Variants of the Two-Dimensional Boussinesq Equation with Compactons, Solitons, and Periodic Solutions

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Abstract—Variants of the two-dimensional Boussinesq equation with positive and negative exponents are studied. The sine-cosine ansatz is fruitfully used to carry out the analysis. Exact solutions of different physical structures: compactons, solitary patterns, solitons, and periodic solutions, are obtained. The quantitative change in the physical structure of the solutions is shown to depend mainly on the exponent of the wave function $u(x, t)$ and on the ratio $a/b$ of the derivatives of $u(x, t)$.

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1. INTRODUCTION

The balance between the weak nonlinear term $uu_x$ and the dispersion term $u_{xxx}$ of the KdV equation

$$u_t + uu_x + u_{xxx} = 0$$

(1)

gives rise to solitons. The nonlinearity and dispersion in the KdV equation dominate, while dissipation effects are small enough to be neglected in the lowest order approximation [1-8]. The KdV equation is therefore incapable of shock waves [1-15].

The best known two-dimensional generalizations of the KdV equations are the Kadomtsev-Petviashvili (KP) equation

$$u_t + uu_x + u_{xxx} + u_{yy} = 0$$

(2)

and the Zakharov-Kuznetsov (ZK) equation [9-11]

$$u_t + uu_x + \left(u_{xx} + u_{yy}\right)_x = 0,$$

(3)

and

$$u_t + uu_x + \left(u_{xx} + u_{yy} + u_{zz}\right)_x = 0,$$

(4)

in two- and three-dimensional spaces. The integrable KP equation characterizes small-amplitude, weakly dispersive waves on a fluid sheet. The ZK equation governs the behavior of weakly
nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [1,2].

The now well-known \( K(n, n) \) equation introduced in [16] is

\[
    u_t + a (u^n)_x + (u^n)_{xxx} = 0, \quad n > 1, \tag{5}
\]

where the convection term in the \( K(n, n) \) equation is nonlinear and the dispersion term \((u^n)_{xxx}\) is genuinely nonlinear as well. The delicate interaction between nonlinear convection with genuine nonlinear dispersion generates solitary waves with exact compact support that are termed compactons. Compactons are solitons with finite wavelength or solitons with the absence of infinite wings. Unlike soliton that narrows as the amplitude increases, the compacton’s width is independent of the amplitude.

The soliton concept has been examined by many mathematical methods such as the inverse scattering method, the Bäcklund transformation, the Darboux transformation, and the Painlevé analysis. However, the compacton concept has been studied by using many analytical and numerical methods in [16–37] such as the pseudo spectral method, finite differences method, sine-cosine ansatz.

The Boussinesq equation is a nonlinear fourth-order partial differential equation defined by

\[
    u_{tt} + au_{xx} + b (u^2)_{xx} + ku_{xxxx} = 0. \tag{6}
\]

The equation is used in the analysis of long waves in shallow water. It is also used in the analysis of many other physical applications such as the percolation of water in porous subsurface of a horizontal layer of material. The \((2 + 1)\)-dimensional Boussinesq equation

\[
    u_{tt} - u_{xx} - u_{yy} - (u^2)_{xx} - u_{xxxx} = 0 \tag{7}
\]

was studied thoroughly in [38].

Motivated by the rich treasure of the Boussinesq equation in \((1 + 1)\)- and \((2 + 1)\)-dimensional spaces, we will focus our study on two variants of the \((2 + 1)\)-dimensional Boussinesq-type of equations defined by:

1. Boussinesq-type of equations with positive exponents given by

\[
    u_{tt} - u_{xx} - u_{yy} - a (u^{2n})_{xx} - b (u^n (u^n)_{xx})_{xx} = 0, \quad n > 1, \tag{8}
\]

and

2. Boussinesq-type of equations with negative exponents given by

\[
    u_{tt} - u_{xx} - u_{yy} - a (u^{-2n})_{xx} - b (u^{-n} (u^{-n})_{xx})_{xx} = 0, \quad n > 1. \tag{9}
\]

It is clear that for \( n = 0 \), both equations read the standard two-dimensional wave equation.

The sine-cosine ansatz will be used to back up our analysis to develop compactons, solitary patterns, plane periodic, and solitary traveling waves solutions. The change of physical structure of the obtained solutions will be examined. In what follows, we highlight the main steps of the sine-cosine ansatz.

\section{THE SINE-COSINE ANSATZ}

1. We seek a formal travelling wave solution

\[
    u(x, y, t) = u(\xi), \quad (\xi = (\mu x + \eta y - ct)), \tag{10}
\]

where the wave variable \( \xi \) is
where $\mu$, $\eta$, and $c$ are constants that will be determined. Equations (10) and (11) carry a nonlinear PDE

$$P(u, u_t, u_x, u_u, u_{xx}, u_{yy}, u_{xxx}, \ldots) = 0,$$  \hspace{1cm} (12)

into a nonlinear ODE

$$Q(u, u_t, u_{xx}, u_{xxx}, \ldots) = 0.$$  \hspace{1cm} (13)

Notice that

$$Q_u, Q_{xx}, Q_{xxx} = 0.$$  \hspace{1cm} (14)

The ordinary differential equation (13) is then integrated as long as all terms contain derivatives, where integration constants are neglected.

2. The sine-cosine ansatz admits the use of the assumption

$$u(x, y, t) = \{\lambda \cos^\beta(\xi)\}, \hspace{1cm} |\xi| \leq \frac{\pi}{2\mu},$$  \hspace{1cm} (15)

or the assumption

$$u(x, y, t) = \{\lambda \sin^\beta(\xi)\}, \hspace{1cm} |\xi| \leq \frac{\pi}{\mu},$$  \hspace{1cm} (16)

where $\lambda$ and $\beta$ are parameters that will be determined. We then use

$$u(\xi) = \lambda \cos^\beta(\xi),$$  
$$u^n(\xi) = \lambda^n \cos^{n\beta}(\xi),$$  
$$(u^n)_{xx} = -n^2 \beta^2 \lambda^n \cos^{n\beta+2}(\xi),$$

and for (16) we use

$$u(\xi) = \lambda \sin^\beta(\xi),$$  
$$u^n(\xi) = \lambda^n \sin^{n\beta}(\xi),$$  
$$(u^n)_{xx} = -n^2 \beta^2 \lambda^n \sin^{n\beta+2}(\xi).$$

3. Substituting (17) or (18) into the reduced ODE gives a trigonometric equation of cosine or sine terms. The constants are then determined by

(i) balancing the exponents of each pair of the resulting trigonometric functions cosine or sine;
(ii) collecting all terms with the same power in $\cos^k(\xi)$ or $\sin^k(\xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknowns $\beta$, $\lambda$, and $\mu$. The problem is now completely reduced to an algebraic one;
(iii) and by determining the constants $\lambda$, $\mu$, and $\beta$ by algebraic calculations, the proposed solutions in (15) and in (16) follow immediately.

3. BOUSSINESQ-TYPE OF EQUATIONS WITH POSITIVE EXPONENTS

3.1. Compacton Solutions

We first consider the two-dimensional Boussinesq-type of equations with positive exponents

$$u_{tt} - u_{xx} - u_{yy} - a (u^{2n})_{xx} - b [u^n(u^n)_{xx}]_{xx} = 0, \hspace{1cm} n > 1.$$  \hspace{1cm} (19)
For \( n = 0 \), equation (19) is the standard two-dimensional wave equation. Using the wave variable 
\( \xi = \mu x + \eta y - ct \) carries (19) into the ODE
\[
(c^2 - \mu^2 - \eta^2) u_{\xi\xi} - \alpha \mu^2 (u^{2n})_{\xi} - b \mu^4 \left( u^n (u^n)_{\xi\xi} \right) = 0. \tag{20}
\]
Integrating (20) twice and setting the constants of integration to be zero we find
\[
(c^2 - \mu^2 - \eta^2) u - \alpha \mu^2 u^{2n} - b \mu^4 u^n (u^n)_{\xi\xi} = 0. \tag{21}
\]
Substituting (15) into (21) gives
\[
(c^2 - \mu^2 - \eta^2) \lambda \cos^2(\xi) - \alpha \mu^2 \lambda^{2n} \cos^{2n}\beta(\xi) + bn^2 \mu^4 \beta^2 \lambda^{2n} \cos^{2n}\beta(\xi) - bn\mu^4 \lambda^{2n} \beta(n\beta - 1) \cos^{2n}\beta - 2(\xi) = 0. \tag{22}
\]
Applying the balance method gives the system of algebraic equations
\[
\begin{align*}
n\beta - 1 &\neq 0, \\
2n\beta - 2 &= \beta, \\
bn^2 \mu^4 \beta^2 &= \alpha \mu^2, \\
bn\lambda^{2n} \mu^4 \beta(n\beta - 1) &= (c^2 - \mu^2 - \eta^2) \lambda.
\end{align*} \tag{23}
\]
Solving system (23) yields
\[
\begin{align*}
\beta &= \frac{2}{2n - 1}, \\
\mu &= \frac{2n - 1}{2n} \sqrt{\frac{a}{b}}, \\
\lambda &= \left( \frac{2n \left( 4bn^2 \left( c^2 - \eta^2 \right) - \alpha (2n - 1)^2 \right)}{a^2 (2n - 1)^2} \right)^{1/(2n-1)}.
\end{align*} \tag{24}
\]
Results (24) can be easily obtained if we also use the sine ansatz (16). In view of (24), the compactons solutions
\[
\begin{align*}
u(x, y, t) &= \begin{cases}
\left\{ \frac{2n \left[ 4bn^2 \left( c^2 - \eta^2 \right) - \alpha (2n - 1)^2 \right]}{a^2 (2n - 1)^2} \sin^2 \left[ \frac{2n - 1}{2n} \sqrt{\frac{a}{b}} x + \eta y - ct \right] \right\}^{1/(2n-1)}, \\
|\xi| \leq \pi, & \frac{a}{b} > 0, \\
0, & \text{otherwise},
\end{cases} \\
u(x, y, t) &= \begin{cases}
\left\{ \frac{2n \left[ 4bn^2 \left( c^2 - \eta^2 \right) - \alpha (2n - 1)^2 \right]}{a^2 (2n - 1)^2} \cos^2 \left[ \frac{2n - 1}{2n} \sqrt{\frac{a}{b}} x + \eta y - ct \right] \right\}^{1/(2n-1)}, \\
|\xi| \leq \frac{\pi}{2}, & \frac{a}{b} > 0, \\
0, & \text{otherwise},
\end{cases}
\end{align*} \tag{25}
\]
and
\[
\begin{align*}
u(x, y, t) &= \begin{cases}
\left\{ \frac{2n \left[ 4bn^2 \left( c^2 - \eta^2 \right) - \alpha (2n - 1)^2 \right]}{a^2 (2n - 1)^2} \sinh^2 \left[ \frac{2n - 1}{2n} \sqrt{-\frac{a}{b}} x + \eta y - ct \right] \right\}^{1/(2n-1)}, \\
|\xi| \leq \pi, & \frac{a}{b} > 0, \\
0, & \text{otherwise},
\end{cases} \\
u(x, y, t) &= \begin{cases}
\left\{ \frac{-2n}{a} \cosh^2 \left[ \left( \frac{2n - 1}{2n} \sqrt{-\frac{a}{b}} x + \eta y - ct \right) \right] \right\}^{1/(2n-1)}, \\
|\xi| \leq \frac{\pi}{2}, & \frac{a}{b} > 0, \\
0, & \text{otherwise},
\end{cases}
\end{align*} \tag{26}
\]
follow immediately, where \( \eta \) and \( c \) are arbitrary constants.

### 3.2. Solitary Patterns Solutions

For \( a/b < 0 \), we found that the solitary patterns solutions exist only if \( \eta = c \), and given in the form
\[
\begin{align*}
u(x, y, t) &= \left\{ \frac{2n}{a} \sinh^2 \left[ \frac{2n - 1}{2n} \sqrt{-\frac{a}{b}} x + \eta y - ct \right] \right\}^{1/(2n-1)}, \\
u(x, y, t) &= \left\{ \frac{-2n}{a} \cosh^2 \left[ \left( \frac{2n - 1}{2n} \sqrt{-\frac{a}{b}} x + \eta y - ct \right) \right] \right\}^{1/(2n-1)},
\end{align*} \tag{27, 28}
\]
4. BOUSSINESQ-TYPE OF EQUATIONS WITH NEGATIVE EXPONENTS

4.1. Periodic Solutions

We first consider the two-dimensional Boussinesq-type of equations with negative exponents

\[ u_{tt} - u_{xx} - u_{yy} - a (u^{-2n})_{xx} - b [u^{-n} (u^{-n})_{xx}] = 0, \quad n > 1. \]  
\[ (29) \]

For \( n = 0 \), equation (29) is the standard two-dimensional wave equation. Proceeding as before, we use the wave variable \( \xi = \mu x + \eta y - ct \) into (29) to read the ODE

\[ (c^2 - \mu^2 - \eta^2) u_{\xi\xi} - a \mu^2 (u^{-2n})_{\xi\xi} - b \mu^4 [u^{-n} (u^{-n})_{\xi\xi}]_{\xi} = 0. \]
\[ (30) \]

Integrating (30) twice gives

\[ (c^2 - \mu^2 - \eta^2) u - a \mu^2 u^{-2n} - b \mu^4 u^{-n} (u^{-n})_{\xi} = 0, \]
\[ (31) \]

where the constants of integration are considered zeros. Substituting (15) into (31) gives

\[ (c^2 - \mu^2 - \eta^2) \lambda \cos^\beta (\xi) - a \mu^2 \lambda^{-2n} \cos^{-2n} \beta (\xi) + bn^2 \mu^4 \lambda^{-2n} \cos^{-2n} \beta (\xi) \]
\[ - bn^2 \mu^4 \lambda^{-2n} \beta (n\beta + 1) \cos^{-2n} \beta - (c^2 - \mu^2 - \eta^2) \lambda. \]
\[ (32) \]

The balance method gives the system of algebraic equations

\[ n\beta + 1 \neq 0, \]
\[ -2n\beta - 2 = \beta, \]
\[ bn^2 \mu^4 \beta^2 = a, \]
\[ (33) \]

Solving system (33) gives

\[ \beta = -\frac{2}{2n + 1}, \]
\[ \mu = \frac{2n + 1}{2n} \sqrt{\frac{a}{b}}, \]
\[ \lambda = \left( -\frac{a^2 (2n + 1)^2}{2n \left[ 4bn^2 (c^2 - \eta^2) - a(2n + 1)^2 \right]} \right)^{1/(2n + 1)}. \]
\[ (34) \]

Results (34) can be easily obtained if we also use the sine ansatz as well. In view of (34), we obtain the periodic solutions

\[ u(x, y, t) = \left\{ -\frac{a^2 (2n + 1)^2}{2n \left[ 4bn^2 (c^2 - \eta^2) - a(2n + 1)^2 \right]} \csc^2 \left[ \frac{2n + 1}{2n} \sqrt{\frac{a}{b}} x + \eta y - ct \right] \right\}^{1/(2n + 1)}, \]
\[ (35) \]

for \( 0 < \xi < \pi \), and

\[ u(x, y, t) = \left\{ -\frac{a^2 (2n + 1)^2}{2n \left[ 4bn^2 (c^2 - \eta^2) - a(2n + 1)^2 \right]} \sec^2 \left[ \frac{2n + 1}{2n} \sqrt{\frac{a}{b}} x + \eta y - ct \right] \right\}^{1/(2n + 1)}, \]
\[ (36) \]

for \( |\xi| < \pi/2 \), where \( \eta \) and \( c \) are arbitrary constants.

4.2. Solitons Solutions

For \( a/b < 0 \), we found that the solitons solutions are generated only if \( \eta = c \), and given in the form

\[ u(x, y, t) = \left\{ -\frac{a}{2n} \csc^2 \left[ \frac{2n + 1}{2n} \sqrt{-\frac{a}{b}} x + \eta y - ct \right] \right\}^{1/(2n + 1)}, \]
\[ (37) \]

and

\[ u(x, y, t) = \left\{ \frac{a}{2n} \sech^2 \left[ \frac{2n + 1}{2n} \sqrt{-\frac{a}{b}} x + \eta y - ct \right] \right\}^{1/(2n + 1)}. \]
\[ (38) \]
5. CONCLUDING REMARKS

Two generalized forms of $(2 + 1)$-dimensional Boussinesq-type of equations were examined. Distinct exact solutions with distinct physical structures were formally obtained. It was formally derived that the solutions may come as compactons, solitary patterns, periodic solutions, or solitons. The effect of exponents, and the change of the ratio $a/b$ of the derivatives of $u(x, t)$, positive or negative, cause a qualitative difference in the resulting physical structures of the resulting solutions.

In Table 1 below, we summarize the physical structures of solutions for the generalized Boussinesq type of equations.

<table>
<thead>
<tr>
<th>Exponents</th>
<th>$a$</th>
<th>Solution Structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positive</td>
<td>Positive</td>
<td>Compacton solutions</td>
</tr>
<tr>
<td>Positive</td>
<td>Negative</td>
<td>Solitary patterns solutions</td>
</tr>
<tr>
<td>Negative</td>
<td>Positive</td>
<td>Periodic solutions</td>
</tr>
<tr>
<td>Negative</td>
<td>Negative</td>
<td>Solitons solutions</td>
</tr>
</tbody>
</table>

It has been revealed that the exponent and the ratio $a/b$ factors cause a qualitative change in the physical structures of the resulting solutions. Obviously, further physical explanations for this change of physical structures are needed. This is beyond the scope of this work.

REFERENCES