A large order asymptotic existence theorem for group divisible 3-designs with index one

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In this paper we prove that group divisible 3-designs exist for sufficiently large order with a fixed number of groups, fixed block size and index one, assuming that the necessary arithmetic conditions are satisfied. Let \( k \) and \( u \) be positive integers, \( 3 \leq k \leq u \). Then there exists an integer \( m_0 = m_0(k, u) \) such that there exists a group divisible 3-design of group type \( m^n \) with block size \( k \) and index one for any integer \( m \geq m_0 \) satisfying the necessary arithmetic conditions

1. \( m(u - 2) \equiv 0 \mod (k - 2) \),
2. \( m^2(u - 1)(u - 2) \equiv 0 \mod (k - 1)(k - 2) \),
3. \( m^3u(u - 1)(u - 2) \equiv 0 \mod k(k - 1)(k - 2) \).

1. Introduction

**Definition 1.1.** Let \( v \) be a non-negative integer, \( \lambda \) and \( t \) be positive integers and \( K \) be a set of positive integers. A group divisible \( t \)-design (or \( t \)-GDD) of order \( v \), index \( \lambda \) and block sizes \( K \) is a triple \((X, \Gamma, A)\) where,

1. \( X \) is a set of \( v \) elements (called points),
2. \( \Gamma = \{G_1, G_2, \ldots\} \) is a set of non-empty subsets of \( X \) which partition \( X \) (called groups),
3. \( A \) is a family of subsets of \( X \) each of cardinality from \( K \) (called blocks) such that each block intersects any given group in at most one point,
4. and each \( t \)-set of points from \( t \) distinct groups is in exactly \( \lambda \) blocks.

**Definition 1.1** is a generalization of the concept of group divisible designs (GDDs) for \( t \geq 2 \), the latter corresponding to \( t = 2 \).

By the group type of a \( t \)-GDD \((X, \Gamma, A)\) we mean the list \((|G|: G \in \Gamma)\) of group sizes. If a \( t \)-GDD has \( n_1 \) groups of size \( g_1 \), \( 1 \leq i \leq r \), then the list contains each \( g_i \) counted \( n_i \) times. In this case we denote its group type by \((g_1^{n_1}, g_2^{n_2}, \ldots, g_r^{n_r})\).

Group divisible \( t \)-designs with equal group sizes are called uniform.

Group divisible designs with block size \( k \) and \( k \) groups of uniform group size \( m \) are called transversal designs and denoted by \( TD_\lambda(t, k, m) \). If \( \lambda = 1 \), then we use \( TD(t, k, m) \). If \( t = 2 \) we use \( TD_\lambda(k, m) \).

The following result proves an asymptotic existence theorem for group divisible \( t \)-designs. This theorem plays an important role in more general asymptotic existence proofs of the large-order group divisible \( t \)-designs.

**Theorem 1.2** ([8]). Let \( t, k, u \) and \( m \) be positive integers, \( t \leq k \leq u \). Then there exists an integer \( d_1 = d_1(t, k, u, m) \) such that there exists a \( t \)-GDD of group type \((ba^d m)^u\) with block size \( k \) and index one for all positive integers \( b, a \geq 2 \) and \( d \geq d_1 \), if
and only if
\[ m^{-1} \left( \frac{u - s}{t - s} \right) \equiv 0 \mod \left( \frac{k - s}{t - s} \right) \text{ for } s = 0, 1, \ldots, t - 1. \]

Theorem 1.3 is a well-known result of Chowla–Erdős–Straus [1] that guarantees the existence for TD(k, m)s of large order.

Theorem 1.3 (See [1]). For any positive integer \( k \), \( 2 \leq k \), there is an integer \( m^* = m^*(k) \) such that for any integer \( m \geq m^* \) there is a TD(k, m).

Blanchard generalizes the above theorem for any strength \( t \) in his unpublished manuscript [2].

Theorem 1.4 (See [2]). For any positive integers \( t \) and \( k \), \( t \leq k \), there is an integer \( m^* = m^*(t, k) \) such that for any integer \( m \geq m^* \) there is a TD\( (t, k, m) \).

Mohácsy in [8] generalizes the Chowla–Erdős–Straus result for uniform GDDs of group type \( m^u \) confirming a conjecture of Wilson.

Theorem 1.5 (See [8]). Let \( k \) and \( u \) be positive integers, \( 2 \leq k \leq u \). Then there exists an integer \( m_0 = m_0(k, u) \) such that there exists a GDD of group type \( m^u \) with block size \( k \) and index one for any integer \( m \geq m_0 \) satisfying the necessary arithmetic conditions
1. \( m(u - 1) \equiv 0 \mod (k - 1) \),
2. \( m^2(u - 1) \equiv 0 \mod k(k - 1) \).

A similar result, but where \( m \) is fixed and \( u \) is large was previously proved by Chang in his thesis [4]. Lamken and Wilson gave a different proof for the same theorem as one of the many applications of the Lamken–Wilson theory of edge-colored graph designs [6].

This paper gives an asymptotic existence result for group divisible 3-designs of sufficiently large order with index one, when the number of groups and the block size are fixed, assuming that the necessary arithmetic conditions are satisfied. It is well known that the necessary arithmetic conditions for the existence of a 3-GDD of group type \( m^u \) with block size \( k \) and index one are \( m(u - 2) \equiv 0 \mod (k - 2) \), \( m^2(u - 1)(u - 2) \equiv 0 \mod (k - 1)(k - 2) \) and \( m^3u(u - 1)(u - 2) \equiv 0 \mod k(k - 1)(k - 2) \). The necessary conditions are not always sufficient for the existence of such 3-GDDs. For example a 3-GDD of group type \( 5^6 \) with block size 7 does not exist. The existence of a 3-GDD of group type \( 5^8 \) with block size 7 would imply the existence of a GDD of group type \( 5^7 \) with block size 6 which contradicts the Bruck–Ryser–Chowla theorem. In this paper we prove that the necessary arithmetic conditions are asymptotically sufficient in group size \( m \), that is for a fixed number of groups \( u \) and fixed block size \( k \) a 3-GDD of group type \( m^u \) with block size \( k \) and index one exists for sufficiently large \( m \) if the necessary arithmetic conditions are satisfied. The author first heard about this problem from Ray-Chaudhuri in 1992.

Theorem 1.6. Let \( k \) and \( u \) be positive integers, \( 3 \leq k \leq u \). Then there exists an integer \( m_0 = m_0(k, u) \) such that there exists a 3-GDD of group type \( m^u \) with block size \( k \) and index one for any integer \( m \geq m_0 \) satisfying the necessary arithmetic conditions
1. \( m(u - 2) \equiv 0 \mod (k - 2) \),
2. \( m^2(u - 1)(u - 2) \equiv 0 \mod (k - 1)(k - 2) \),
3. \( m^3u(u - 1)(u - 2) \equiv 0 \mod k(k - 1)(k - 2) \).

Mohácsy in [7] proved a partial asymptotic existence result for 3-GDDs of group type \( m^u \) with block size \( k \) and index one for sufficiently large group size \( m \) if the necessary arithmetic conditions are satisfied and \( m \) and \( k(k - 1)(k - 2) \) are relatively prime. This paper together with paper [7] gives the complete proof for the asymptotic existence of large-order group divisible 3-designs.

2. Group divisible subdesigns and incomplete group divisible designs

Definition 2.1. Let \( G \) be a \( t \)-GDD \( (X, \Gamma', A) \) with index \( \lambda \). A group divisible subdesign \( H \) (or sub \( t \)-GDD) of \( G \) is a \( t \)-GDD \( (X', \Gamma'', A') \) such that
1. \( X' \subseteq X \),
2. \( \Gamma'' \) consists of non-empty groups \( G \cap X' \), where \( G \) is a group of \( \Gamma' \),
3. \( A' \) consists of the blocks \( A \in A \), which are completely contained in \( X' \).

The fundamental construction of Wilson [10] for GDDs is generalized by Mohácsy and Ray-Chaudhuri in [9] for \( t \)-GDDs containing sub \( t \)-GDDs, where \( t \geq 2 \).
Theorem 2.2 (See [9]). Let \((X, \Gamma', A)\) be a \(t\)-GDD of index one and let \(s_x\) be a positive integral weight assigned to each point \(x \in X\). Let \((S_x : x \in X)\) be pairwise disjoint sets with \(|S_x| = s_x\). With the notation
\[
S_Y = \bigcup_{x \in Y} S_x
\]
for \(Y \subseteq X\), put \(X^* = X_s\) and \(I^* = \{S_G : G \in \Gamma'\}\). Suppose that for each block \(A \in A\), a \(t\)-GDD \((A, (S_x : x \in A), \mathcal{B}_A)\) of index one exists and denote \(A^* = \bigcup_{A \in A} \mathcal{B}_A\). Then \((X^*, I^*, A^*)\) is a \(t\)-GDD of index one which contains a sub \(t\)-GDD \((S_x : x \in A), \mathcal{B}_A\) for each \(A \in A\).

Using Wilson’s terminology, we refer to a \(t\)-GDD with a weighting as a recipe and the additional \(t\)-GDDs for each block as ingredients.

The following definition is a generalization of the concept of incomplete group divisible designs described in [5, p. 235] for \(t \geq 2\).

**Definition 2.3.** Let \(v\) be a non-negative integer, \(\lambda\) and \(t\) and \(l\) be positive integers with \(1 \leq l \leq t - 1\) and let \(K\) be a set of positive integers. An incomplete group divisible \((t, l)\)-design (or \((t, l)\)-IGDD) of order \(v\), index \(\lambda\) and block sizes \(K\) is a quadruple \((X, \Gamma', \mathcal{H}, A)\) where,

1. \(X\) is a set of \(v\) elements (called points),
2. \(\Gamma' = \{G_1, G_2, \ldots\}\) is a set of non-empty subsets of \(X\) which partition \(X\) (called groups),
3. \(\mathcal{H} = \{H_1, H_2, \ldots\}\) is a set of non-empty subsets of \(X\) with the property \(H_i \subseteq G_i\) for \(i = 1, 2, \ldots\) (called holes),
4. \(A\) is a family of subsets of \(X\) each of cardinality from \(K\) (called blocks) such that each block intersects any given group in at most one point and does not intersect more than \(l\) holes,
5. and each \(t\)-set of points from \(t\) distinct groups that intersects not more than \(l\) holes is in exactly \(\lambda\) blocks.

We say a \((t, l)\)-IGDD is a type of \((g_1, h_1)^{n_1}, (g_2, h_2)^{n_2}, \ldots, (g_r, h_r)^{n_r}\) if there are \(n_i\) groups of size \(g_i\) which contain a hole of size \(h_i\) for \(i = 1, 2, \ldots, r\). Incomplete group divisible \(t\)-designs with equal group sizes and equal hole sizes are called uniform. The fundamental construction for \(t\)-GDDs can be extended to \((t, l)\)-IGDDs as follows.

**Theorem 2.4 (See [7]).** Let \(t, k, l\) and \(u\) be positive integers with \(1 \leq l \leq t \leq k \leq u\). If there exists a \((t, l)\)-IGDD of group type \((n + w, w)^u\) with block size \(u\) and a \(t\)-GDD of group type \(a^u\) with block size \(k\) then there exists a \((t, l)\)-IGDD of group type \((na + wa, wa)^u\) with block size \(k\).

**Theorem 2.5** follows from **Definitions 2.3** and **2.1**.

**Theorem 2.5 (See [7]).** Let \(t, k\) and \(u\) be positive integers with \(t \leq k \leq u\). If there exists a \((t, t - 1)\)-IGDD of group type \((n + w, w)^u\) with block size \(k\) and a \(t\)-GDD of group type \(w^u\) with block size \(k\) then there exists a \((t, t - 1)\)-IGDD of group type \((n + w)^u\) with block size \(k\). Conversely, if there exists a \(t\)-GDD of group type \((n + w)^u\) with block size \(k\) containing a sub \(t\)-GDD of group type \(w^u\) then there is a \((t, t - 1)\)-IGDD of group type \((n + w, w)^u\) with block size \(k\).

**Theorems 2.6 and 2.7** proved in [7] give additive constructions for 3-GDDs and IGDDs for the \(t = 3\) and \(\lambda = 1\) case. These theorems are generalizations of Blanchard’s constructions for transversal 3-designs [3].

**Theorem 2.6 (See [7]).** If there is

1. a 3-GDD of group type \(n^u\) with block size \(k\),
2. a \((3, 1)\)-IGDD of group type \((n + v, v)^u\) with block size \(k\),
3. a \((3, 1)\)-IGDD of group type \((n + w, w)^u\) with block size \(k\), and
4. a TD\((3, u + 1, s)\),

then there is a \((3, 1)\)-IGDD of group type \((sn + av + bw, av + bw)^u\) with block size \(k\) for any integers \(a\) and \(b\) with \(a, b \geq 0\) and \(a + b \leq s\).

**Theorem 2.7 (See [7]).** If there is

1. a 3-GDD of group type \(n^u\) with block size \(k\),
2. a \((3, 1)\)-IGDD of group type \((n + w, w)^u\) with block size \(k\),
3. a \((3, 2)\)-IGDD of group type \((n + w, w)^u\) with block size \(k\),
4. a TD\((3, u + 2, p)\),
5. a TD\((3, u + 2, s_j)\) and a TD\((3, u + 2, s_j)\) with \(s_i \geq p\) for \(i = 1, 2, \ldots\),
6. a 3-GDD of group type \(w^u\) with block size \(k\)

then there exists a 3-GDD of group type \((s_1s_2n + pw)^u\) with block size \(k\) containing a sub 3-GDD of group type \((pw)^u\).
Blanchard [3] gives a construction for a (3, 1)-IGDD of group type \((ba^\beta + q^d, q^d)^u\) with block size \(u\) for any prime power \(q\) assuming the existence of a TD\((3, u + 1, b)\). Blanchard’s construction can be generalized for an arbitrary power and for block size \(k\) as stated in **Theorem 2.8**. The \(c = 1\) case of **Theorem 2.8** was proved in [7, Theorem 3.4].

**Theorem 2.8.** Let \(k, u\) and \(c\) be positive integers with \(3 \leq k \leq u\) satisfying the conditions

1. \(c(u - 2) \equiv 0 \mod (k - 2)\),
2. \(c^2(u - 1)(u - 2) \equiv 0 \mod (k - 1)(k - 2)\),
3. \(c^3u(u - 1)(u - 2) \equiv 0 \mod k(k - 1)(k - 2)\).

If there exists a TD\((3, u + 1, b)\) with \(b \geq 2u\), then there exists a positive integer \(d_3 = d_3(k, u, b, c)\) such that for any positive integers \(a \geq 2\) and \(d \geq d_3\) there exists a (3, 1)-IGDD of group type \((cba^d + ca^d, ca^d)^u\) with block size \(k\) and index one.

**Proof.** First we apply the \(c = 1\) case of **Theorem 2.8** with \(k = u\) to construct a (3, 1)-IGDD of type \((ba^\beta + a^d, a^d)^u\) with block size \(u\) and index one for all positive integers \(a \geq 2\), where \(d_2 = d_2(u, b, u, b) = d_3(u, u, b, 1)\). By **Theorem 1.2** there is an integer \(d_1 = d_1(k, u, c)\) such that a 3-GDD of group type \((ca^d)^u\) with block size \(k\) exists for all positive integers \(a \geq 2\) and \(e \geq d_1\). Let \(d_3 = d_3(k, u, b, c) = d_2(u, u, b) + d_1(k, u, c)\). When \(d \geq d_3\), then we can write \(d = d_2 + f\), where \(f \geq d_1\).

Weight each point of the previously constructed (3, 1)-IGDD of group type \((ba^\beta + a^d, a^d)^u\) with block size \(u\) by the uniform weight \(ca^d\). For each block of the (3, 1)-IGDD there is a 3-GDD of group type \((ca^d)^u\) with block size \(k\). Thus, by **Theorem 2.4**, we can construct a (3, 1)-IGDD of group type \((cba^d + ca^d, ca^d)^u\) with block size \(k\) and the proof is complete. 

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3. A construction for group divisible 3-designs

**Theorem 3.1.** Let \(k, u\) and \(c\) be positive integers, \(3 \leq k \leq u\). Then there exist positive integers \(n = n(k, u, c)\) and \(w = w(k, u, c)\) such that \(gcd(n, w) = c\) and there exist a 3-GDD of group type \(n^u\), a 3-GDD of group type \(w^u\), a (3, 1)-IGDD and a (3, 2)-IGDD of group type \((n + w, w)^u\) all having block size \(k\) if and only if

1. \(c(u - 2) \equiv 0 \mod (k - 2)\),
2. \(c^2(u - 1)(u - 2) \equiv 0 \mod (k - 1)(k - 2)\),
3. \(c^3u(u - 1)(u - 2) \equiv 0 \mod k(k - 1)(k - 2)\).

The \(c = 1\) case of **Theorem 3.1** was proved by Mohácsy in [7]. The proof for general \(c\) is similar to the previous proof.

**Proof.** First we prove the necessity of the conditions. If a 3-GDD of group type \(n^u\) with block size \(k\) and a 3-GDD of group type \(w^u\) with block size \(k\) exist then

1. \(n(u - 2) \equiv w(u - 2) \equiv 0 \mod (k - 2)\),
2. \(n^2(u - 1)(u - 2) \equiv w^2(u - 1)(u - 2) \equiv 0 \mod (k - 1)(k - 2)\),
3. \(n^3u(u - 1)(u - 2) \equiv w^3u(u - 1)(u - 2) \equiv 0 \mod k(k - 1)(k - 2)\).

\(gcd(n, w) = c\) implies that \(gcd(n^i, w^i) = c^i\), \(i = 1, 2, 3\). Thus there are integers \(a_i\) and \(b_i\) such that \(a_in^i + b_iw^i = c^i\), \(i = 1, 2, 3\). This way we have

\[
\begin{align*}
1. a_1n(u - 2) + b_1w(u - 2) & \equiv 0 \mod k - 2, \\
2. a_2n^2(u - 1)(u - 2) + b_2w^2(u - 1)(u - 2) & \equiv 0 \mod (k - 1)(k - 2), \\
3. a_3n^3u(u - 1)(u - 2) + b_3w^3u(u - 1)(u - 2) & \equiv 0 \mod k(k - 1)(k - 2).
\end{align*}
\]

Now we prove the sufficiency of the conditions. Let \(Q_1 = 3^{2u}\) and \(Q_2 = 7^{2u}\). Then a TD\((3, u + 1, Q_i)\) exists for \(i = 1, 2\). Now we apply **Theorem 2.8** to construct a (3, 1)-IGDD of group type \((3Q_1Q_2^u + Q_1Q_2^u, Q_1Q_2^u)^u\) and a (3, 1)-IGDD of group type \((Q_1Q_2^u + Q_1Q_2^u, Q_1Q_2^u)^u\) with block size \(k\) for all \(\alpha \geq d_4(k, u, c) = \max(d_1(k, u, Q_1, c), d_3(k, u, Q_2, c))\). Let us choose \(\alpha\) to be a fixed integer such that \(\alpha \geq \max(d_4(k, u, c), d_1(k, u, c), d_1(k, u, m))\), where \(d_1 = d_1(k, u, m)\) is from **Theorem 1.2**.

Let us choose \(\beta\) to be a fixed integer such that \(5^{\beta - 1} \leq (Q_1Q_2)^u < 5^\beta\). Then \((Q_1Q_2)^u < 5^\beta \leq 5(Q_1Q_2)^u\). Since \((Q_1Q_2)^u \leq 5^\beta\) and \(Q_1Q_2 \geq 5\), \(\beta > \alpha > d_1(u + 2, u + 2, 1)\). Thus, there exists a TD\((3, u + 2, 5^\beta)\). Let \(p = 5^\beta\).

Now choose a prime number \(s_1\) such that \(s_1 \geq 5(2u, 5^\beta)\). This way a TD\((3, u + 2, s_1)\) exists and \(s_1 \geq 5^\beta = p\), \((s_1, 5) = 1\) and \((s_1, Q_i) = 1\) for \(i = 1, 2\).

Let \(d_0(k, u, c) = \max(d_1(k, u, c), d_1(u + 1, u + 1, 1))\). Choose a prime number \(R\) such that \(R \geq 5Q_1^u + 1, R \equiv -1 \mod 5, R \equiv -1 \mod 3, R \equiv -1 \mod 7\) and \(R \equiv 1 \mod 2^d_0\). According to the Chinese remainder theorem and the Dirichlet theorem such a prime \(R\) exists. Then a TD\((3, u, R)\) exists.
It is easy to check that the conditions of Theorem 1.2 for \( t = 3 \) are satisfied. Thus, from Theorem 1.2 there exists an integer \( d_1 = d_1(k, u, c) \) such that if \( d \geq d_1 \) then there exists a 3-GDD of group type \((cR^d)^u\) with block size \( k \).

We can construct a 3-GDD of group type \((cR^{d+1})^u\) with block size \( k \) containing a sub 3-GDD of group type \((cR^d)^u\) for all \( d \geq d_1 \) by applying the fundamental construction theorem, Theorem 2.2, with the TD\((3, u, R)\) as the recipe and the 3-GDDs of group type \((cR^d)^u\) with block size \( k \) as ingredients.

It follows from Theorem 1.2 that there exists a 3-GDD of group type \((cR^d(R - 1))^u\) with block size \( k \) for all integers \( d \geq d_1 = d_1(k, u, c) \). Theorem 2.5 guarantees the existence of a \((3, 2)\)-IGDD of group type \((cR^d(R - 1) + cR^d, cR^d)^u\) with block size \( k \) for all integers \( d \geq d_1 \). Since \( R - 1 \equiv 0 \mod 4^{20} \), a TD\((3, u + 2, R)\) exists by Theorem 1.2. Therefore a \((3, 1)\)-IGDD of group type \((cR^{d(R - 1)} + cR^d, cR^d)^u\) with block size \( k \) exists by Theorem 2.8 for all \( d \geq d_1 \). Let us choose the value of \( d \) such that \( d \geq \max(d_1(k, u, c), d_1(k, u, R - 1, c)) \).

We can write \((R^d - 1) = q(R - 1)\) where \( q \) is a positive integer and \( \gcd(q, 3) = 1 \) and \( \gcd(q, 7) = 1 \). Finally we choose a prime number \( s_2 \) such that \( s_2 \geq \max(2u, 5^\beta) \) and \( s_2s_1R^d + 5^\beta q \equiv 0 \mod Q^a \) for \( i = 1, 2 \). According to the Chinese remainder theorem and the Dirichlet theorem such a prime \( s_2 \) exists, since \( \gcd(s_1R^d, Q_1) = 1 \), \( \gcd(Q_1, Q_2) = 1 \) and \( \gcd(q, 21) = 1 \). With this choice of \( s_2 \) a TD\((3, u + 2, s_2)\) exists and \( s_2 \geq 5^\beta = p \), \( \gcd(s_2, 5) = 1 \) and \( \gcd(s_2, Q_1) = 1 \) for \( i = 1, 2 \). (Note that \( Q_1 = 3^{2^u} \) and \( Q_2 = 7^{2^u} \).

Now, applying Theorem 2.7 with \( s_1, s_2, n = cR^d(R - 1), p = 5^\beta \) and \( w = cR^d \) we construct a 3-GDD of group type \((cs_1s_2R^d(R - 1) + c5^\beta R^d)^u\) with block size \( k \) containing a sub 3-GDD of group type \((c5^\beta R^d)^u\). We construct the sub 3-GDD of group type \((c5^\beta R^d)^u\) to contain a sub 3-GDD of group type \((c5^\beta)^u\). This can be achieved by applying the fundamental construction theorem, Theorem 2.2. We take the TD\((3, u, R^d)\) as the recipe and 3-GDDs of group type \((c5^\beta)^u\) with block size \( k \) as ingredients. The existence of a 3-GDD of group type \((c5^\beta)^u\) with block size \( k \) follows from Theorem 2.2.

We have constructed a 3-GDD of group type \((cs_1s_2R^d(R - 1) + c5^\beta q(R - 1) + c5^\beta)^u\) with block size \( k \) containing a sub 3-GDD of group type \((c5^\beta)^u\). Let \( n = cs_1s_2R^d(R - 1) + c5^\beta q(R - 1) \) and \( w = c5^\beta \). With this choice of \( n \) and \( w \) we have the following.

1. \( \gcd(n, w) = c \), since \( \gcd(s_1, 5) = 1 \) for \( i = 1, 2 \) and \( R \equiv -1 \mod 5 \).

2. Since \( R - 1 \equiv 0 \mod 2^{20} \), a 3-GDD of group type \( n^u \) with block size \( k \) exists by Theorem 1.2. The existence of a 3-GDD of group type \( w^u \) with block size \( k \) follows from Theorem 1.2 as well.

3. The existence of a \((3, 2)\)-IGDD of group type \( (n + w, w)^u \) with block size \( k \) follows from Theorem 2.5.

4. We still need to justify the existence of a \((3, 1)\)-IGDD of group type \((n + w, w)^u \) with block size \( k \). As we previously mentioned, there exist a \((3, 1)\)-IGDD of group type \((cQ_1Q_2 + cQ_2^q, cQ_2^q)^u \) and a \((3, 1)\)-IGDD of group type \((cQ_2Q_1^q + cQ_1^q, cQ_1^q)^u \) with block size \( k \). Applying Theorem 2.6 with \( s = Q_1^{a_1} \) for \( i = 1, 2, a = 1 \) and \( b = 0 \) we can construct a \((3, 1)\)-IGDD of group type \((cQ_1Q_2^q + cQ_2^q, cQ_2^q)^u \) and a \((3, 1)\)-IGDD of group type \((cQ_2Q_1^q + cQ_1^q, cQ_1^q)^u \) with block size \( k \). Since \( s_2s_1R^d + 5^\beta q \equiv 0 \mod Q_1^q Q_1^q \), we can write \( s_2s_1R^d + 5^\beta q = A \cdot Q_1^q Q_1^q \) for some positive integer \( A \). Now, we apply Theorem 2.6 with \( s = A(R - 1), n = cQ_2Q_1^q, v = cQ_2Q_1^q \) and \( w = cQ_2Q_1^q \) to construct a \((3, 1)\)-IGDD of group type \((A(R - 1) - cQ_1Q_2^q + acQ_2^q + bcQ_2^q, acQ_2^q + bcQ_2^q)^u \) with block size \( k \) for any non-negative integers \( a \) and \( b \) with \( s + b = A(R - 1) \). We claim that we can find \( a \) and \( b \), \( a, b \geq 0 \), such that \( aQ_1^q + bQ_2^q = 5^\beta \) and \( a + b \leq A(R - 1) \).

Since \((Q_1, Q_2) = 5^\beta \) and \((Q_1, Q_2) = 1 \), there exist non-negative solutions \( a \) and \( b \) for \( aQ_1^q + bQ_2^q = 5^\beta \). Furthermore \((a + b) \cdot \min(Q_1^q, Q_2^q) \leq aQ_1^q + bQ_2^q = 5^\beta \leq 5 \cdot (Q_1Q_2)^u \) implies that \( a + b \leq 5 \cdot \max(Q_1^q, Q_2^q) = 5Q_1^q \leq R - 1 \leq A(R - 1) \) and the proof is complete. □

### 4. Proof of the main theorem

In the final section we will obtain our main result, Theorem 1.6. The following large integer representation theorem and Lemma 4.2 will be needed in the proof.

**Theorem 4.1.** Let \( n' \) and \( w' \) be positive integers such that \( (n', w') = c \). Then there exists an integer \( m_1 = m_1(n', w') \) such that any integer \( m \) which is divisible by \( c \) and \( m \geq m_1 \) can be represented as \( m = s_i s_i n' + p' w' \) where \( s_i \) and \( p' \) are non-negative integers and \( p' \leq s_i \) for \( i = 1, 2 \).

**Proof.** The \( c = 1 \) case of this theorem was already proved in [7]. The general case is an immediate consequence of the \( c = 1 \) case. □

Let \( k \) and \( u \) be positive integers, \( 3 \leq k \leq u \). Let \( m_{\text{min}} = \min\{m|m(u - 2) \equiv 0 \mod (k - 2), m^2(u - 1)(u - 2) \equiv 0 \mod (k - 1)(k - 2), m > 1\} \), that is \( m_{\text{min}} \) is the smallest positive integer for which the congruences \( m_{\text{min}}(u - 2) \equiv 0 \mod (k - 2), m_{\text{min}}^2(u - 1)(u - 2) \equiv 0 \mod (k - 1)(k - 2) \) and \( m_{\text{min}}^2u(u - 1)(u - 2) \equiv 0 \mod k(k - 1)(k - 2) \) are satisfied.
Lemma 4.2. Let \( k \) and \( u \) be positive integers, \( 3 \leq k \leq u \). Let us suppose that

1. \( m(u-2) \equiv 0 \mod (k-2) \),
2. \( m^2(u-1)(u-2) \equiv 0 \mod (k-1)(k-2) \),
3. \( m^3u(u-1)(u-2) \equiv 0 \mod k(k-1)(k-2) \)

for some positive integer \( m \). Then \( m \) is a multiple of \( m_{min} \).

Proof. Any \( m \geq m_{min} \), satisfying the conditions of Lemma 4.2, can be written in \( m = qm_{min} + r \) form, where \( q \geq 1 \) and \( 0 \leq r < m_{min} \). Then obviously \( r \) satisfies the first condition, \( r(u-2) \equiv 0 \mod (k-2) \).

Let \( g = \gcd((u-1)(u-2), (k-1)(k-2)) \). From \( m^2(u-1)(u-2) \equiv 0 \mod (k-1)(k-2) \), it follows that \( \frac{(k-1)(k-2)}{g} \) divides \( m^2 \). Let us write \( \frac{(k-1)(k-2)}{g} = A^2B \), where \( A \) and \( B \) are positive integers and \( B \) is square free or \( B = 1 \). Thus \( m^2 \equiv 0 \mod A^2B \) implies \( m \equiv 0 \mod AB \). Similarly, \( m_{min} \equiv 0 \mod AB \) and consequently, \( r \equiv 0 \mod AB \). This way \( r^2 \equiv 0 \mod A^2B \) and \( r^2(u-1)(u-2) \equiv 0 \mod (k-1)(k-2) \).

Let \( f = \gcd(u((u-1)(u-2), (k-1)(k-2)) \). Then \( \frac{k(k-1)(k-2)}{f} \) divides \( m^3 \). We can write \( \frac{k(k-1)(k-2)}{f} = A^3B_1^2B_2 \) where \( A, B_1 \) and \( B_2 \) are positive integers such that \( \gcd(B_1, B_2) = 1 \) and \( B_i \) is square free or 1, \( i = 1, 2 \). \( m^3 \equiv 0 \mod A^3B_1^2B_2 \) implies \( m \equiv 0 \mod AB_1B_2 \). Similarly, \( m_{min} \equiv 0 \mod AB_1B_2 \) and consequently, \( r \equiv 0 \mod AB_1B_2 \). This way \( r^3 \equiv 0 \mod A^3B_1^2B_2 \) and \( r^3(u-1)(u-2) \equiv 0 \mod k(k-1)(k-2) \). Since \( 0 \leq r < m_{min} \), \( r = 0 \).

Proof of Theorem 1.6. It is well known that the necessary conditions for the existence of a 3-GDD of group type \( m^u \) with block size \( k \) and index one are \( m(u-2) \equiv 0 \mod (k-2) \), \( m^2(u-1)(u-2) \equiv 0 \mod (k-1)(k-2) \) and \( m^3u(u-1)(u-2) \equiv 0 \mod k(k-1)(k-2) \). Now we prove that the conditions are sufficient for the existence of such 3-GDDS for sufficiently large \( m \). From Theorem 3.1, there exist positive integers \( n = m(n, u, m_{min}) \) and \( w = w(k, u, m_{min}) \) such that \( \gcd(n, w) = m_{min} \) and there exist a 3-GDD of group type \( n^u \), a 3-GDD of group type \( w^u \), a \((3, 1)\)-IGDD and a \((3, 2)\)-IGDD of group type \((n + w, u)^u \) all having block size \( k \).

Let us choose two different prime powers \( q_1 \) and \( q_2 \) such that \( q_1 > q_2 \), \((q_1, w) = 1\) and \((q_2, n) = 1\). Set \( n' = n(k, u, m_{min}) = q_1^2d \), \( w' = w(k, u, m_{min}) = q_2d \), where \( d = d(k, u) \) is a fixed integer such that a \( TD(3, u + 2, ba^d) \) exists for any positive integers \( b \) and \( a \geq 2 \). Note that such a \( d \) exists by Theorem 1.2.

Let us set \( m_0 = m_0(k, u) = m_1(n(k, u, m_{min}), w(k, u, m_{min})) \), where \( m_1 \) is from Theorem 4.1. Note that \( m_{min} \) depends on the values of \( k \) and \( u \) and \( \gcd(n', w') = m_{min} \). Let \( m \geq m_0 \) and let \( m \) satisfy the necessary arithmetic conditions. Then \( m \) is divisible by \( m_{min} \) and by Theorem 4.1 if \( m \geq m_0 \) then there exist non-negative integers \( s_1, s_2 \) and \( p' \) such that \( m = s_1s_2n' + p'w' \) and \( p' \leq s_1 \) for \( i = 1, 2 \). Set \( s_1 = s_1'q_1^d \), \( s_2 = s_2'q_1^d \) and \( p = p'q_2^d \), then we have \( m = s_1s_2n' + p'w' = (s_1'q_1^d)(s_2'q_1^d) + (p'q_2^d)w \). Hence Theorem 2.7 guarantees the existence of a 3-GDD of group type \((s_1s_2n + pw)^u \) with block size \( k \) and the proof is complete.

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References