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NOTE

PERMANENTS OF DOUBLY STOCHASTIC MATRICES

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A conjecture on the permanents of doubly stochastic matrices is proposed. Some results supporting it are presented.

Let Ω_n be the set of all $n \times n$ doubly stochastic matrices, and let Λ_n^k be the set of all $n \times n$ (0, 1)-matrices whose row and column sums are all equal to k. For any $n \times n$ matrix $A = (a_{ij})$, the permanent of A is defined by

$$\mathrm{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where S_n is the symmetric group of order n.

Let $\mu_k(n) = \min\{\mathrm{per}(A)\colon A \in \Lambda_n^k\}$, and let $\theta_k = \lim_{n\to\infty}[\mu_k(n)]^{1/n}$.

An interesting and unsolved conjecture of Schrijver and Valiant [1, 2] asserts that

$$\theta_k = \frac{(k-1)^{k-1}}{k^{k-2}}. \tag{1}$$

It is proved in [2] that $\mu_k(n) \leq k^{2n}/\binom{nk}{n}$, and $\theta_k \leq (k-1)^{k-1}/k^{k-2}$. The conjecture (1) is trivially true for $k = 1$ or 2, and was proved for $k = 3$ by Voorhoeve in [3].

In this note, we propose a conjecture on the permanents of doubly stochastic matrices, and we show that if this conjecture is valid, the above conjecture of Schrijvar and Valiant follows easily. Also, some results supporting the proposed conjecture are presented.

The permanent per(A) of a doubly stochastic matrix $A = (a_{ij}) \in \Omega_n$ has the following probabilistic interpretation, see [4]. Suppose there are n balls $a_1, a_2, \ldots, a_n$ and n boxes $b_1, b_2, \ldots, b_n$. For any $1 \leq i, j \leq n$, let a_{ij} be the probability that the ball a_i will move into box b_j. Thus, per(A) is the probability that each box b_j, $1 \leq j \leq n$, will receive exactly one ball. For any $1 \leq j \leq n$, let $q_j(A)$ be the probability of the event E_j that the box b_j will receive exactly one ball. Thus,

$$q_j(A) = P(E_j) = \sum_{i=1}^{n} a_{ij} \prod_{\substack{k=1 \\ k \neq i}}^{n} (1 - a_{kj}).$$

Conjecture. *For any $A \in \Omega_n$,*

$$q(A) = \prod_{j=1}^{n} q_j(A) \leqslant \operatorname{per}(A). \tag{2}$$

By the multiplication rule for conditional probabilities, one has

$$\begin{aligned}\operatorname{per}(A) &= P(E_1E_2 \cdots E_n) \\ &= P(E_1)P(E_2 \mid E_1) \cdots P(E_n \mid E_1E_2 \cdots E_{n-1}).\end{aligned}$$

Although this formula motivated the proposal of the conjecture, it seems impossible to utilize it to prove the conjecture.

Proposition 1. *If* (2) *holds for all $A \in \Lambda_n^k$, then* (1) *holds.*

Proof. Let $A \in \Lambda_n^k$. Using (2), one has

$$\begin{aligned}\operatorname{per}(A) &= k^n \operatorname{per}\left(\frac{1}{k}A\right) \geqslant k^n \prod_{j=1}^{n} q_j\left(\frac{1}{k}A\right) \\ &= k^n\left(\left(1-\frac{1}{k}\right)^{k-1}\right)^n = \left[\frac{(k-1)^{k-1}}{k^{k-2}}\right]^n.\end{aligned}$$

Thus, $\theta_k \geqslant (k-1)^{k-1}/k^{k-2}$. □

Now we present some results supporting the above conjecture.

Proposition 2. *If $A \in \Omega_n$ is such that*

$$\left[1 - \frac{2}{n} + \frac{2}{n}\operatorname{per}(A)\right]^n \leqslant \operatorname{per}(A), \tag{3}$$

then (2) *holds for A.*

Proof. For each j, $1 \leqslant j \leqslant n$, $q_j(A)$ is the sum of n terms of the form

$$a_{ij} \prod_{\substack{k=1 \\ k \neq i}}^{n} (a_{k1} + a_{k2} + \cdots + a_{k(j-1)} + a_{k(j+1)} + \cdots + a_{kn}), \quad 1 \leqslant i \leqslant n.$$

This expression can be expanded as a sum of terms, each is a product of n entries from all n rows of A. Any such product is either a summand in per(A), or, there are at least two entries from the same column of A. One can see that

$$q_1(A) + q_2(A) + \cdots + q_n(A) \leqslant n \operatorname{per}(A) + (n-2)(1 - \operatorname{per}(A)). \tag{4}$$

Thus, by hypothesis,

$$q(A) \leqslant \left[\operatorname{per}(A) + \frac{n-2}{n}(1 - \operatorname{per}(A))\right]^n$$

$$= \left[1 - \frac{2}{n} + \frac{2}{n}\operatorname{per}(A)\right]^n \leqslant \operatorname{per}(A). \quad \square$$

Corollary 3. *Inequality* (2) *holds for any* $A \in \Omega_4$, *and for any* $A \in \Omega_n$ *with* $\operatorname{per}(A) \geqslant \frac{1}{4}$.

Proof. Let $n = 4$. The condition (3) becomes

$$(1 + \operatorname{per}(A))^4 \leqslant 16 \operatorname{per}(A).$$

Since $4!/4^4 \leqslant \operatorname{per}(A) \leqslant 1$, and since the function

$$f(t) = (1+t)^4 - 16t \leqslant 0$$

for $t \in [4!/4^4, 1]$, the condition (3) is satisfied for $A \in \Omega_4$.

For any $n > 4$, let $g(t) = (1 - 2/n + 2t/n)^n - t$. Since $g(1) = 0$, $g''(t) > 0$ for $t \in [0, 1]$, and $g(\frac{1}{4}) < e^{-3/2} - \frac{1}{4} < 0$ for $n > 4$, the condition (3) is satisfied for A with $\frac{1}{4} \leqslant \operatorname{per}(A) \leqslant 1$. This completes the proof. $\square$

Let $J_n = (a_{ij}) \in \Omega_n$, $n \geqslant 3$, where $a_{ij} = 1/n$, $1 \leqslant i, j \leqslant n$. Then, $\operatorname{per}(J_n) = n!/n^n$, and $q(J_n) = (\frac{n-1}{n})^{n(n-1)}$. By Stirling's formula, $n! \geqslant \sqrt{2\pi n}\, n^{n+1/2} e^{-n}$. It is easy to see that (2) holds for $A = J_n$.

The above results show that the conjecture (2) holds for $A \in \Omega_n$ in a small neighborhood of J_n, and for A with the value of per(A) close to 1.

References

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