2 & \Delta Z_2 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \Delta X_2 & \Delta Y_2 & \Delta Z_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Delta X_2 & \Delta Y_2 & \Delta Z_2 & 0 \\
\end{bmatrix}
\]  
(18)

\(J(x_2^0)\) is a 9x10 matrix from initial parameters, and described as:

\[
J(x_2^0) = [\text{diag}([s^0, \ldots, s^9]), r^0]
\]  
(19)

Owing to the matrix \(G(x_2)\) meets the identity equation \[20\]

\[
E_f(x_2) = G(x_2) \delta x_2
\]  
(20)

where the initial \(G(x_2^0)\) is computed by

\[
G(x_2^0) = \text{kron}(I, s^0 R^0)
\]  
(21)

On the other hand, the constraints \(g(x_2) = 0\) also should be linearized. From equation (15), it can be rewritten as

\[
g(x_2^0) + M(x_2^0) \delta x_2 = 0
\]  
(22)

From equations (17) and (22), we can readily obtain the new formulates as follows,

\[
\begin{bmatrix}
\delta x_1 \\
\delta x_2 \\
\delta e_2
\end{bmatrix} =
\begin{bmatrix}
-A_1 & -(A_2 - E(e_2^0)) J(x_2^0) & G(x_2^0) \\
O & O & I \\
- \delta y + A_1 x_1^0 + (A_2 - E(e_2^0)) f(x_2^0) & -e_2^0 & e_2^0 \\
\end{bmatrix}
\]  
(23)

It is apparently that the NWTLSE-EC model becomes a classical LS approximate problem. Therefore, according to the theory of classical weighted least squares adjustment with constraints we readily have normal function as

\[
\begin{bmatrix}
B^T P B + C^T \hat{e} \\
C & 0
\end{bmatrix}
\begin{bmatrix}
\hat{e} \\
W
\end{bmatrix} = 0
\]  
(25)

Hence, a possible formula for estimating variance component of unit weight can also be expressed as

\[
\hat{\sigma}^2 = \frac{V^TPV}{m-n}
\]  
(26)

It is notable that for a nonlinear model we do not claim that the result carried out by equation (26) is an unbiased and optimal estimate. Furthermore, as an iterative solution of nonlinear problem the initial value is a crucial plot for our algorithm to converge to a stable value. Therefore, a simple algorithm described by Felus (13) (where called algorithm 2) can be used to compute the initial parameters for the following algorithm.

4 An iterative algorithm for NWTLSE-EC

Based on the formulas derived in section 3, a possible iterative algorithm can be designed as follows:

Input: \(y, A_1, A_2, Q_1, Q_2, \hat{R}^{(k)}, z^{(k)}, x_1^0, e^{(k)}\) (a given “acceptable” tolerance)

Step 1, calculate \(B^{(k)}, L^{(k)}, C^{(k)},\) and \(W^{(k)}\) by equations (23) and (24), respectively.

Step 2, set

\[
N^{(k)} = \begin{bmatrix}
(B^{(k)})^T P B^{(k)} & (C^{(k)})^T \\
C^{(k)} & 0
\end{bmatrix}
\]

\[b^{(k)} = \begin{bmatrix}
(B^{(k)})^T P L^{(k)} \\
W^{(k)}
\end{bmatrix}
\]

and calculate
Step 3, calculate

\[
\begin{bmatrix}
\hat{\xi}^{(4)} \\
\hat{k}^{(4)} \\
-1
\end{bmatrix} = -S[; , \text{end}] / S[\text{end}, \text{end}]
\]

Step 4, calculate

\[
\hat{x}_1^{(k+1)} = \hat{x}_1^{(4)} + \hat{\xi}^{(4)} [1:3, ;]
\]

\[
\hat{x}_2^{(k+1)} = \hat{x}_2^{(4)} + \hat{\xi}^{(4)} [4:13, ;]
\]

\[
\hat{e}_2^{(k+1)} = \hat{\xi}^{(4)} + \hat{\xi}^{(4)} [14: \text{end}, ;]
\]

Then reconstruct \( \hat{E}^{(k+1)} (\hat{e}_2^{(k+1)}) \) with equation (18), \( \hat{R}^{(k+1)} \) with equation (3) and \( \hat{e}^{(k+1)} \) from \( \hat{e}_2^{(k+1)} \).

Step 5, repeat the first step to the forth until \( \hat{\xi}^T \hat{\xi} < \hat{\xi}^0 \), and calculate \( \hat{\sigma}^2_0 \) according to equation (26).

Output: Translation parameters vector \( \hat{x} \), rotation matrix \( \hat{R} \), scale \( \hat{s} \), the estimated residual vector of \( \hat{e} \), and estimated variance component \( \hat{\sigma}_0^2 \).

5 Experiments and analysis

Owing to the developed algorithm mainly used to compute the transformation parameters under the assumption that all the coordinates measured in two 3D systems and contaminated by random error with large rotation angle, a simulation experiment is designed to test the performance of our algorithm. As is illustrated in Figure 1, there are 36 points determined between latitude from 24°N to 29°N and longitude from 110°E to 115°E with one-degree increment. The WGS-84 ellipsoid parameters are used to calculate the original coordinates, and the designed transformations parameters (translation parameters \([-120 737 8714] \) (km), rotation angles \([175 168 280] \) (degree) and scale 1.135) are employed to compute the target coordinates.

In order to capture a cross-covariance matrix, the observations are carried out as follows. In the original coordinate system, if the distance between two points, excluding the compare points, is less than 300 km, it (\( od \)) will be perturbed by the random errors which is normally distributed with zero mean and variance

\[
\sigma^2 = (12 \text{mm} + 0.02 \text{mm/km} \times od \text{km})^2
\]

Similarly, the distance \( td \) in target coordinate system will be noised by the random errors which is also zero mean and variance

\[
\sigma^2 = (6 \text{mm} + 0.01 \text{mm/km} \times td \text{km})^2
\]

Then, according to the simulated data, the adjusted coordinates and the adjusted cross-covariance in two coordinate systems can be obtained by the nonlinear weighted LS adjustments with the measured distances, where the weight matrix defined by the reciprocal of distance variance\(^{[21]}\). Once the variance-covariance \( Q_1 \) and \( Q_2 \) being estimated, the weights matrix can be defined by equation (9).

The nonlinear weighted LS (NWLS) method\(^{[15]}\) and the NWTLS-EC method are used to compute the transformations parameters, and equation (26) is employed to estimate the variance component of unit weight. By running NWLS and NWTLS-EC with 1000 independent simulated data sets, the estimated variance component are shown in figure 2. The priori variance component of unit weight in each simulation is always equal to one. Obviously, the mean of the estimated variance component of unit weight from NWLS is seriously biased.
Nevertheless, the mean from NWLS-EC is closer to the priori variance component of unit weight. Therefore, NWLS-EC approach is more reasonable than NWLS adjustment in this aspect.

In order to demonstrate the efficiency of NWLS-EC algorithm in dealing with the case that the observations are heteroscedastic and correlated in 3D coordinate transformations, the author made a comparison of transformation parameters between the estimated value and the given (designed) value.

Similarly, the 1000 simulations are performed by using NWLS-EC algorithm. The difference between the estimated and the given translations is shown in figure 3, their means are 0.2104 m, -0.5324 m and...
The relative error are \(1.7534 \times 10^{-4}\)\% and \(7.2244 \times 10^{-3}\)\% respectively.

To perform an 3D datum transformations, not the rotation angle but the rotation matrix is required directly. Therefore, we compared \(\hat{R}\) with \(R\) to instead the rotation angles. Figure 4 shows the difference between the elements of \(\hat{\alpha}_1\), \(\hat{\alpha}_2\) and \(\hat{\alpha}_3\) in \(\hat{R}\) and its corresponding elements \(\alpha_1\), \(\alpha_2\) and \(\alpha_3\) in \(R\). Their means of difference are 0.00000140836, 0.00000013898 and -0.00000205413, respectively. The difference between the elements of \(\hat{\beta}_1\), \(\hat{\beta}_2\) and \(\hat{\beta}_3\) in \(\hat{R}\) and its corresponding elements \(\beta_1\), \(\beta_2\) and \(\beta_3\) in \(R\) are illustrated in figure 5, and their means are -0.00000239966, 0.00000183879 and 0.00000024832, i.e., relative errors are \(2.4838 \times 10^{-3}\)\% and \(1.1829 \times 10^{-2}\)\% respectively. The difference of the last row between \(\hat{c}_1\), \(\hat{c}_2\), \(\hat{c}_3\), \(c_1\), \(c_2\) and \(c_3\) are presented in figure 6. Their corresponding means of differences are 0.00000004313, 0.00000172350, -0.0000015999 and relative errors are \(2.0742 \times 10^{-3}\)\%, \(2.0217 \times 10^{-3}\)\%, \(1.6418 \times 10^{-3}\)\% respectively. By studying equation (2), we can easily to explain that why \(\hat{c}_1\) is closer to given value \(c_1\) than other elements, its function is simpler than others.

Furthermore, the difference of scale is shown in figure 7, its means and relative errors are 0.000000000006161 and (5.4 \times 10^{-10})\%, respectively. The decimal places in vertical axis of figure 7 is 10^{-3}, which is the same as the threshold of each simulation.

From the above discussions and experiment results, apparently, the suggested iterative algorithm for NWTLS-EC method is an effective approximation approach for three dimensional datum transformation with large rotation angle if the initial value is included in the convergence radius.

Finally, we employ the estimated parameters to transform extrapolation and interpolation points, and computed their mean square error of a point by running NWLS and NWTLS-EC with 1000 independent simulated data set, respectively. From the figure 8 and 9, the mean square error of a point computed from NWTLS-EC is smaller than it from NWLS.
Figure 5  The difference between the estimated $b_1$, $b_2$, $b_3$, and the given value, where the estimated values are computed from NWTLS-EC with 1000 independent simulated data set; the black line in each sub-figure denotes the mean of differences.

Figure 6  The difference between the estimated $c_1$, $c_2$, $c_3$, and the given value, where the estimated results are computed from NWTLS-EC with 1000 independent simulated data set; the black line in each sub-figure denotes the mean of differences.
The difference of scale between the estimated value and given value, where the estimated values are computed from NWILS-EC with 1000 independent simulated data set; the black line denotes the mean of difference.

The blue and purple line are the mean square error of a point from NWLS and NWILS-EC method, respectively; the black dash and the green dot line are the corresponding means within 1000 simulations; they are extrapolation points.

6 Conclusions

We investigated the nonlinear weighted total least squares adjustment with equality constraints in estimating the parameters of 3D coordinate transformations with large rotation angle, and introduced the classical weighted least squares with equality constraints to solve this model once it being linearized. The simulation experiment demonstrated the proposed algorithm for NWILS-EC method can be used for 3D Cartesian coordinate transformations with large rotation angle. Of course, this method is also adapted for small rotation angle case.
The blue and purple line are the mean square error of a point from NWLS and NWTLS-EC, respectively; the black dash and the green dot line are the corresponding mean within 1000 simulations; they are interpolation points.

Consequently, some conclusions can be drawn from the discussion above as follows:

1) The proposed NWTLS-EC algorithm can successfully obtain the reasonable transformation parameters when the observations are heteroscedastic and correlated, even the coordinate components are correlated;

2) The proposed NWTLS-EC algorithm is well suited for 3D orthogonal transformation with large rotation angle.

Furthermore, a practical application such as ITRF coordinate transformation will be introduced in the forthcoming publication.

References


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