Conservative Systems of Semi-Linear Wave Equations with Periodic-Dirichlet Boundary Conditions

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1. INTRODUCTION

Let \( J = [0, 2\pi] \times [0, \pi], \ n \geq 1, \) be an integer, \( V: \mathbb{R}^n \to \mathbb{R} \) a function of class \( C^2 \) whose gradient and Hessian matrix are respectively denoted by \( V' \) and \( V'' \), and let \( h \in H \) with \( H = (L^2(J))^n \) be given, with the usual inner product \( (\cdot, \cdot) \) and corresponding norm \( |\cdot| \). We consider the system of semi-linear wave equations

\[
\begin{align*}
  u_{tt} - u_{xx} - V'(u) = h(t, x),
\end{align*}
\]

where subscripts denote the partial derivatives. By generalized solution of the periodic-Dirichlet problem on \( J \) for Eq. (1) (shortly GPDS on \( J \)) we mean a function \( u \in H \) such that the equality

\[
(v_{tt} - v_{xx}, u) - (V'(u), v) = (h, v)
\]

holds for all \( v \in (C^2(J))^n \) which satisfy the conditions

\[
\begin{align*}
  v(2\pi, x) - v(0, x) = v_t(2\pi, x) - v_t(0, x) = 0, & \quad x \in [0, \pi], \\
v(t, 0) = v(t, \pi) = 0, & \quad t \in [0, 2\pi].
\end{align*}
\]

If we write this problem as an abstract semi-linear equation in \( H \) and apply the result of [9], we easily find that system (1) has a unique GPDS on \( J \) for each \( h \in H \) if we can find real numbers \( \alpha \leq \beta \) such that \( [\alpha, \beta] \cap \sigma(L) = \emptyset \) and

\[
\alpha I \leq V''(u) \leq \beta I
\]

for all \( u \in \mathbb{R}^n \). Here \( L \) is the abstract realization in \( H \) of the wave operator with the periodic-Dirichlet conditions on \( J \) (see Section 2), so that its spectrum \( \sigma(L) \) is the set \( \{m^2 - l^2; l \in \mathbb{Z}, \ m \in \mathbb{N}^* \} \), and the relation \( A \leq B \)
between \((n \times n)\)-matrices in (2) means that \(B - A\) is positive semi-definite. Notice that this result extends to semi-linear wave systems a theorem of Lazer and Sanchez [8] on the existence of periodic solutions of ordinary differential systems of the form

\[ u''(t) - V'(u(t)) = h(t). \quad (3) \]

Condition (2), in the case of Eq. (3), was improved by Lazer [7] for the uniqueness problem, and by Ahmad [1] for the existence. They replace (2) by the condition

\[ A \leq V''(u) \leq B \]

for all \(u \in \mathbb{R}^n\), where \(A\) and \(B\) are symmetric \((n \times n)\)-matrices whose respective eigenvalues \(\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n\) and \(\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n\) are such that

\[ \bigcup_{k=1}^{n} [\alpha_k, \beta_k] \cap \{\ell^2 : \ell \in \mathbb{N}\} = \emptyset. \]

Notice that \(\{-\ell^2 : \ell \in \mathbb{N}\}\) is the spectrum of \((d^2/dt^2)\) together with the 2\(\pi\)-periodic boundary conditions. Subsequent proofs and extensions of this result were given by Brown and Lin [5] and Ward [12], and an abstract version also applicable to semi-linear elliptic systems was introduced by Bates [2].

Recently, Bates and Castro [3] have considered the system (1) under Dirichlet boundary conditions on \([0, \pi] \times [0, n]\). By a combination of a minimax argument and Galerkin's method, they have proved that this problem has a unique solution if condition (4) holds with

\[ \bigcup_{k=1}^{n} [\alpha_k, \beta_k] \cap \sigma(L) = \emptyset, \]

(with here for \(L\) the abstract realization of the wave operator with Dirichlet boundary conditions on \(([0, \pi])^2\) and if \(h : ([0, \pi])^2 \to \mathbb{R}^n\) has a first derivative with respect to \(t\) belonging to \((L^2([0, \pi])^2))^n\) and satisfies the condition

\[ h(0, x) = h(\pi, x) = 0, \quad x \in [0, \pi]. \]

Those conditions on \(h\) are not very natural and the aim of this paper is to obtain an existence and uniqueness theorem for the GPDS of (1) on \(J\) (and Dirichlet conditions could have been treated as well) which avoids those restrictions on \(h\). More precisely, we shall prove the following

**Theorem 1.** Let \(V : \mathbb{R}^n \to \mathbb{R}^n\) be a function of class \(C^2\) and let \(J = [0, 2\pi] \times [0, \pi]\). Assume that there exist two \((n \times n)\)-symmetric matrices
A and B, with respective eigenvalues \( \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \) and \( \beta_1 \leq \beta_2 \leq \cdots \leq \beta_n \) such that one has
\[
A \leq V''(u) \leq B
\]
for every \( u \in \mathbb{R}^n \) and
\[
\bigcup_{k=1}^{n} [\alpha_k, \beta_k] \cap \{m^2 - l^2 : l \in \mathbb{Z}, m \in \mathbb{N}^*\} = \emptyset.
\]

Then Eq. (1) with the periodic-Dirichlet boundary conditions on \( J \) has a unique generalized solution \( u \in (L^2(J))^n \) for every \( h \in (L^2(J))^n \).

The proof will use a Galerkin's type argument like in Bates and Castro's paper [3] but the approximate equations will be solved by a global inverse function theorem and monotonicity-type properties will replace compactness in the limit process.

2. Abstract Formulation
and the Corresponding Galerkin Approximate Equations

If \( \{c_k : 1 \leq k \leq n\} \) denotes an orthonormal basis in \( \mathbb{R}^n \) and if we set
\[
v_{lm}(t, x) = \exp(ilt) \sin mx, \quad l \in \mathbb{Z}, m \in \mathbb{N}^*,
\]
then every \( u \in H \) has a Fourier series
\[
u = \sum_{k=1}^{n} \sum_{(l, m) \in \mathbb{Z} \times \mathbb{N}^*} u_{klm} v_{lm} c_k,
\]
where the \( u_{klm} \) satisfy \( u_{klm} = u_{k,-l,m} \) to make the series real. If we define
\[
dom L = \left\{ u \in H : u \text{ is given by } (6) \text{ with } \sum_{k=1}^{n} \sum_{(l, m) \in \mathbb{Z} \times \mathbb{N}^*} (m^2 - l^2)^2 |u_{klm}|^2 < \infty \right\},
\]
and
\[
L : \dom L \subset H \to H, u \mapsto \sum_{k=1}^{n} \sum_{(l, m) \in \mathbb{Z} \times \mathbb{N}^*} (m^2 - l^2) u_{klm} v_{lm} c_k,
\]
it is easy to check that $L$ is a self-adjoint operator such that

$$\text{ker } L = \text{span}\{\cos mt \sin mx, \sin mt \sin mx : m \in \mathbb{N}^*, 1 \leq k \leq n\},$$

$$\text{Im } L = (\text{ker } L)^\perp,$$

$$\sigma(L) = \{m^2 - l^2 : l \in \mathbb{Z}, m \in \mathbb{N}^*\}.$$

Moreover, for every $h \in H$, $u$ is a GPDS on $J$ of the system

$$u_{tt} - u_{xx} = h$$

if and only if $u \subset \text{dom } L$ and $Lu = h$ (see, e.g., [6, 10] for details). Therefore, if we assume the existence of a constant $C > 0$ such that, for all $u \in \mathbb{R}^n$, one has

$$|V''(u)| \leq C,$$  \hspace{1cm} (9)

it is well known that the mapping $N$ defined on $H$ by

$$(Nu)(t, x) = V'(u(t, x)) \quad \text{a.e. on } J$$

maps continuously $H$ into itself, and then the existence of GPDS on $J$ for (1) is equivalent to the existence of a solution $u \in \text{dom } L$ for the equation in $H$

$$Lu - Nu = h. \hspace{1cm} (10)$$

We shall now construct Galerkin's approximate equations for (10) in a way similar to that used in [3] and motivated by Lazer's initial constructions in [7]. For the $(n \times n)$-symmetric matrices $A$ and $B$ introduced in (4), let $\{a_k : 1 \leq k \leq n\}$ and $\{b_k : 1 \leq k \leq n\}$ be orthonormal bases in $\mathbb{R}^n$ such that

$$Aa_k = a_k a_k, \hspace{1cm} Bb_k = \beta_k b_k \quad (1 \leq k \leq n).$$

For every $j \in \mathbb{N}$, define the subspace $H_j$ of $H$ by

$$H_j = \left\{ \sum_{k=1}^n \sum_{(l,m) \in \mathbb{Z} \times \mathbb{N}^*} u_{klm} v_{lm} b_k : u_{klm} \in \mathbb{R}, u_{klm} = u_{k,-l,m} \right\},$$

where $(\mathbb{Z} \times \mathbb{N}^*)_j = \{(l, m) \in \mathbb{Z} \times \mathbb{N}^* : |m^2 - l^2| \leq j, m^2 \leq j\}$. Notice that by this construction, the restriction of $L$ to $\text{dom } L \cap H_j$ has, in contrast with $L$, a spectrum bounded below and above and made of eigenvalues having finite multiplicity. Moreover, $\bigcup_{j \in \mathbb{N}} H_j$ is dense in $H$ and if we denote by $P_j : H \rightarrow H$ the orthogonal projector onto $H_j (j \in \mathbb{N})$, the Galerkin's approximate equations for (10) will be

$$Lu_j - P_j Nu_j = P_j h, \quad u_j \in \text{dom } L \cap H_j = H_j, \quad j \in \mathbb{N}, \hspace{1cm} (12)$$
which will be studied using a global inverse function theorem (see [5] for a different use of this theorem to the study of (3)).

3. THE GLOBAL INVERSE FUNCTION THEOREM AND THE EXISTENCE OF SOLUTIONS FOR THE GALERKIN'S EQUATIONS

We state here the used version of the global inverse function theorem for reader's convenience.

**Lemma 1.** If $X$ and $Y$ are Banach spaces and if $F: X \to Y$ is a mapping which is continuously Fréchet-differentiable on $X$ with a Fréchet differential $F'$ satisfying the following conditions:

(i) $F'(u): X \to Y$ is bijective for all $u \in X$;
(ii) there exists $K > 0$ such that, for all $u \in X$, one has

$$|(F'(u))^{-1}| \leq K,$$

then $F$ is a homeomorphism.

We shall refer to [4] for a proof of this lemma and for its historical development starting with Hadamard's version for finite-dimensional spaces, which will be used here. We are indebted to the referee for the simplified version of Section 3 given here.

Let $j \in \mathbb{N}$ be fixed. To apply Lemma 1 to the corresponding Galerkin's appropriate equation (12) we have to introduce a direct sum decomposition of $H_j$ which is due to Bates and Castro [3].

Let

$$X_j = \left\{ \sum_{k=1}^{n} \sum_{(l,m) \in (Z \times N^r)_j \mid m^2 - l^2 > \beta_k} u_{k\ell m} v_{lm} b_k : u_{k\ell m} \in \mathbb{R}, u_{k\ell m} = u_{k,-l,m} \right\},$$

$$Y_j = \left\{ \sum_{k=1}^{n} \sum_{(l,m) \in (Z \times N^r)_j \mid m^2 - l^2 < \beta_k} u_{k\ell m} v_{lm} b_k : u_{k\ell m} \in \mathbb{R}, u_{k\ell m} = u_{k,-l,m} \right\},$$

$$Z_j = \left\{ \sum_{k=1}^{n} \sum_{(l,m) \in (Z \times N^r)_j \mid m^2 - l^2 < \alpha_k} u_{k\ell m} v_{lm} b_k : u_{k\ell m} \in \mathbb{R}, u_{k\ell m} = u_{k,-l,m} \right\}.$$

Clearly, $H_j = X_j \oplus Y_j$ (orthogonal direct sum) and, because of condition (5), one has

$$\dim Y_j = \dim Z_j < \infty. \quad (13)$$

We now prove the existence of Galerkin's approximate solutions.
Lemma 2. Under the assumptions of Theorem 1, Eq. (12) has for each \( j \in \mathbb{N} \) and each \( h \in H \) a unique solution \( u_j \in \text{dom } L \cap H_j \) and there exists a constant \( C = C(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k, |h|) \) such that, for all \( j \in \mathbb{N} \), we have

\[
|u_j| \leq C.
\]

Proof. We shall show that the mapping \( F_j : H_j \to H_j \) defined by

\[
F_j u_j = Lu_j - P_j Nu_j
\]

for every \( u_j \in H_j \) satisfies all the conditions of Lemma 1.

The continuous Fréchet differentiability of \( F_j \) is trivial. We need some estimates to prove that conditions (i) and (ii) of Lemma 1 are satisfied.

If \( u_j \in H_j \) and \( x_j \in \text{dom } L \cap X_j = X_j \), with

\[
x_j = \sum_{k=1}^{n} \sum_{(l,m) \in (\mathbb{Z} \times 
abla^+)_{j}}^{m^2 - l^2 > \beta_k} x_{klm} v_{lm} b_k,
\]

we have

\[
(L x_j - P_j N'(u_j) x_j, x_j) \geq \sum_{k=1}^{n} \sum_{(l,m) \in (\mathbb{Z} \times \nabla^+)_{j}}^{m^2 - l^2 > \beta_k} (m^2 - l^2) |x_{klm}|^2 - (B x_j, x_j)
\]

\[
= \sum_{k=1}^{n} \sum_{(l,m) \in (\mathbb{Z} \times \nabla^+)_{j}}^{m^2 - l^2 > \beta_k} (m^2 - l^2 - \beta_k) |x_{klm}|^2
\]

\[
\geq \min_{1 < k < n} \left( \frac{m^2 - l^2 - \beta_k}{m^2 - l^2 - \beta_k} |x_j|^2 + m_1 |x_j|^2 \right). \quad (14)
\]

Similarly, if \( z_j \in \text{dom } L \cap Z_j = Z_j \), we find

\[
(L z_j - P_j N'(u_j) z_j, z_j) \leq -\min_{1 < k < n} \left( \frac{\alpha_k - m^2 + l^2}{m^2 - l^2 - \beta_k} |z_j|^2 \right) = -m_2 |z_j|^2. \quad (15)
\]

Inequalities (14) and (15) imply that \( X_j \cap Z_j = \{0\} \) which, together with (13) and a lemma of Lazer [7] imply that \( H_j = X_j \oplus Z_j \) algebraically and hence topologically.

Consequently, if \( u_j \in H_j, \ v_j \in H_j, \ v_j = x_j + z_j \) with \( x_j \in X_j, \ z_j \in Z_j \), we obtain, using (14), (15) and the symmetry of \( L \) and \( P_j N'(u_j) \),

\[
(F'(u_j) v_j, x_j - z_j) - (F'(u_j) x_j, x_j) - (F'(u_j) z_j, z_j)
\]

\[
\geq m_1 |x_j|^2 + m_2 |z_j|^2 \geq m_0 (|x_j|^2 + |z_j|^2)
\]

\[
\geq (m_0/2)(|x_j| + |z_j|)^2,
\]
where 

\[ m_0 = \min\{m_1, m_2\}. \]

As a consequence, we obtain

\[ (m_0/2)(|x_j| + |z_j|)^2 \leq |F'(u_j)v_j| (|x_j| + |z_j|), \]

and hence

\[ (m_0/2)|v_j| \leq (m_0/2)(|x_j| + |z_j|) \leq |F_j'(u_j)v_j|. \]

This implies that \( F_j(u_j): H_j \to H_j \) is bijective (since \( \dim H_j < \infty \)) and

\[ |(F_j'(u_j))^{-1}| \leq 2/m_0 \]

for all \( u_j \in H_j \), with \( m_0 \) independent of \( j \). By Lemma 1, \( F_j: H_j \to H_j \) is a homeomorphism. To obtain the estimate for the unique solution \( u_j \) of equation in \( H_j \), with \( h \in H \),

\[ F_j(u_j) = P_j h, \]

we notice that

\[ u_j = F_j^{-1}(P_j h) - F_j^{-1}(F_j(0)); \]

hence, using the integral mean value theorem, we get

\[ |u_j| \leq (2/m_0) |P_j h - F_j(0)| \leq (2/m_0) |h + N(0)|, \]

with a right-hand member independent of \( j \). The proof is complete.

4. CONVERGENCE OF GALERKIN'S METHOD FOR SOME SEMI-LINEAR EQUATIONS

Let \( H \) be a real Hilbert space, with inner product \((\cdot,\cdot)\) and corresponding norm \(|\cdot|\), and let \( \tilde{L}: \text{dom} \tilde{L} \subset H \to H \) be a linear, closed, densely defined operator such that

\[ \text{Im} \tilde{L} = (\ker \tilde{L})^\perp \]

and whose right inverse on \( \text{Im} \tilde{L} \) defined by

\[ \tilde{K} = (\tilde{L}|_{\text{dom} \tilde{L} \cap \text{Im} \tilde{L}})^{-1} \]
CONSERVATIVE SYSTEMS OF SEMI-LINEAR WAVE EQUATIONS

is compact. Denoting by $P: H \to H$ the orthogonal projector onto $\ker \tilde{L}$, we shall say that the sequence $(v_k)$ in $\text{dom } \tilde{L}$ converges to $v \in H$, and we shall write

$$v_k \overset{P}{\to} v,$$

if

$$Pv_k \to Pv \quad \text{and} \quad (I - P)v_k \to (I - P)v$$

for $k \to \infty$, where $\to$ denotes the weak convergence in $H$. We then say (see, e.g., [11]) that the mapping $\tilde{N}: H \to H$ is of type $m(\tilde{L})$ if, for each sequence $(v_k)$ in $\text{dom } \tilde{L}$ such that, for $k \to \infty,$

$$v_k \overset{P}{\to} v \quad \text{and} \quad (\tilde{N}v_k, v_k - v) \to 0,$$

one has

$$\tilde{N}v_k \to \tilde{N}v.$$

One can show, using Minty's trick, that every continuous monotone mapping which takes bounded sets into bounded sets is of type $m(\tilde{L})$ for every $\tilde{L}$ satisfying the properties listed above.

We now state and prove a convergence result for Galerkin's method associated to nonlinear perturbations of $\tilde{L}$, a variant of a result given in [11].

**Lemma 3.** Assume that there exists a sequence $(H_j)$ of finite-dimensional vector subspaces of $H$ such that

$$H_j \subset H_{j+1}, \quad \tilde{L}(\text{dom } \tilde{L} \cap H_j) \subset H_j \quad (j \in \mathbb{N}), \quad H = \bigcup_{j \in \mathbb{N}} H_j,$$

and let $P_j: H \to H$ be the orthogonal projector onto $H_j \quad (j \in \mathbb{N})$. Let $\tilde{N}: H \to H$ be a mapping of type $m(\tilde{L})$ taking bounded sets into bounded sets. Assume that for some $h \in H$ and some $r > 0$ the equation

$$\tilde{L}v_j - P_j\tilde{N}v_j = P_jh$$

has a solution $v_j \in \text{dom } \tilde{L} \cap H_j$ such that $|v_j| \leq r \quad (j \in \mathbb{N})$.

Then, equation

$$\tilde{L}v - \tilde{N}v = h$$

has at least one solution $v \in \text{dom } \tilde{L}$ such that $|v| \leq r$. 
Proof. By the reflexivity of $H$, there exists $v \in H$ and a subsequence $(v_{jk})$ of $(v_j)$ such that

$$v_{jk} \to v$$

for $k \to \infty$, and $|v| \leq r$. Equation (16) is equivalent to the system

$$PP_j \tilde{N}v_j + PP_j h = 0, \quad \tilde{L}Qv_j - QP_j \tilde{N}v_j = QP_j h,$$  \hfill (17)

where $Q = I - P$, and hence to the system

$$PP_j(\tilde{N}v_j + h) = 0, \quad Qv_j = \tilde{R}QP_j(\tilde{N}v_j + h).$$  \hfill (18)

From the second equation in (18), the boundedness property of $\tilde{N}$ and the compactness of $\tilde{R}$, we deduce that

$$Qv_{jk} \to Qv$$

and hence

$$v_{jk} \to v$$

as $k \to \infty$. On the other hand, using the first equation in (17), we obtain

$$(\tilde{N}v_{jk}, v_{jk} - v) = (P_{jk} \tilde{N}v_{jk}, v_{jk} - v) - (\tilde{N}v_{jk}, (I - P_{jk})v))$$

$$= -(PP_j h, v_{jk} - v) + (QP_j \tilde{N}v_{jk}, Q(v_{jk} - v)) - (\tilde{N}v_{jk}, (I - P_{jk})v),$$

so that, letting $k \to \infty$ and using the boundedness properties of $\tilde{N}$, we have

$$(\tilde{N}v_{jk}, v_{jk} - v) \to 0 \quad \text{if} \quad k \to \infty.$$  \hfill (19)

$N$ being of the type $m(\mathcal{L})$, this implies that

$$\tilde{N}v_{jk} \to \tilde{N}v \quad \text{if} \quad k \to \infty,$$  \hfill (20)

and, using the second equation in (17), the boundedness properties of $\tilde{N}$ and the weak closedness of the graph of $\tilde{L}$, we obtain, going, if necessary, to a subsequence.

$$v \in \text{dom } \tilde{L}, \quad \tilde{L}v_{jk} \to \tilde{L}v \quad \text{if} \quad k \to \infty.$$  \hfill (20)

Finally, let $m \in \mathbb{N}$ and $f_m \in H_m$. Then, by (16) we have, for all $k$ such that $j_k \geq m$,

$$0 = (\tilde{L}v_{jk} - P_{jk} \tilde{N}v_{jk} - P_{jk} h, f_m) = (\tilde{L}v_{jk} - \tilde{N}v_{jk} - h, f_m),$$

where
which gives, if \( k \to \infty \), using (19) and (20),

\[
(\tilde{L}v - \tilde{N}v - h, f_m) = 0.
\]

As \( n \in \mathbb{N} \) is arbitrary and \( \bigcup_{m \in \mathbb{N}} H_m \) dense in \( H \), this implies that

\[
\tilde{L}v - \tilde{N}v - h = 0
\]

and completes the proof.

5. PROOF OF THEOREM 1

The application of Lemma 3 to the situation in Sections 2 and 3 requires first rewriting Eq. (10) in an equivalent form in which the nonlinear perturbation is of type \( m \) with respect to the linear part. As \( 0 \in \sigma(L) \), it follows from condition (5) that there exists an integer \( 0 \leq p \leq n \) such that

\[
\beta_k \geq \alpha_k > 0, \quad 1 \leq k \leq p, \quad \alpha_k \leq \beta_k < 0, \quad p + 1 \leq k \leq n.
\]

Now, define the operators \( S_+ \) and \( S_- \) on \( \mathbb{R}^n \) as follows:

\[
S_+ x = \sum_{k=1}^{p} \xi_k a_k, \quad S_- x = \sum_{k=p+1}^{n} \xi_k b_k,
\]

for every \( x = \sum_{k=1}^{n} \xi_k a_k \) in \( \mathbb{R}^n \). Using again Lazer's lemma [7], we shall have

\[
\mathbb{R}^n = \text{Im} \ S_+ \oplus \text{Im} \ S_-
\]

if we can show that \( \text{Im} \ S_+ \cap \text{Im} \ S_- = \{0\} \). But, for \( x \in \text{Im} \ S_+ \), we have

\[
(Bx, x) \geq (Ax, x) = \sum_{k=1}^{p} \alpha_k \xi_k^2 \geq \left( \min_{1 \leq k \leq p} \alpha_k \right) \left( \sum_{k=1}^{p} \xi_k^2 \right)
\]

and, for \( x \in \text{Im} \ S_- \), we have

\[
(Ax, x) \leq (Bx, x) = \sum_{k=p+1}^{n} \beta_k \xi_k^2 \leq \left( \max_{p+1 \leq k \leq n} \beta_k \right) \left( \sum_{k=p+1}^{n} \xi_k^2 \right),
\]

and, as \( \min_{1 \leq k \leq p} \alpha_k > 0 \) and \( \max_{p+1 \leq k \leq n} \beta_k < 0 \), we have \( \text{Im} \ S_+ \cap \text{Im} \ S_- = \{0\} \). If we now define the operators \( S_+ \) and \( S_- \) on \( H \) by

\[
(S_\pm u)(t, x) = S_\pm (u(t, x)) \quad \text{a.e. on } J,
\]
then \( H = \text{Im} \, \mathcal{S}_+ \oplus \text{Im} \, \mathcal{S}_- \) (topologically), \( \mathcal{S}_\pm (\text{dom} \, L) \subset \text{dom} \, L \), \( \mathcal{S}_+ - \mathcal{S}_- \) is a linear homeomorphism on \( H \) with \( (\mathcal{S}_+ - \mathcal{S}_-)^{-1} = \mathcal{S}_+ - \mathcal{S}_- \), and, on \( \text{dom} \, L \), we have
\[
L \mathcal{S}_\pm = \mathcal{S}_\pm L.
\]

Consequently, if we set in Eq. (10)
\[
u = (\mathcal{S}_+ - \mathcal{S}_-)\nu, \quad \text{so that} \quad v = (\mathcal{S}_+ - \mathcal{S}_-)u,
\]
we obtain the equivalent equation
\[
L(\mathcal{S}_+ - \mathcal{S}_-)v - N((\mathcal{S}_+ - \mathcal{S}_-)v) = h.
\]
Moreover, as \( \mathcal{S}_\pm \) trivially commute with the \( P_j (j \in \mathbb{N}) \), \( u_j \in H_j \) will be a solution of (12) if and only if \( v_j = (\mathcal{S}_+ - \mathcal{S}_-)u_j \in H_j \) satisfies the equation
\[
L(\mathcal{S}_+ - \mathcal{S}_-)v_j - P_j N((\mathcal{S}_+ - \mathcal{S}_-)v_j)) = P_j h.
\]
If we set
\[
\widetilde{L} = L(\mathcal{S}_+ - \mathcal{S}_-), \quad \widetilde{N} = N \circ (\mathcal{S}_+ - \mathcal{S}_-),
\]
the \( \widetilde{L} \) has the same domain, kernel, range and spectrum as \( L \). Now, for every \( w \in H \), \( \widetilde{N} \) is also of class \( C^1 \) at \( w \), and
\[
\widetilde{N}'(w) = N'((\mathcal{S}_+ - \mathcal{S}_-)w) \circ (\mathcal{S}_+ - \mathcal{S}_-).
\]
Consequently, for every \( w \) and \( v \) in \( H \), we obtain, using the symmetry of \( N'(u) \),
\[
(\widetilde{N}'(w)v, v) = (N'((\mathcal{S}_+ - \mathcal{S}_-)w)(\mathcal{S}_+ - \mathcal{S}_-)v, (\mathcal{S}_+ + \mathcal{S}_-)v)
= (N'((\mathcal{S}_+ - \mathcal{S}_-)w) \mathcal{S}_+ v, \mathcal{S}_+ v) - (N'((\mathcal{S}_+ - \mathcal{S}_-)w) \mathcal{S}_- v, \mathcal{S}_- v)
\geq (A \mathcal{S}_+ v, \mathcal{S}_+ v) - (B \mathcal{S}_- v, \mathcal{S}_- v) > 0.
\]
This implies that \( \widetilde{N} \) is monotone; being also Lipschitzian, it is therefore of type \( m(\widetilde{L}) \) and it takes bounded sets into bounded sets. As \( \sigma(\widetilde{L}) \setminus \{0\} \) is made of eigenvalues with finite multiplicity with no finite accumulation point, its right inverse \( \widetilde{R} \) will be compact and we can apply Lemma 2 and Lemma 3 to obtain the existence of \( v \in \text{dom} \, \widetilde{L} \) such that
\[
\widetilde{L} v - \widetilde{N} v = h,
\]
and hence the existence of the solution \( u = (\mathcal{S}_+ - \mathcal{S}_-)v \) for (10). For the uniqueness, let \( u_1 \) and \( u_2 \) be two solutions of (10) and set
\[
u_j = P_j u_j, \quad \xi_j = x_j + z_j (x_j \in X_j, z_j \in Z_j), \quad v_j - x_j - z_j, \quad x_j - z_j, \quad w_j = z_j - z_j,
\]
so that

$$u_i^j - u_j^j = v_j + w_j$$

($i = 1, 2; j \in \mathbb{N}$). Therefore, using the notations of Lemma 2, we have

$$0 = (L(u^1 - u^2) - (Nu^1 - Nu^2), v_j - w_j)$$

$$= (L(u_j^i - u_j^i), v_j - w_j) - \left( \int_0^1 N'(u_j^2 + s(u^1 - u^2))(u^1 - u^2) \, ds, v_j - w_j \right)$$

$$- (L(v_j + w_j), v_j - w_j) - \left( \int_0^1 N'(u_j^2 + s(u^1 - u^2))(v_j + w_j) \, ds, v_j - w_j \right)$$

$$- \left( \int_0^1 N'(u_j^2 + s(u^1 - u^2))(u^1 - u_j^1 + u_j^2 - u^2) \, ds, v_j - w_j \right)$$

$$= (Lv_j, v_j) - (Lw_j, w_j) - \left( \int_0^1 N'(u_j^2 + s(u^1 - u^2)) \, v_j \, ds, v_j \right)$$

$$+ \left( \int_0^1 N'(u_j^2 + s(u^1 - u^2)) \, w_j \, ds, w_j \right)$$

$$- \left( \int_0^1 N'(u_j^2 + s(u^1 - u^2))(u^1 - u_j^1 + u_j - u^2) \, ds, v_j - w_j \right)$$

$$\geq m_1 |v_j|^2 + m_2 |w_j|^2 - C(|u^1 - u_j^1| + |u^2 - u_j^2|),$$

where $C > 0$ is some constant depending only on $|A|, |B|$ and $r$. Consequently,

$$v_j \to 0, \quad w_j \to 0 \quad \text{as} \quad j \to \infty,$$

so that

$$u^1 - u^2 = \lim_{j \to \infty} (u_j^1 - u_j^2) = \lim_{j \to \infty} (v_j + w_j) = 0,$$

and the proof of Theorem 1 is now complete.

**Remark.** It would be interesting to find a proof of Theorem 1 which does not require a Galerkin's argument. The difficulty lies in the fact that Lazer's lemma used in Section 3 requires that one of the spaces in the direct sum has a finite dimension. This is not the case if one tries to apply directly Lemma 1 to (10) by using the subspaces $X$ and $Z$ of $\mathcal{H}$ defined like $X_j$ and $Z_j$ but with $(\mathbb{Z} \times \mathbb{N}^*)_j$ replaced by $\mathbb{Z} \times \mathbb{N}^*$. Then $X$ and $Z$ are both infinite dimensional and Lazer's lemma cannot be used to prove that $\mathcal{H} = X \oplus Z$, a condition required by Lemma 1.
REFERENCES