Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 339 (2008) 876–888
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Dissipative solutions for equations of viscoelastic diffusion in polymers ☆

Dmitry A. Vorotnikov

Faculty of Mathematics, Voronezh State University, Universitetskaya pl. 1, 394006, Voronezh, Russia

Received 2 May 2007
Available online 27 July 2007
Submitted by Steven G. Krantz

Abstract

We study the system of differential equations which describes the diffusion of a penetrant liquid in a polymer. We construct dissipative solutions to the Cauchy problem for this system in the space.

Keywords: Evolution PDEs; Dissipative solution; Non-Fickian diffusion; Non-linearity; Polymer-penetrant systems; Topological degree

1. Introduction

Widespread applications of polymeric materials in industry, medicine, and everyday life require theoretical understanding and mathematical modelling of their behaviour. One of the basic distinguishing properties of polymers is viscoelasticity. It shows itself in the peculiar dynamics of polymers which can be modelled via non-classical (non-Newtonian) differential relations between the stress tensor and the strain velocity tensor, and this leads to challenging mathematical problems (see e.g. [7]). Furthermore, there is a growing consensus that viscoelastic mechanisms play an important role in diffusion processes in polymers [4].

It is well known that diffusion in continuums is described by the following conservation law:

$$\frac{\partial u}{\partial t} = -\operatorname{div} J \tag{1.1}$$

where $u = u(t,x)$ is the concentration and $J = J(t,x)$ is the concentration flux vector (they depend on time t and the spatial point x).

The classical Fick's law states that the flux is proportional to the concentration gradient

$$J = -D(u)\nabla u \tag{1.2}$$

where $D(u)$ is the diffusion coefficient (generally speaking, it is a positive-definite tensor). Formulas (1.1) and (1.2) yield the classical diffusion equation

☆ The work was partially supported by BRHE Program of Ministry of Education and Science of Russia and CRDF and by RFBR.
E-mail address: mitvorot@math.vsu.ru.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.07.048

$$\frac{\partial u}{\partial t} = \operatorname{div}\big(D(u)\nabla u\big). \tag{1.3}$$

If $D(u) \equiv DI$ (where I is the unit tensor and D is a positive number), then (1.3) becomes the heat equation

$$\frac{\partial u}{\partial t} = D\Delta u. \tag{1.4}$$

Experiments show that the concentration behaviour in diffusion processes in polymers cannot be described by (1.3) or (1.4) (see e.g. [17]). Let us mention two examples of such phenomena. The first one is so-called "case II diffusion" where concentration fronts can move with constant speed (the Fick's law implies that a front should propagate with speed proportional to $\frac{1}{\sqrt{t}}$). The second one is called "sorption overshoot." It means that the mass of penetrant absorbed by the polymer increases sharply until some point and then decreases, little by little, to a steady-state value [3].

Thus, Fick's law (1.2) should be replaced by another relation in order to explain the observed phenomena. One of such relations (based on the relaxation (viscoelastic) mechanism) was proposed by Cohen et al. [3,4] for the diffusion of a penetrant liquid in a polymer

$$J = -D(u)\nabla u - E(u)\nabla \int_{-\infty}^{t} \exp\left(\int_{t}^{s} \beta\big(u(\xi,x)\big)\,d\xi\right) f\left(u(s,x), \frac{\partial u(s,x)}{\partial s}\right) ds. \tag{1.5}$$

Here E, β, D are scalar functions of a scalar argument, f is a scalar function of two scalar arguments, D and E are called the diffusion and stress-diffusion coefficients, respectively. The function β is the inverse of the relaxation time. A typical example of β is [3]

$$\beta(u) = \frac{1}{2}(\beta_R + \beta_G) + \frac{1}{2}(\beta_R - \beta_G)\tanh\left(\frac{u - u_{RG}}{\delta}\right)$$

where $\beta_R, \beta_G, \delta, u_{RG}$ are positive constants, $\beta_R > \beta_G$.

The constitutive law (1.5) may be rewritten as a system of two differential equations using the new variable $\sigma(t,x) = \int_{-\infty}^{t} \exp(\int_t^s \beta(u(\xi,x))\,d\xi) f(u(s,x), \frac{\partial u(s,x)}{\partial s})\,ds$ which is called stress (but has no exact connection with the classical stress tensor)

$$J(t,x) = -D(u)\nabla u - E(u)\nabla\sigma, \tag{1.6}$$

$$\frac{\partial \sigma}{\partial t} + \beta(u)\sigma = f\left(u, \frac{\partial u}{\partial t}\right). \tag{1.7}$$

We assume (as in [3]) that $D(u) \equiv D > 0$ and $E(u) \equiv E > 0$. Then (1.6) and (1.1) yield the diffusion equation

$$\frac{\partial u}{\partial t} = D\Delta u + E\Delta\sigma. \tag{1.8}$$

A typical (but simple) form for the function f is

$$f(u,u') = \mu u + \nu u' \tag{1.9}$$

where μ and ν are constants. This relation was used, for instance, in [4,5,15] (and in [3] with $\nu = 0$). In this paper we assume, however, that μ is not a constant but depends on u.

Initial-boundary value problems for some particular cases of the general system (1.1), (1.6), (1.7) were studied in [1,2,8]. H. Amann [2] considers a wide class of these particular cases and shows existence of maximal (not global in time) solutions. A global existence result is given in [8] but only for the one-dimensional case ($0 < x < 1$). Another result on global in time solvability is presented in [1] for $f = \mu u$ and $D = E = \text{const}$. It is formulated for $0 < x < 1$, but the technique used there seems to be applicable for $x \in \Omega$ where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary.

Observe that if we take the gradient of (1.7), we get

$$\frac{\partial \nabla\sigma}{\partial t} + \nabla\big(\beta(u)\sigma\big) = \nabla f(u,u'). \tag{1.10}$$

In [12] the term $\nabla(\beta(u)\sigma)$ was replaced by the term $\beta(u)\nabla\sigma$, and $f(u,u')$ was taken in the form $f(u,u')=\mu u$ with a constant $\mu>0$. These simplifications provide extra a priori estimates, but the goal of our paper is to study the original model.

The lack of suitable a priori estimates impedes investigation of system (1.7)–(1.8) even if the equalities are considered in the sense of distributions. A possible way out is to study so-called dissipative solutions.

The idea of dissipative solutions for PDEs is due to P.-L. Lions [10], who proposed it for the Euler equations of ideal fluid motion. It consists in replacement of an equation by a system of inequalities depending on functional parameters. The solutions of this system are called dissipative solutions to the original equation. Any regular (strong) solution of the equation should be a dissipative solution. All dissipative solutions should coincide with the regular solution (with the same initial data) as long as the latter one exists.

The aim of our paper is to construct dissipative solutions to the Cauchy problem for system (1.7)–(1.8) in the n-dimensional space, $n\in\mathbb{N}$. The proof of existence is based on approximating and topological arguments (cf. [18]).

2. Statement of the problem, notations, and the main result

We consider a polymer filling the whole space $\mathbb{R}^n$, $n\in\mathbb{N}$. The most important particular cases are $n=2$ (diffusion in polymer films) and $n=3$. We study the diffusion of a penetrant in this polymer which is described by the following Cauchy problem:

$$\frac{\partial u}{\partial t}=D\Delta u+E\Delta\sigma, \tag{2.1}$$

$$\frac{\partial\sigma}{\partial t}+\beta(u)\sigma=\mu(u)u+\nu\frac{\partial u}{\partial t}, \tag{2.2}$$

$$u|_{t=0}=a,\qquad \sigma|_{t=0}=\sigma_0. \tag{2.3}$$

Here $u=u(t,x):[0,+\infty)\times\mathbb{R}^n\to\mathbb{R}$ is the unknown concentration of the penetrant (at the spatial point x at the moment of time t), $\sigma=\sigma(t,x):[0,+\infty)\times\mathbb{R}^n\to\mathbb{R}$ is the unknown stress, $a=a(x)$, $\sigma_0=\sigma_0(x):\mathbb{R}^n\to\mathbb{R}$ are given initial data, D,E,ν are given positive constants, $\mu,\ \beta:\mathbb{R}\to\mathbb{R}$ are given functions. We assume that these functions are continuous, and that there are constants $K_\mu,K_\beta,L_\mu,L_\beta$ such that

$$\bigl|\mu(\xi)\bigr|\leqslant K_\mu,\qquad \bigl|\mu(\xi)-\mu(\eta)\bigr|\leqslant L_\mu|\xi-\eta|,\quad \xi,\eta\in\mathbb{R}, \tag{2.4}$$

$$\bigl|\beta(\xi)\bigr|\leqslant K_\beta,\qquad \bigl|\beta(\xi)-\beta(\eta)\bigr|\leqslant L_\beta|\xi-\eta|,\quad \xi,\eta\in\mathbb{R}. \tag{2.5}$$

Denote $\tau=\sigma-\nu u$, $\tau_0=\sigma_0-\nu a$, $\gamma(\cdot)=\mu(\cdot)-\nu\beta(\cdot)$, $d=D+E\nu$. Then we can rewrite (2.1)–(2.3) in the following form:

$$\frac{\partial u}{\partial t}=d\Delta u+E\Delta\tau, \tag{2.6}$$

$$\frac{\partial\tau}{\partial t}+\beta(u)\tau=\gamma(u)u, \tag{2.7}$$

$$u|_{t=0}=a,\qquad \tau|_{t=0}=\tau_0. \tag{2.8}$$

By (2.4) and (2.5), there are constants K_γ,L_γ such that

$$\bigl|\gamma(\xi)\bigr|\leqslant K_\gamma,\qquad \bigl|\gamma(\xi)-\gamma(\eta)\bigr|\leqslant L_\gamma|\xi-\eta|,\quad \xi,\eta\in\mathbb{R}. \tag{2.9}$$

Let us introduce the necessary notations.

We use the standard notations $L_p(\Omega)$, $W_p^m(\Omega)$, $H^m(\Omega)=W_2^m(\Omega)$ $(m\in\mathbb{Z},1\leqslant p\leqslant\infty)$, $H_0^m(\Omega)=\mathring{W}_2^m(\Omega)$ $(m\in\mathbb{N})$ for Lebesgue and Sobolev spaces of functions defined on an open domain $\Omega\subset\mathbb{R}^n$.

Hereafter we write $(u,v)_m$ for the scalar product $(u,v)_{H^m(\Omega)}$ and $\|u\|_m$ for the Euclidean norm $\|u\|_{H^m(\Omega)}$, $m\in\mathbb{N}$. The scalar product and the norm in $L_2(\Omega)^k=L_2(\Omega,\mathbb{R}^k)$, is denoted by (u,v) and $\|u\|$, respectively (k is equal to 1 or n).

As usual, we identify the space $H^{-m}(\Omega)$, $m\in\mathbb{N}$, with the space of linear continuous functionals on $H_0^m(\Omega)$ (the dual space). The value of a functional from $H^{-m}(\Omega)$ on an element from $H_0^m(\Omega)$ is denoted by $\langle\cdot,\cdot\rangle$ (the "bra-ket" notation). We recall that $\|\varphi\|_{-m}=\sup_{\|w\|_m=1}|\langle\varphi,w\rangle|$.

We shall write simply L_p, H^m for $L_p(\mathbb{R}^n)$, $H^m(\mathbb{R}^n)$, etc.

The operator $I-\Delta$ (identity operator minus Laplacian) is an isomorphism from H^m onto H^{m-2}, $m\in\mathbb{Z}$. The scalar product in H^m may be defined as

$$(u,v)_m=\int\limits_{\mathbb{R}^n}(I-\Delta)^{m/2}u(x)(I-\Delta)^{m/2}v(x)\,dx.$$

Here

$$(I-\Delta)^{m/2}u=\mathcal{F}^{-1}\bigl(1+\|\xi\|^2_{\mathbb{R}^n}\bigr)^{m/2}\mathcal{F}(u)$$

where $\mathcal{F}_{x\to\xi}$ is the Fourier transform of $\mathbb{R}^n$.

Note that

$$\|u\|_m\leqslant\|u\|_r\quad(m<r,\ u\in H^r).$$

The operators Δ, $I-\Delta: H^m\to H^{m-2}$ are self-adjoint.

We recall that

$$\|u\|_m=\|\hat{u}\|_m$$

where $u\in H^m_0(\Omega)$, $m\in\mathbb{N}$, and $\hat{u}\in H^m$ coincides with u in Ω and vanishes outside Ω.

The symbols $C(\mathcal{J};X)$, $C_w(\mathcal{J};X)$, $L_2(\mathcal{J};X)$, etc., denote the spaces of continuous, weakly continuous, quadratically integrable, etc., functions on an interval $\mathcal{J}\subset\mathbb{R}$ (which may be unbounded) with values in a Banach space X. We recall that a function $u:\mathcal{J}\to X$ is *weakly continuous* if for any linear continuous functional g on X the function $g(u(\cdot)):\mathcal{J}\to\mathbb{R}$ is continuous. The symbol $L_{2,\mathrm{loc}}([0,\infty);X)$ stands for the space of functions $u:(0,\infty)\to X$ which belong to $L_2(0,T;X)$ for all $T>0$.

If X is a function space ($L_2(\Omega)$, $H^m(\Omega)$, etc.), then we identify the elements of $C(\mathcal{J};X)$, $L_2(\mathcal{J};X)$, etc., with scalar functions defined on $\mathcal{J}\times\Omega$ according to the formula

$$u(t)(x)=u(t,x),\quad t\in\mathcal{J},\ x\in\Omega.$$

For sufficiently regular functions $v,\zeta:\mathcal{J}\times\Omega\to\mathbb{R}$, denote

$$\mathbf{E}_1(v,\zeta)=-\frac{\partial v}{\partial t}+d\Delta v+E\Delta\zeta,$$
$$\mathbf{E}_2(v,\zeta)=-\frac{\partial\zeta}{\partial t}-\beta(v)\zeta+\gamma(v)v.$$

We shall also use the function spaces (T is a positive number)

$$W=W(\Omega,T)=\bigl\{\tau\in L_2\bigl(0,T;H^1_0(\Omega)\bigr),\ \tau'\in L_2\bigl(0,T;H^{-1}(\Omega)\bigr)\bigr\},$$
$$\|\tau\|_W=\|\tau\|_{L_2(0,T;H^1_0(\Omega))}+\|\tau'\|_{L_2(0,T;H^{-1}(\Omega))};$$
$$W_2=W_2(\Omega,T)=\bigl\{\tau\in L_2\bigl(0,T;H^2_0(\Omega)\bigr),\ \tau'\in L_2\bigl(0,T;H^{-2}(\Omega)\bigr)\bigr\},$$
$$\|\tau\|_{W_2}=\|\tau\|_{L_2(0,T;H^2_0(\Omega))}+\|\tau'\|_{L_2(0,T;H^{-2}(\Omega))}.$$

Lemma III.1.2 from [16] implies the embeddings W, $W_2\subset C([0,T];L_2(\Omega))$.

Now we can give the definition of dissipative solution for problem (2.6)–(2.8).

Definition. Let $a\in H^{-1}$, $\tau_0\in L_2$. A pair of functions (u,τ) from the class

$$u\in L_{2,\mathrm{loc}}\bigl([0,\infty);L_2\bigr)\cap C_w\bigl([0,\infty);H^{-1}\bigr),\tag{2.10}$$
$$\tau\in C_w\bigl([0,\infty);L_2\bigr)\tag{2.11}$$

is called a *dissipative* solution for problem (2.6)–(2.8) if, for all sufficiently regular functions $v,\zeta\in C([0,\infty);L_2\cap L_\infty)$ and all non-negative numbers t, one has

$$\frac{1}{2}\|u(t)-v(t)\|_{-1}^2+\frac{1}{2}\|\tau(t)-\zeta(t)\|^2+\frac{d}{5}\int_0^t\|u(s)-v(s)\|^2\,ds$$
$$\leqslant\exp\left(\int_0^t\Gamma(s)\,ds\right)\left\{\frac{1}{2}\|a-v(0)\|_{-1}^2+\frac{1}{2}\|\tau_0-\zeta(0)\|^2\right.$$
$$\left.+\int_0^t\exp\left(\int_s^0\Gamma(\xi)\,d\xi\right)\left[\left(\mathbf{E}_1(v,\zeta)(s),u(s)-v(s)\right)_{-1}+\left(\mathbf{E}_2(v,\zeta)(s),\tau(s)-\zeta(s)\right)\right]ds\right\} \tag{2.12}$$

where

$$\Gamma(s)=\frac{1}{2}+\frac{10E^2}{d}+2K_\beta+2d+\frac{5}{2d}K_\gamma^2+\frac{5}{2d}L_\beta^2\|\zeta(s)\|_{L_\infty}^2+\frac{5}{2d}L_\gamma^2\|v(s)\|_{L_\infty}^2. \tag{2.13}$$

The main result of our paper is that these solutions have three basic merits [10] of dissipative solutions:

Theorem 2.1.

(a) *Given $a\in H^{-1}$, $\tau_0\in L_2$, there is a dissipative solution for problem* (2.6)–(2.8).
(b) *If, for some $a\in H^{-1}$, $\tau_0\in L_2$, there exist $T>0$ and a sufficiently regular solution $(u_1,\tau_1):[0,T]\times\mathbb{R}^n\to\mathbb{R}\times\mathbb{R}$ to problem* (2.6)–(2.8)*, then the restriction of any dissipative solution* (*with the same initial data*) *to* $[0,T]$ *coincides with* (u_1,τ_1).
(c) *Every sufficiently regular solution $(u,\tau):[0,\infty)\times\mathbb{R}^n\to\mathbb{R}\times\mathbb{R}$ is a* (*unique*) *dissipative solution.*

Note that a dissipative solution is not a priori unique (for given initial data). Theorem 2.1(c) means that if there is a global in time regular solution, then the dissipative solution is unique and coincides with it.

Now we turn to the proof of the theorem.

3. Proof of the main result

Before proving Theorem 2.1 we have to make some auxiliary observations. Let us begin with the following simple inequality:

Lemma 3.1. *Let $f,\chi,L,M:[0,T]\to\mathbb{R}$ be scalar functions, $\chi,L,M\in L_1(0,T)$, and $f\in W_1^1(0,T)$ (i.e. f is absolutely continuous). If*

$$\chi(t)\geqslant 0,\qquad L(t)\geqslant 0 \tag{3.1}$$

and

$$f'(t)+\chi(t)\leqslant L(t)f(t)+M(t) \tag{3.2}$$

for a.a. $t\in(0,T)$, then

$$f(t)+\int_0^t\chi(s)\,ds\leqslant\exp\left(\int_0^t L(s)ds\right)\left[f(0)+\int_0^t\exp\left(\int_s^0 L(\xi)\,d\xi\right)M(s)\,ds\right] \tag{3.3}$$

for all $t\in[0,T]$.

Proof. Denote the left-hand side of (3.3) by $g(t)$. Then

$$g'(t)\leqslant L(t)f(t)+M(t)\leqslant L(t)g(t)+M(t). \tag{3.4}$$

Let $G(t)=\exp(\int_t^0 L(\xi)\,d\xi)g(t)$. Multiplying (3.4) with $\exp(\int_t^0 L(\xi)\,d\xi)$, we get

$$G'(t) \leqslant \exp\left(\int_t^0 L(\xi)\,d\xi\right) M(t). \tag{3.5}$$

Integration of this inequality gives

$$G(t) \leqslant G(0) + \int_0^t \exp\left(\int_s^0 L(\xi)\,d\xi\right) M(s)\,ds. \tag{3.6}$$

It remains to observe that $G(t) = \exp(\int_t^0 L(s)\,ds)(f(t) + \int_0^t \chi(s))$ and $G(0) = f(0)$. $\square$

Now, consider the following auxiliary problem:

$$\frac{\partial u}{\partial t} = d\Delta u + \lambda E \Delta\tau, \tag{3.7}$$

$$\frac{\partial \tau}{\partial t} + \varepsilon\Delta^2\tau + \lambda\beta(u)\tau = \lambda\gamma(u)u, \tag{3.8}$$

$$u|_{t=0} = a, \qquad \tau|_{t=0} = \tau_0. \tag{3.9}$$

Here $\varepsilon > 0$, $\lambda \in [0, 1]$ are parameters.

Let Ω be a domain in $\mathbb{R}^n$ (it may be $\mathbb{R}^n$ itself).

Definition. Given $a, \tau_0 \in L_2(\Omega)$, a pair of functions (u, τ) from the class

$$u \in W(\Omega, T), \qquad \tau \in W_2(\Omega, T) \tag{3.10}$$

is a *weak* solution of problem (3.7)–(3.9) if equality (3.7) holds in the space $H^{-1}(\Omega)$ a.e. on $(0, T)$, (3.8) holds in the space $H^{-2}(\Omega)$ a.e. on $(0, T)$, and (3.9) holds in $L_2(\Omega)$.

The last condition makes sense due to the embeddings $W, W_2 \subset C([0, T]; L_2(\Omega))$.

Lemma 3.2. *Let (u, τ) be a weak solution to problem* (3.7)–(3.9). *Then the following a priori estimate holds*:

$$\|u\|_W + \|\tau\|_{W_2} + \|u\|_{L_\infty(0,T;L_2(\Omega))} + \|\tau\|_{L_\infty(0,T;L_2(\Omega))} \leqslant C \tag{3.11}$$

where C is independent of λ and Ω, but depends on $\|a\|$, $\|\tau_0\|$, T, ε.

Proof. Take the "bra-ket" of the terms of (3.7) (as elements of $H^{-1}(\Omega)$) and $u(t) \in H_0^1(\Omega)$ at a.a. $t \in [0, T]$,

$$\left\langle \frac{\partial u}{\partial t}, u \right\rangle = d\langle \Delta u, u\rangle + \lambda E\langle \Delta\tau, u\rangle.$$

Take the "bra-ket" of the terms of (3.8) (as elements of $H^{-2}(\Omega)$) and $\tau(t) \in H_0^2(\Omega)$ at a.a. $t \in [0, T]$,

$$\left\langle \frac{\partial \tau}{\partial t}, \tau \right\rangle + \varepsilon\big\langle \Delta^2\tau, \tau\big\rangle + \lambda\big(\beta(u)\tau, \tau\big) = \lambda\big(\gamma(u)u, \tau\big).$$

Note (see e.g. [16, Lemma III.1.2]) that

$$\big(\|u\|^2\big)' = 2\langle u', u\rangle, \qquad \big(\|\tau\|^2\big)' = 2\langle \tau', \tau\rangle.$$

Adding the results and making some simple transformations, we get

$$\begin{aligned}
&\frac{1}{2}\big(\|u\|^2\big)' + \frac{1}{2}\big(\|\tau\|^2\big)' + d(\nabla u, \nabla u) + \varepsilon(\Delta\tau, \Delta\tau) + (\nabla\tau, \nabla\tau) \\
&\quad = \lambda E\langle \Delta\tau, u\rangle - \lambda\big(\beta(u)\tau, \tau\big) + \lambda\big(\gamma(u)u, \tau\big) - (\Delta\tau, \tau).
\end{aligned}$$

Integration of this equality yields

$$\begin{aligned}
&\frac{1}{2}\|u(t)\|^2+\frac{1}{2}\|\tau(t)\|^2+d\int_0^t\big(\nabla u(s),\nabla u(s)\big)\,ds+\varepsilon\int_0^t\big(\Delta\tau(s),\Delta\tau(s)\big)\,ds+\int_0^t\big(\nabla\tau(s),\nabla\tau(s)\big)\,ds\\
&\quad\leqslant\frac{1}{2}\|a\|^2+\frac{1}{2}\|\tau_0\|+\lambda E\int_0^t\|\Delta\tau(s)\|\|u(s)\|\,ds+\lambda K_\beta\int_0^t\|\tau(s)\|^2\,ds\\
&\qquad+\lambda K_\gamma\int_0^t\|u(s)\|\|\tau(s)\|\,ds+\int_0^t\|\Delta\tau(s)\|\|\tau(s)\|\,ds.
\end{aligned}$$

Applying Cauchy's inequality $ab\leqslant ca^2+\frac{1}{4c}b^2$ to the last three terms, we obtain

$$\begin{aligned}
&\frac{1}{2}\|u(t)\|^2+\frac{1}{2}\|\tau(t)\|^2+d\int_0^t\|\nabla u(s)\|^2\,ds+\varepsilon\int_0^t\|\Delta\tau(s)\|^2+\int_0^t\|\nabla\tau(s)\|^2\,ds\\
&\quad\leqslant\frac{1}{2}\|a\|^2+\frac{1}{2}\|\tau_0\|+\frac{\varepsilon}{3}\int_0^t\|\Delta\tau(s)\|^2\,ds+\frac{3\lambda^2E^2}{4\varepsilon}\int_0^t\|u(s)\|^2\,ds+\lambda K_\beta\int_0^t\|\tau(s)\|^2\,ds+\lambda K_\gamma\int_0^t\|u(s)\|^2\,ds\\
&\qquad+\frac{\lambda K_\gamma}{4}\int_0^t\|\tau(s)\|^2\,ds+\frac{\varepsilon}{3}\int_0^t\|\Delta\tau(s)\|^2\,ds+\frac{3}{4\varepsilon}\int_0^t\|\tau(s)\|^2\,ds.
\end{aligned}$$

Thus, since $\lambda\leqslant 1$,

$$\begin{aligned}
&\frac{1}{2}\|u(t)\|^2+\frac{1}{2}\|\tau(t)\|^2+d\int_0^t\|\nabla u(s)\|^2\,ds+\frac{\varepsilon}{3}\int_0^t\|\Delta\tau(s)\|^2+\int_0^t\|\nabla\tau(s)\|^2\,ds\\
&\quad\leqslant\frac{1}{2}\|a\|^2+\frac{1}{2}\|\tau_0\|+\left(\frac{3E^2}{4\varepsilon}+K_\gamma\right)\int_0^t\|u(s)\|^2\,ds+\left(K_\beta+\frac{K_\gamma}{4}+\frac{3}{4\varepsilon}\right)\int_0^t\|\tau(s)\|^2\,ds.
\end{aligned}\tag{3.12}$$

Now Lemma 3.1 (with $f(t)=\int_0^t\|u(s)\|^2\,ds+\int_0^t\|\tau(s)\|^2\,ds$) implies a bound for the right-hand and the left-hand sides of (3.12), so we get an estimate of $\|u\|_{L_\infty(0,T;L_2(\Omega))}$, $\|\tau\|_{L_\infty(0,T;L_2(\Omega))}$, $\|u\|_{L_2(0,T;H_0^1(\Omega))}$ and $\|\tau\|_{L_2(0,T;H_0^2(\Omega))}$ (here one can use the Gronwall lemma instead of Lemma 3.1 as well). It remains to estimate the time derivatives.

Since (3.7) holds in $H^{-1}(\Omega)$, for any $\varphi\in H_0^1(\Omega)$ one has

$$\begin{aligned}
\big\|\langle u',\varphi\rangle\big\|_{L_2(0,T)}&=\big\|d\langle\Delta u,\varphi\rangle+\lambda E\langle\Delta\tau,\varphi\rangle\big\|_{L_2(0,T)}\\
&=\big\|d\langle\nabla u,\nabla\varphi\rangle+\lambda E\langle\Delta\tau,\varphi\rangle\big\|_{L_2(0,T)}\\
&\leqslant d\|u\|_{L_2(0,T;H^1(\Omega))}\|\varphi\|_1+E\|\Delta\tau\|_{L_2(0,T;L_2(\Omega))}\|\varphi\|\leqslant C\|\varphi\|_1,
\end{aligned}$$

and we get the required estimate for u'.

Similarly, since (3.8) holds in $H^{-2}(\Omega)$, for any $\varphi\in H_0^2(\Omega)$ one has

$$\begin{aligned}
\big\|\langle\tau',\varphi\rangle\big\|_{L_2(0,T)}&\leqslant\big\|\langle\varepsilon\Delta^2\tau,\varphi\rangle\big\|_{L_2(0,T)}+\big\|\langle\lambda\beta(u)\tau,\varphi\rangle\big\|_{L_2(0,T)}+\big\|\langle\lambda\gamma(u)u,\varphi\rangle\big\|_{L_2(0,T)}\\
&\leqslant\big\|\langle\varepsilon\Delta\tau,\Delta\varphi\rangle\big\|_{L_2(0,T)}+K_\beta\big\|\langle\tau,\varphi\rangle\big\|_{L_2(0,T)}+K_\gamma\big\|\langle u,\varphi\rangle\big\|_{L_2(0,T)}\\
&\leqslant\varepsilon\|\tau\|_{L_2(0,T;H^2(\Omega))}\|\varphi\|_2+K_\beta\|\tau\|_{L_2(0,T;L_2(\Omega))}\|\varphi\|+K_\gamma\|u\|_{L_2(0,T;L_2(\Omega))}\|\varphi\|\\
&\leqslant C\|\varphi\|_2.\qquad\square
\end{aligned}$$

Lemma 3.3. *Given $T > 0$, a domain Ω, and initial data $a, \tau_0 \in L_2(\Omega)$, there exists a weak solution to problem* (3.7)–(3.9) *in class* (3.10).

Proof. Assume first that the domain Ω is bounded.

Let us introduce auxiliary operators by the following formulas:

$$\begin{aligned}
&Q_1 : W_2 \to L_2\big(0, T; H^{-1}(\Omega)\big), \qquad Q_1(\tau) = -E\Delta\tau,\\
&Q_2 : W \times W_2 \to L_2\big(0, T; H^{-2}(\Omega)\big), \qquad Q_2(u, \tau) = \beta(u)\tau - \gamma(u)u,\\
&Q : W \times W_2 \to L_2\big(0, T; H^{-1}(\Omega)\big) \times L_2\big(0, T; H^{-2}(\Omega)\big) \times L_2(\Omega) \times L_2(\Omega),\\
&Q(u, \tau) = \big(Q_1(\tau), Q_2(u, \tau), 0, 0\big),\\
&\widetilde{A} : W \times W_2 \to L_2\big(0, T; H^{-1}(\Omega)\big) \times L_2\big(0, T; H^{-2}(\Omega)\big) \times L_2(\Omega) \times L_2(\Omega),\\
&\widetilde{A}(u, \tau) = \big(u' - d\Delta u, \tau' + \varepsilon\Delta^2\tau, u|_{t=0}, \tau|_{t=0}\big).
\end{aligned}$$

Then problem (3.7)–(3.9) is equivalent to the operator equation

$$\widetilde{A}(u, \tau) + \lambda Q(u, \tau) = (0, 0, a, \tau_0). \tag{3.13}$$

The embeddings $W \subset L_2(0, T; L_2(\Omega))$, $W_2 \subset L_2(0, T; H_0^1(\Omega))$ are compact (by e.g. [16, Theorems III.2.1 and II.1.1]). Let us show that the operators Q_1, Q_2 are compact. Really, the operator Q_1 may be considered as a superposition of the compact embedding operator

$$j : W_2 \to L_2\big(0, T; H_0^1(\Omega)\big)$$

and of the operator

$$Q_1 : L_2\big(0, T; H_0^1(\Omega)\big) \to L_2\big(0, T; H^{-1}(\Omega)\big),$$

which is continuous. The operator Q_2 may be considered as a superposition of the embedding operator

$$j_1 : W \times W_2 \to L_2\big(0, T; L_2(\Omega)\big) \times L_2\big(0, T; L_2(\Omega)\big),$$

the Nemytskii operator

$$Q_2 : L_2\big(0, T; L_2(\Omega)\big) \times L_2\big(0, T; L_2(\Omega)\big) \to L_2\big(0, T; L_2(\Omega)\big),$$

and of the embedding operator

$$j_2 : L_2\big(0, T; L_2(\Omega)\big) \to L_2\big(0, T; H^{-2}(\Omega)\big).$$

The first operator is compact, the third is continuous, so it suffices to observe that the second one is also continuous by Krasnoselskii's theorem [9,14] on continuity of Nemytskii operators.

Hence, the operator Q is also compact. But the operator $\widetilde{A}$ is invertible (e.g. by Theorem 1.1 from [6, Chapter VI]). Rewrite Eq. (3.13) as

$$(u, \tau) + \lambda\widetilde{A}^{-1}Q(u, \tau) = \widetilde{A}^{-1}(0, 0, a, \tau_0). \tag{3.14}$$

By Lemma 3.2, Eq. (3.14) has no solutions on the boundary of a sufficiently large ball B in $W \times W_2$, independent of λ. Without loss of generality $a_0 = \widetilde{A}^{-1}(0, 0, a, \tau_0)$ belongs to this ball. Then we can consider the Leray–Schauder degree (see e.g. [11]) of the map $I + \lambda\widetilde{A}^{-1}Q$ (I is the identity map) on the ball B with respect to the point a_0,

$$\deg_{\mathrm{LS}}\big(I + \lambda\widetilde{A}^{-1}Q, B, a_0\big).$$

By the homotopic invariance property of the degree we have

$$\deg_{\mathrm{LS}}\big(I + \lambda\widetilde{A}^{-1}Q, B, a_0\big) = \deg_{\mathrm{LS}}(I, B, a_0) = 1 \neq 0.$$

Thus, Eq. (3.14) (and therefore, problem (3.7)–(3.9)) has a solution in the ball B for every λ.

Now, let Ω be unbounded. Denote by Ω_m the intersection of Ω with the ball B_m of radius $m \in \mathbb{N}$ centered at the origin in the space $\mathbb{R}^n$. As we have just proved, there exists a weak solution $(u_m, \tau_m) \in W(\Omega_m, T) \times W_2(\Omega_m, T)$ to

problem (3.7)–(3.9). All these solutions are bounded by estimate (3.11). Denote by $\tilde{u}_m$ and $\tilde{\tau}_m$ the functions which coincide with u_m and τ_m, respectively, in Ω_m, and are identically zero in $\Omega\backslash\Omega_m$. Then without loss of generality (passing to a subsequence if necessary) we may assume that there exists a pair (u_*, τ_*) such that

$$\begin{aligned}
&\tilde{u}_m \to u_* \quad \text{weakly in } L_2\big(0,T;H_0^1(\Omega)\big),\\
&\tilde{u}_m \to u_* \quad \text{*-weakly in } L_\infty\big(0,T;L_2(\Omega)\big),\\
&\tilde{\tau}_m \to \tau_* \quad \text{weakly in } L_2\big(0,T;H_0^2(\Omega)\big),\\
&\tilde{\tau}_m \to \tau_* \quad \text{*-weakly in } L_\infty\big(0,T;L_2(\Omega)\big).
\end{aligned}$$

For any natural number k, Lemma 3.2 implies that the sequence $\{u_m'\}$, $m \geqslant k$, is bounded in $L_2(0,T;H^{-1}(\Omega_k))$, and the sequence $\{\tau_m'\}$, $m \geqslant k$, is bounded in $L_2(0,T;H^{-2}(\Omega_k))$. Hence (by e.g. [16, Theorem III.2.1])

$$\begin{aligned}
&u_m|_{\Omega_k} = \tilde{u}_m|_{\Omega_k} \to u_*|_{\Omega_k} \quad \text{strongly in } L_2\big(0,T;L_2(\Omega_k)\big),\\
&\tau_m|_{\Omega_k} = \tilde{\tau}_m|_{\Omega_k} \to \tau_*|_{\Omega_k} \quad \text{strongly in } L_2\big(0,T;L_2(\Omega_k)\big)
\end{aligned}$$

for every k. Furthermore (by [13, Corollary 6])

$$\begin{aligned}
&u_m|_{\Omega_k} = \tilde{u}_m|_{\Omega_k} \to u_*|_{\Omega_k} \quad \text{strongly in } C\big([0,T];H^{-1}(\Omega_k)\big),\\
&\tau_m|_{\Omega_k} = \tilde{\tau}_m|_{\Omega_k} \to \tau_*|_{\Omega_k} \quad \text{strongly in } C\big([0,T];H^{-1}(\Omega_k)\big).
\end{aligned}$$

Let us show that (u_*, τ_*) is a weak solution to problem (3.7)–(3.9) on Ω, i.e. for any $\varphi \in H_0^1(\Omega)$, $\Phi \in H_0^2(\Omega)$ one has

$$\big\langle u_*', \varphi\big\rangle = \langle d\Delta u_*, \varphi\rangle + \langle \lambda E \Delta \tau_*, \varphi\rangle, \tag{3.15}$$

$$\big\langle \tau_*', \Phi\big\rangle + \big\langle \varepsilon\Delta^2\tau_*, \Phi\big\rangle + \big\langle \lambda\beta(u_*)\tau_*, \Phi\big\rangle - \big\langle \lambda\gamma(u_*)u_*, \Phi\big\rangle = 0, \tag{3.16}$$

$$u_*|_{t=0} = a, \qquad \tau_*|_{t=0} = \tau_0. \tag{3.17}$$

Strong convergence in $C([0,T];H^{-1}(\Omega_k))$ implies (3.17) immediately, since

$$a|_{\Omega_k} = u_m|_{\Omega_k}(0) \to u_*|_{\Omega_k}(0), \qquad \tau_0|_{\Omega_k} = \tau_m|_{\Omega_k}(0) \to \tau_*|_{\Omega_k}(0).$$

Take arbitrary $\varphi \in H_0^1(\Omega)$, $\Phi \in H_0^2(\Omega)$. Since the functions with compact supports are dense in $H_0^1(\Omega)$ and $H_0^2(\Omega)$, it suffices to consider the case when φ and Φ have compact supports. Fix k large enough (such that the supports of φ and Φ are contained in Ω_k). Then for each $m \geqslant k$ one has

$$\big\langle u_m', \varphi\big\rangle = \langle d\Delta u_m, \varphi\rangle + \langle \lambda E \Delta \tau_m, \varphi\rangle, \tag{3.18}$$

$$\big\langle \tau_m', \Phi\big\rangle + \big\langle \varepsilon\Delta^2\tau_m, \Phi\big\rangle + \big\langle \lambda\beta(u_m)\tau_m, \Phi\big\rangle - \big\langle \lambda\gamma(u_m)u_m, \Phi\big\rangle = 0. \tag{3.19}$$

Passing to the limit as $m \to \infty$ in (3.18), (3.19), we obtain (3.15), (3.16). More precisely, we can pass to the limit in the terms with time derivatives at least in the sense of scalar distributions on $(0,T)$; passage to the limit in the linear terms is immediate since a linear image of weakly converging sequence is again weakly converging; passage to the limit in the non-linear terms is possible since the Nemytskii operator

$$\begin{aligned}
&Q_2 : L_2\big(0,T;L_2(\Omega_k)\big) \times L_2\big(0,T;L_2(\Omega_k)\big) \to L_2\big(0,T;L_2(\Omega_k)\big),\\
&Q_2(u,\tau) = \beta(u)\tau - \gamma(u)u
\end{aligned}$$

is continuous (see the proof of Lemma 3.2).

Finally, repeating the arguments from the proof of Lemma 3.2, one easily checks that $u_*' \in L_2(0,T;H^{-1}(\Omega))$, $\tau_*' \in L_2(0,T;H^{-2}(\Omega))$. □

Lemma 3.4. *Let $(u,\tau) \in W(\mathbb{R}^n,T) \times W_2(\mathbb{R}^n,T)$ be a weak solution to problem* (3.7)–(3.9) *with $\lambda = 1$. Then, for all sufficiently regular functions $v, \zeta \in C([0,\infty); L_2 \cap L_\infty)$ and $0 \leqslant t \leqslant T$, one has*

$$\frac{1}{2}\|u(t)-v(t)\|_{-1}^2+\frac{1}{2}\|\tau(t)-\zeta(t)\|^2+\frac{d}{5}\int_0^t\|u(s)-v(s)\|^2\,ds+\varepsilon\int_0^t\|\Delta\tau(s)-\Delta\zeta(s)\|^2\,ds$$
$$\leqslant\exp\left(\int_0^t\Gamma(s)\,ds\right)\left\{\frac{1}{2}\|a-v(0)\|_{-1}^2+\frac{1}{2}\|\tau_0-\zeta(0)\|^2\right.$$
$$+\int_0^t\exp\left(\int_s^0\Gamma(\xi)\,d\xi\right)\left[\big(\mathbf{E}_1(v,\zeta)(s),u(s)-v(s)\big)_{-1}\right.$$
$$\left.\left.+\big(\mathbf{E}_2(v,\zeta)(s),\tau(s)-\zeta(s)\big)+\varepsilon^2\|\Delta^2\zeta(s)\|^2\right]ds\right\}\tag{3.20}$$

where Γ is as in (2.13).

Proof. Let $v,\zeta\in C([0,\infty);L_2\cap L_\infty)$ be some sufficiently regular functions. Adding $\mathbf{E}_1(v,\zeta)$ to both sides of (3.7), we get

$$\frac{\partial(u-v)}{\partial t}=d\Delta(u-v)+E\Delta(\tau-\zeta)+\mathbf{E}_1(v,\zeta).\tag{3.21}$$

Similarly, from (3.8) we have

$$\frac{\partial(\tau-\zeta)}{\partial t}+\varepsilon\Delta^2\tau+\beta(u)\tau-\beta(v)\zeta=\gamma(u)u-\gamma(v)v+\mathbf{E}_2(v,\zeta).\tag{3.22}$$

Rewrite (3.21) in the following form:

$$\frac{\partial(u-v)}{\partial t}-d(u-v)-E(\tau-\zeta)+d(I-\Delta)(u-v)+E(I-\Delta)(\tau-\zeta)=\mathbf{E}_1(v,\zeta).\tag{3.23}$$

Applying the operator $(I-\Delta)^{-1}$ to both members of (3.23), we arrive at

$$\frac{\partial(I-\Delta)^{-1}(u-v)}{\partial t}-d(I-\Delta)^{-1}(u-v)-E(I-\Delta)^{-1}(\tau-\zeta)+d(u-v)+E(\tau-\zeta)$$
$$=(I-\Delta)^{-1}\mathbf{E}_1(v,\zeta).\tag{3.24}$$

Take the L_2-scalar product of (3.24) and $(u-v)(t)$ for a.a. $t\in(0,T)$,

$$\left(\frac{\partial(u-v)}{\partial t},u-v\right)_{-1}-d(u-v,u-v)_{-1}-E(\tau-\zeta,u-v)_{-1}+d(u-v,u-v)+E(\tau-\zeta,u-v)$$
$$=\big(\mathbf{E}_1(v,\zeta),u-v\big)_{-1}.\tag{3.25}$$

Take the L_2-scalar product of (3.22) and $(\tau-\zeta)(t)$ for a.a. $t\in(0,T)$ (in some terms the scalar product should be replaced by the "bra-ket" of a functional from H^{-2} and an element from H_0^2),

$$\left\langle\frac{\partial(\tau-\zeta)}{\partial t},\tau-\zeta\right\rangle+\varepsilon\big(\Delta^2\zeta,\tau-\zeta\big)+\varepsilon\big\langle\Delta^2(\tau-\zeta),\tau-\zeta\big\rangle+\big(\beta(u)\tau-\beta(v)\zeta,\tau-\zeta\big)$$
$$=\big(\gamma(u)u-\gamma(v)v,\tau-\zeta\big)+\big(\mathbf{E}_2(v,\zeta),\tau-\zeta\big).\tag{3.26}$$

Adding (3.25) and (3.26), we get

$$\frac{1}{2}\big(\|u-v\|_{-1}^2\big)'+\frac{1}{2}\big(\|\tau-\zeta\|^2\big)'+d\|u-v\|^2+\varepsilon\big(\Delta(\tau-\zeta),\Delta(\tau-\zeta)\big)$$
$$=-\varepsilon\big(\Delta^2\zeta,\tau-\zeta\big)+d(u-v,u-v)_{-1}+E(\tau-\zeta,u-v)_{-1}$$
$$-E(\tau-\zeta,u-v)-\big(\beta(u)(\tau-\zeta),\tau-\zeta\big)-\big(\big(\beta(u)-\beta(v)\big)\zeta,\tau-\zeta\big)$$
$$+\big(\gamma(u)(u-v),\tau-\zeta\big)+\big(\big(\gamma(u)-\gamma(v)\big)v,\tau-\zeta\big)+\big(\mathbf{E}_1(v,\zeta),u-v\big)_{-1}+\big(\mathbf{E}_2(v,\zeta),\tau-\zeta\big).$$

Note that

$$\begin{aligned}\left|E(\tau-\zeta,u-v)_{-1}-E(\tau-\zeta,u-v)\right| &\leqslant E\big(\|\tau-\zeta\|_{-1}\|u-v\|_{-1}+\|\tau-\zeta\|\|u-v\|\big)\\ &\leqslant 2E\|\tau-\zeta\|\|u-v\|.\end{aligned}$$

Taking into account (2.5), (2.9) we conclude

$$\begin{aligned}&\frac{1}{2}\big(\|u-v\|_{-1}^2\big)'+\frac{1}{2}\big(\|\tau-\zeta\|^2\big)'+d\|u-v\|^2+\varepsilon\big\|\Delta(\tau-\zeta)\big\|^2\\ &\quad\leqslant \varepsilon\big\|\Delta^2\zeta\big\|\|\tau-\zeta\|+d\|u-v\|_{-1}^2+2E\|\tau-\zeta\|\|u-v\|+K_\beta\|\tau-\zeta\|^2+L_\beta\|u-v\|\|\zeta\|_{L_\infty}\|\tau-\zeta\|\\ &\qquad+K_\gamma\|u-v\|\|\tau-\zeta\|+L_\gamma\|u-v\|\|v\|_{L_\infty}\|\tau-\zeta\|+\big(\mathbf{E}_1(v,\zeta),u-v\big)_{-1}+\big(\mathbf{E}_2(v,\zeta),\tau-\zeta\big).\end{aligned}$$

Thus, by Cauchy's inequality

$$\begin{aligned}&\frac{1}{2}\big(\|u-v\|_{-1}^2\big)'+\frac{1}{2}\big(\|\tau-\zeta\|^2\big)'+d\|u-v\|^2+\varepsilon\big\|\Delta(\tau-\zeta)\big\|^2\\ &\quad\leqslant \varepsilon^2\big\|\Delta^2\zeta\big\|^2+\frac{1}{4}\|\tau-\zeta\|^2+d\|u-v\|_{-1}^2+\frac{5E^2}{d}\|\tau-\zeta\|^2+\frac{d}{5}\|u-v\|^2\\ &\qquad+K_\beta\|\tau-\zeta\|^2+\frac{5}{4d}L_\beta^2\|\zeta\|_{L_\infty}^2\|\tau-\zeta\|^2+\frac{d}{5}\|u-v\|^2\\ &\qquad+\frac{5}{4d}K_\gamma^2\|\tau-\zeta\|^2+\frac{d}{5}\|u-v\|^2+\frac{5}{4d}L_\gamma^2\|v\|_{L_\infty}^2\|\tau-\zeta\|^2+\frac{d}{5}\|u-v\|^2\\ &\qquad+\big(\mathbf{E}_1(v,\zeta),u-v\big)_{-1}+\big(\mathbf{E}_2(v,\zeta),\tau-\zeta\big).\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{1}{2}\big(\|u-v\|_{-1}^2\big)'+\frac{1}{2}\big(\|\tau-\zeta\|^2\big)'+\frac{d}{5}\|u-v\|^2+\varepsilon\big\|\Delta(\tau-\zeta)\big\|^2\\ &\quad\leqslant \varepsilon^2\big\|\Delta^2\zeta\big\|^2+\big(\mathbf{E}_1(v,\zeta),u-v\big)_{-1}+\big(\mathbf{E}_2(v,\zeta),\tau-\zeta\big)\\ &\qquad+\|\tau-\zeta\|^2\left[\frac{1}{4}+\frac{5E^2}{d}+K_\beta+\frac{5}{4d}L_\beta^2\|\zeta\|_{L_\infty}^2+\frac{5}{4d}K_\gamma^2+\frac{5}{4d}L_\gamma^2\|v\|_{L_\infty}^2\right]+d\|u-v\|_{-1}^2\\ &\quad\leqslant \varepsilon^2\big\|\Delta^2\zeta\big\|^2+\big(\mathbf{E}_1(v,\zeta),u-v\big)_{-1}+\big(\mathbf{E}_2(v,\zeta),\tau-\zeta\big)+\Gamma\left(\frac{1}{2}\|u-v\|_{-1}^2+\frac{1}{2}\|\tau-\zeta\|^2\right)\end{aligned}$$

where Γ is as in (2.13). Applying Lemma 3.1 with $f=\frac{1}{2}\|u-v\|_{-1}^2+\frac{1}{2}\|\tau-\zeta\|^2$, $\chi=\frac{d}{5}\|u-v\|^2+\varepsilon\|\Delta(\tau-\zeta)\|^2$, $L=\Gamma$, $M=\varepsilon^2\|\Delta^2\zeta\|^2+(\mathbf{E}_1(v,\zeta),u-v)_{-1}+(\mathbf{E}_2(v,\zeta),\tau-\zeta)$, we get (3.20). □

Corollary 3.1. *Let $(u,\tau)\in W(\mathbb{R}^n,T)\times W_2(\mathbb{R}^n,T)$ be a weak solution to problem* (3.7)–(3.9) *with $\lambda=1$. The following estimates are valid*:

$$\|u\|_{L_\infty(0,T;H^{-1})}+\|\tau\|_{L_\infty(0,T;L_2)}+\|u\|_{L_2(0,T;L_2)}<K_1, \tag{3.27}$$

$$\varepsilon\|\Delta\tau\|_{L_2(0,T;L_2)}^2<K_2, \tag{3.28}$$

$$\|u'\|_{L_2(0,T;H^{-2})}+\|\tau'\|_{L_2(0,T;H^{-2})}<K_3(1+\sqrt{\varepsilon}). \tag{3.29}$$

The constants $K_1,\dots,K_3$ are independent of ε, but depend on $\|a\|_{-1}$, $\|\tau_0\|$, T.

Proof. Estimates (3.27), (3.28) follow from (3.20) with $v\equiv 0$, $\zeta\equiv 0$ (note that $\mathbf{E}_1(0,0)=\mathbf{E}_2(0,0)=0$).

Let us estimate the first term in (3.29). Since (3.7) holds in H^{-1}, it holds also in H^{-2}. Thus, for any $\varphi\in H_0^2$ one has

$$\begin{aligned}\big\|\langle u',\varphi\rangle\big\|_{L_2(0,T)}&=\big\|d\langle\Delta u,\varphi\rangle+E\langle\Delta\tau,\varphi\rangle\big\|_{L_2(0,T)}\\ &=\big\|d\langle u,\Delta\varphi\rangle+E\langle\tau,\Delta\varphi\rangle\big\|_{L_2(0,T)}\\ &\leqslant d\|u\|_{L_2(0,T;L_2)}\|\varphi\|_2+E\|\tau\|_{L_2(0,T;L_2)}\|\varphi\|_2\\ &\leqslant d\|u\|_{L_2(0,T;L_2)}\|\varphi\|_2+E\sqrt{T}\|\tau\|_{L_\infty(0,T;L_2)}\|\varphi\|_2\leqslant C\|\varphi\|_2\end{aligned}$$

where C depends only on K_1 and T. Similarly, we can estimate the second term in (3.29): for any $\varphi \in H_0^2$ one has

$$\begin{aligned}\left\|\langle\tau',\varphi\rangle\right\|_{L_2(0,T)} &\leqslant \left\|\langle\varepsilon\Delta^2\tau,\varphi\rangle\right\|_{L_2(0,T)} + \left\|\langle\beta(u)\tau,\varphi\rangle\right\|_{L_2(0,T)} + \left\|\langle\gamma(u)u,\varphi\rangle\right\|_{L_2(0,T)}\\ &\leqslant \left\|\langle\varepsilon\Delta\tau,\Delta\varphi\rangle\right\|_{L_2(0,T)} + K_\beta\left\|\langle\tau,\varphi\rangle\right\|_{L_2(0,T)} + K_\gamma\left\|\langle u,\varphi\rangle\right\|_{L_2(0,T)}\\ &\leqslant \sqrt{\varepsilon}\|\sqrt{\varepsilon}\Delta\tau\|_{L_2(0,T;L_2)}\|\varphi\|_2 + K_\beta\|\tau\|_{L_2(0,T;L_2)}\|\varphi\| + K_\gamma\|u\|_{L_2(0,T;H^{-1})}\|\varphi\|_1\\ &\leqslant C\|\varphi\|_2(1+\sqrt{\varepsilon})\end{aligned}$$

where C depends only on K_1, K_2, T. □

Proof of the claim (a) in Theorem 2.1. Take an increasing sequence of positive numbers $T_m \to \infty$ and a decreasing sequence of positive numbers $\varepsilon_m \to 0$. Take a sequence of functions from L_2: $a_m \to a$ in H^{-1}. By Lemma 3.3, there is a pair (u_m, τ_m) which is a weak solution to problem (3.7)–(3.9) with $\lambda = 1$, $T = T_m$, $\varepsilon = \varepsilon_m$, $\Omega = \mathbb{R}^n$, $a = a_m$. Denote by $\tilde{u}_m$ and $\tilde{\tau}_m$ the functions which are equal to u_m and τ_m in $[0, T_m]$ and are equal to zero on $(T_m, +\infty)$.

Lemma 3.4 implies that, for all sufficiently regular functions $v, \zeta \in C([0,\infty); L_2 \cap L_\infty)$ and $0 \leqslant t \leqslant T_m$, one has

$$\begin{aligned}&\frac{1}{2}\left\|u_m(t)-v(t)\right\|_{-1}^2 + \frac{1}{2}\left\|\tau_m(t)-\zeta(t)\right\|^2 + \frac{d}{5}\int_0^t \left\|u_m(s)-v(s)\right\|^2 ds\\ &\quad\leqslant \exp\left(\int_0^t \Gamma(s)\,ds\right)\left\{\frac{1}{2}\left\|a_m - v(0)\right\|_{-1}^2 + \frac{1}{2}\left\|\tau_0-\zeta(0)\right\|^2\right.\\ &\qquad + \int_0^t \exp\left(\int_s^0 \Gamma(\xi)\,d\xi\right)\left[\left(\mathbf{E}_1(v,\zeta)(s), u_m(s)-v(s)\right)_{-1}\right.\\ &\qquad \left.\left. + \left(\mathbf{E}_2(v,\zeta)(s), \tau_m(s)-\zeta(s)\right) + \varepsilon_m^2\left\|\Delta^2\zeta(s)\right\|^2\right]ds\right\}.\end{aligned} \tag{3.30}$$

Fix an arbitrary interval $[0, T]$. Due to a priori estimate (3.27), without loss of generality (passing to a subsequence if necessary) one may assume that there exist limits

$$\begin{aligned}u &= \lim_{m\to\infty}\tilde{u}_m, \quad \text{which is } *\text{-weak in } L_\infty(0,T;H^{-1}) \text{ and weak in } L_2(0,T;L_2);\\ \tau &= \lim_{m\to\infty}\tilde{\tau}_m, \quad \text{which is } *\text{-weak in } L_\infty(0,T;L_2) \text{ and weak in } L_2(0,T;L_2).\end{aligned}$$

Moreover, by (3.29), without loss of generality one may assume that $\tilde{u}_m' \to u'$, $\tilde{\tau}_m' \to \tau'$ in $L_2(0,T;H^{-2})$. This gives that $u, \tau \in C([0,T]; H^{-2})$, and, by Lemma III.1.4 from [16], $u \in C_w([0,T]; H^{-1})$, $\tau \in C_w([0,T]; L_2)$.

Take the scalar product in $L_2(0,T)$ of inequality (3.30) with a smooth scalar function ψ with compact support in $(0,T)$ and with non-negative values

$$\begin{aligned}&\int_0^T\left\{\frac{1}{2}\left\|u_m(t)-v(t)\right\|_{-1}^2 + \frac{1}{2}\left\|\tau_m(t)-\zeta(t)\right\|^2 + \frac{d}{5}\int_0^t \left\|u_m(s)-v(s)\right\|^2 ds\right\}\psi(t)\,dt\\ &\quad\leqslant \int_0^T \exp\left(\int_0^t \Gamma(s)\,ds\right)\left\{\frac{1}{2}\left\|a_m - v(0)\right\|_{-1}^2 + \frac{1}{2}\left\|\tau_0-\zeta(0)\right\|^2\right.\\ &\qquad + \int_0^t \exp\left(\int_s^0 \Gamma(\xi)\,d\xi\right)\left[\left(\mathbf{E}_1(v,\zeta)(s), u_m(s)-v(s)\right)_{-1}\right.\\ &\qquad \left.\left. + \left(\mathbf{E}_2(v,\zeta)(s), \tau_m(s)-\zeta(s)\right) + \varepsilon_m^2\left\|\Delta^2\zeta(s)\right\|^2\right]ds\right\}\psi(t)\,dt.\end{aligned} \tag{3.31}$$

Passing to the inferior limit as $m \to \infty$ in (3.31), and using the fact that the norm of a weak limit of a sequence does not exceed the inferior limit of the norms, we arrive at

$$\begin{aligned}
&\int_0^T \left\{ \frac{1}{2}\left\|u(t)-v(t)\right\|_{-1}^2 + \frac{1}{2}\left\|\tau(t)-\zeta(t)\right\|^2 + \frac{d}{5}\int_0^t \left\|u(s)-v(s)\right\|^2 ds \right\} \psi(t)\,dt \\
&\quad \leqslant \int_0^T \exp\left(\int_0^t \Gamma(s)\,ds\right) \left\{ \frac{1}{2}\left\|a-v(0)\right\|_{-1}^2 + \frac{1}{2}\left\|\tau_0-\zeta(0)\right\|^2 \right. \\
&\qquad \left. + \int_0^t \exp\left(\int_s^0 \Gamma(\xi)\,d\xi\right) \left[\left(\mathbf{E}_1(v,\zeta)(s), u(s)-v(s)\right)_{-1} + \left(\mathbf{E}_2(v,\zeta)(s), \tau(s)-\zeta(s)\right)\right] ds \right\} \psi(t)\,dt. \qquad (3.32)
\end{aligned}$$

Since ψ and T were chosen arbitrarily, (3.32) yields (2.12), and the proof is complete. □

Proof of the claims (b) and (c). Let (u,τ) be a dissipative solution with the same initial data as the sufficiently regular solution (u_1,τ_1). Putting $v = u_1$, $\zeta = \tau_1$ in (2.12) for $t \in [0,T]$, and taking into account that $\mathbf{E}_1(u_1,\tau_1) \equiv \mathbf{E}_2(u_1,\tau_1) \equiv 0$ on $[0,T]$, we get that the right-hand side of (2.12) vanishes there, and we arrive at (b). Observe that (c) is a direct consequence of (a) and (b): any sufficiently regular solution should coincide with all dissipative solutions as long as it exists. □

References

[1] H. Amann, Global existence for a class of highly degenerate parabolic systems, Japan J. Indust. Appl. Math. 8 (1991) 143–151.
[2] H. Amann, Highly degenerate quasilinear parabolic systems, Ann. Sc. Norm. Super. Pisa Cl. Sci. 18 (1991) 135–166.
[3] D.S. Cohen, A.B. White Jr., T.P. Witelski, Shock formation in a multidimensional viscoelastic diffusive system, SIAM J. Appl. Math. 55 (2) (1995) 348–368.
[4] D.A. Edwards, D.S. Cohen, A mathematical model for a dissolving polymer, AIChE J. 18 (1995) 2345–2355.
[5] D.A. Edwards, A mathematical model for trapping skinning in polymers, Stud. Appl. Math. 99 (1997) 49–80.
[6] H. Gajewski, K. Groeger, K. Zacharias, Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen, Akademie-Verlag, Berlin, 1974.
[7] C. Guillopé, J.-C. Saut, Mathematical problems arising in differential models for viscoelastic fluids, in: Mathematical Topics in Fluid Mechanics, Lisbon, 1991, in: Pitman Res. Notes Math. Ser., vol. 274, Longman Sci. Tech., Harlow, 1992, pp. 64–92.
[8] Hu Bei, Zhang Jianhua, Global existence for a class of non-Fickian polymer-penetrant systems, J. Partial Differential Equations 9 (1996) 193–208.
[9] M. Krasnoselskii, Topological Methods in the Theory of Nonlinear Integral Equations, Gostehizdat, 1956 (in Russian); Engl. transl., Macmillan, 1964.
[10] P.-L. Lions, Mathematical Topics in Fluid Mechanics, vol. 1: Incompressible Models, Clarendon Press, 1996.
[11] N.G. Lloyd, Degree Theory, Cambridge Univ. Press, 1978.
[12] B. Riviere, S. Shaw, Discontinuous Galerkin finite element approximation of nonlinear non-Fickian diffusion in viscoelastic polymers, SIAM J. Numer. Anal. 44 (6) (2006) 2650–2670.
[13] J. Simon, Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura Appl. 146 (1987) 65–96.
[14] I.V. Skrypnik, Methods for Analysis of Nonlinear Elliptic Boundary Value Problems, Transl. Math. Monogr., vol. 139, Amer. Math. Soc., 1994.
[15] S. Swaminathan, D.A. Edwards, Travelling waves for anomalous diffusion in polymers, Appl. Math. Lett. 17 (2004) 7–12.
[16] R. Temam, Navier–Stokes Equations. Theory and Numerical Analysis, Stud. Math. Appl., vol. 2, North-Holland Publishing Co., Amsterdam, 1977.
[17] N. Thomas, A.H. Windle, Transport of methanol in poly-(methyl-methocrylate), Polymer 19 (1978) 255–265.
[18] V.G. Zvyagin, V.T. Dmitrienko, Approximating-Topological Approach to Investigation of Problems of Hydrodynamics. Navier–Stokes System, Editorial URSS, Moscow, 2004.