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Second order parabolic Hamilton–Jacobi–Bellman equations in Hilbert spaces and stochastic control: L^2_μ approach

B. Goldys[a], F. Gozzi[b,*]

[a] *School of Mathematics, The University of New South Wales, Sydney 2052, Australia*
[b] *Dipartimento di Scienze Economiche ed Aziendali, Università LUISS, Viale Pola 12, 00198 Roma, Italy*

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Abstract

We study a Hamilton–Jacobi–Bellman equation related to the optimal control of a stochastic semilinear equation on a Hilbert space X. We show the existence and uniqueness of solutions to the HJB equation and prove the existence and uniqueness of feedback controls for the associated control problem via dynamic programming. The main novelty is that we look for solutions in the space $L^2(X, \mu)$, where μ is an invariant measure for an associated uncontrolled process. This allows us to treat controlled systems with degenerate diffusion term that are not covered by the existing literature. In particular, we prove the existence and uniqueness of solutions and obtain the optimal feedbacks for controlled stochastic delay equations and for the first order stochastic PDE's arising in economic and financial models.

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1. Introduction

The aim of this paper is to study the following Hamilton–Jacobi–Bellman (HJB from now on) equation

* Corresponding author. Tel.: +39 0685225723; fax: +39 0686506506.
E-mail addresses: B.Goldys@unsw.edu.au (B. Goldys), gozzi@mail.dm.unipi.it (F. Gozzi).

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$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) + \dfrac{1}{2}\mathrm{Tr}\,(Qv_{xx}(t,x)) + \langle Ax + F(x), v_x(t,x)\rangle - H_0\,(v_x(t,x)) + f(x) = 0, \\ v(T,x) = \varphi(x), \quad x \in X, T \geq 0 \end{cases} \tag{1}$$

on a real separable Hilbert space X with the norm $|\cdot|$. We assume that A is a generator of the strongly continuous semigroup $\left(e^{tA}\right)$ on X, $Q : X \to X$ is a nonnegative and selfadjoint operator (not necessarily nuclear), $H_0 : X \to \mathbb{R}$ is a suitable Lipschitz continuous function, $F : X \to X$ is continuous with bounded Gateaux derivative.

We will show that, under some additional assumptions, this equation has a unique solution, its gradient v_x may be well defined and therefore the optimal feedback control can be found for an associated stochastic control problem.

It is well known that the Hamilton–Jacobi–Bellman equation has no classical solutions in general, even if $\dim(X) < \infty$. This difficulty has been circumvented in the finite dimensional case by introducing the concept of viscosity solutions, see [16,26] and the references therein. Due to some basic measure theoretic problems (see [16, Appendix]) the viscosity solution approach can not be easily adapted to an infinite dimensional case unless Q is of trace class; the first work on this case is [43], see also [36,39,40,42,50] for more recent results. A first attempt to deal with the case when $\mathrm{tr}\,(Q) = \infty$ has been made in [35]. The viscosity method assures the uniform continuity of the solution of the HJB equation and its identification as the value function of a certain stochastic control problem. It does not provide however, at the present stage, the existence of the gradient v_x, hence the existence of optimal control in a feedback form needs another approach.

Another approach to the HJB equation (1) has been initiated in [7,8] and studied later in [32,33] by the second author of this paper (see also [9,10,18–20,25,30,34] for other results in this direction). This approach (that we call the "strong solution approach" in the following) uses perturbations of solutions of the associated linear equation and is based on the assumptions that

- the data φ and f are continuous and bounded,
- F is a bounded function,
- H_0 is a Lipschitz function (or simply locally Lipschitz but with globally Lipschitz Fréchet derivative),
- the solution to the linearized version of Eq. (1) obtained for $F = H_0 = f = 0$ satisfies the condition

$$\int_0^T |v_x(t)|\,\mathrm{d}t < \infty, \tag{2}$$

for any bounded Borel φ. This means that the Ornstein–Uhlenbeck semigroup associated to (A, Q) is strongly Feller and the minimum energy operator $\Gamma(t) = Q_t^{-\frac{1}{2}} e^{tA}$ (where Q_t is given as in (19) has integrable norm in a neighborhood of $t = 0$ (in the finite dimensional setting this would imply the uniform ellipticity of the differential operator

$$\mathcal{L}v = \frac{1}{2}\mathrm{Tr}\,[Qv_{xx}] + \langle Ax, v_x\rangle, \tag{3}$$

see [22, Appendix B] for explanations).

These assumptions for the couple (A, Q) are quite restrictive as showed in [32,33] (we may roughly say that Q cannot be very far from a boundedly invertible operator). This approach allows us to find continuously differentiable solutions, to identify the solution with the value function of a certain stochastic control problem and to provide optimal controls in the feedback form (9). However, the cases when Q is degenerate or when the Ornstein–Uhlenbeck semigroup associated to (A, Q) is not strong Feller (or it does not satisfy (2)), are not covered by this setting.

Let us note that in the two approaches discussed above the HJB equation is studied in the space of continuous functions, thus imposing quite strong assumptions on the data of the HJB equation.

The main goal of this paper is to develop an L^2 theory for second order HJB equations in Hilbert spaces by perturbation of solutions corresponding to the equation:

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) = \dfrac{1}{2}\,\mathrm{Tr}\,\left(Q v_{xx}(t,x)\right) + \langle Ax + F(x), v_x(t,x)\rangle\,, \\ v(0,x) = \phi(x). \end{cases} \tag{4}$$

We may say that we develop a "strong solution approach" but in a different underlying space. The crucial assumption is that solutions to (4) generate a strongly continuous semigroup $P_t\phi(x) = v(t,x)$ in the space $L^2(X,\mu)$, where μ is an invariant measure for (P_t) that is

$$\int_X P_t\phi(x)\mu(\mathrm{d}x) = \int_X \phi(x)\mu(\mathrm{d}x).$$

This approach allows us to treat a large variety of stochastic optimal control problems with irregular data and strongly degenerated operator Q. The price paid is lower regularity of solutions, but we are still able to prove the verification theorem and to obtain the existence of the optimal control in feedback form. The results obtained allow us to solve the optimal control problem in many important cases not covered by the existing theory, like stochastic delay equations, first order stochastic PDE's arising in financial and economic models and stochastic PDE's in unbounded domains.

We would like also to emphasize that our approach can be adapted to treat more general problems, including the case of nonlinear state dependent diffusion coefficients (but independent of the control) and nonlinear state dependent control coefficients, or some boundary control problems, provided the existence of an invariant measure for an uncontrolled system is assumed.

The only attempts to build a theory of HJB equations in spaces $L^2(X,\mu)$, we are aware of, have been made in [15] and [1,2] under assumptions much stronger than ours. In particular, they assume closability of the operator D_Q (see Section 2.3) and therefore some interesting problems, like the control of stochastic delay equations (see Section 6.1) are not covered by those papers.

We recall finally the works [4,6,17,37] where some results on strong solutions are proven in the case of nuclear Q, [30,34] where the strong solution approach is extended to the elliptic case (infinite horizon case). In [15] a first attempt to exploit the existence of the invariant measure was made but without any connection with stochastic control. Let us note that formulations and results similar to ours appear also in some works motivated by stochastic quantization, see e.g. [14].

Recently in a series of papers (see e.g. [27,28]) a deep application of Malliavin Calculus and of the theory of forward–backward systems has been developed to obtain very general results on the existence of smooth solutions to the HJB equation. Those papers cover our main examples, (see Section 6.2) but under stronger conditions on the regularity of data. Indeed, they always need to work with globally Lipschitz continuous data f and φ while we need square integrability with respect to the invariant measure μ only. If μ is Gaussian then f and φ may be of exponential growth.

In the remaining part of the introduction we will present the main motivation and features of our approach.

1.1. The motivation: Stochastic control problems

It is well known that the solution to (1) may be interpreted as the value function of the following stochastic control problem with finite horizon $T \geq 0$ and initial time $t \in [0, T]$. Consider a controlled stochastic system

$$\begin{cases} \mathrm{d}y(s) = \left(Ay(s) + F(y(s)) - Q^{1/2}z(s)\right)\mathrm{d}s + Q^{1/2}\mathrm{d}W(s), & t \leq s \leq T, \\ y(t) = x \in X, \end{cases} \tag{5}$$

on X, driven by the white noise W, where $z(\cdot)$ stands for the control process and $y(\cdot) = y(\cdot; t, x, z)$ is the solution of (5). If

$$J(t, x; z) = \mathbb{E}\left\{\int_t^T [f(y(s; t, x, z)) + h(z)]\mathrm{d}s + \varphi(y(T; t, x, z))\right\} \tag{6}$$

is a cost functional to minimize then the value function of the control problem above is given by

$$V(t, x) = \inf_{z \in M_W^2(t,T;X)} J(t, x; z), \tag{7}$$

where $M_W^2(t, T; X)$ stands for the set of all progressively measurable processes $z : [t, T] \mapsto X$ such that

$$\mathbb{E}\int_t^T |z(s)|^2 \,\mathrm{d}s < +\infty.$$

The classical argument of the Dynamic Programming Principle (see e.g. [26, p.137] for the finite dimensional case) shows that, if the value function V is sufficiently regular, then it is a classical solution of (1) with the Hamiltonian H_0 given by

$$H_0(p) = \sup_{z \in X}\left\{\left\langle Q^{1/2}z, p\right\rangle - h(z)\right\} = h^*\left(Q^{1/2}p\right) \tag{8}$$

where h^* is the Légendre transform of h. Vice versa, if v is the unique classical solution of Eq. (1) one can prove, by the so-called dynamic programming method (see Section 5) that $v = V$ and that there exists a unique optimal control z^* given (when H_0 is differentiable) by the formula

$$z^*(s) = \frac{\mathrm{d}H_0}{\mathrm{d}p}(v_x(s, y^*(s))) \tag{9}$$

where y^* is the optimal state given by the solution of the closed loop equation

$$\begin{cases} \mathrm{d}y(s) = \left[Ay(s) + F(y(s)) - Q^{\frac{1}{2}}z^*(s)\right]\mathrm{d}s + Q^{1/2}\mathrm{d}W(s), & t \leq s \leq T \\ y(t) = x, \quad x \in X. \end{cases} \tag{10}$$

This fact turns out to be very useful for applications and is one of the main goals of this work. In fact this result is obtained in the so-called relaxed control setting in Section 5.

1.2. The L^2 approach

Our main assumption is that the uncontrolled system

$$\begin{cases} \mathrm{d}y(s) = [Ay(s) + F(y(s))]\,\mathrm{d}s + Q^{1/2}\mathrm{d}W(s), & t \leq s \leq T, \\ y(t) = x \in X, \quad t \leq T \end{cases} \tag{11}$$

possesses an invariant measure μ which will be used as the reference measure. Under this assumption we will study Eq. (1) in the space $L^2(X, \mu)$ using the perturbation method. Then quite general cases of data A, Q, F, φ, f, can be treated. More precisely:

- $\varphi, f \in L^2(X, \mu)$, so they are not necessarily continuous and bounded;
- F is of linear growth so not necessarily bounded;
- we do not assume any smoothing properties of the linearized version of (1) and therefore we do not impose any restrictions on Q; it is possible to take $Q = I$ but it may be also a one dimensional projection. This means that the Ornstein–Uhlenbeck semigroup associated to (A, Q) need not to be strongly Feller (no "uniform ellipticity" of the operator $\mathcal{L}$ in (3)).

This generality comes at a price. We can deal only with a class of Hamiltonians of the form $H_0(p) = H\left(Q^{1/2}p\right)$, which correspond to the control process in (5) taken in the form $Q^{1/2}z$. This assumption may seem restrictive but in fact it is quite natural in many control problems, when the operator Q is degenerate. This condition says that the system should be controlled by feedbacks taking values in the same space in which lives the noise disturbing the system (see Section 6 for more detailed discussion and examples, see also the introduction of [18]). Let us note that, if $Q^{1/2} = 0$, then both the control and the noise disappear. So, a possible, quite natural, interpretation of Eq. (11) is that the uncontrolled system is in fact deterministic and the noise is brought into the system by the control only.

Our main idea of solving Eq. (1) derives from a classical property of diffusion processes that allows us to apply the perturbation method without using the strong Feller property of the linear part and which we describe briefly below. Consider a Kolmogorov equation

$$\begin{cases} \dfrac{\partial w}{\partial t} = \dfrac{1}{2}\mathrm{Tr}[Qw_{xx}] + \langle Ax + F(x), w_x \rangle, & t \in [0, T[, x \in D(A) \\ w(0, x) = \varphi(x), & x \in X. \end{cases} \tag{12}$$

The solution to this equation may be identified as the transition semigroup (P_t) of the process $y(\cdot; x)$ defined by Eq. (11), i.e.

$$w(t, x) = P_t\varphi(x) = \mathbb{E}\varphi(y(t, x)) \tag{13}$$

for a bounded continuous φ. If there exists an invariant measure μ for y then (P_t) extends to a strongly continuous semigroup of contractions on $L^2(X, \mu)$ with the generator $\mathcal{N}$, which on nice functions takes the form of the differential operator

$$\mathcal{N}\phi(x) = \frac{1}{2}\mathrm{Tr}[Q\phi_{xx}] + \langle Ax + F(x), \phi_x \rangle . \tag{14}$$

Moreover the following fundamental identity holds for every $T > 0$:

$$\|P_T\phi\|_\mu^2 + \int_0^T \left\| Q^{1/2} (P_t\phi)_x \right\|_\mu^2 \mathrm{d}t = \|\phi\|_\mu^2, \tag{15}$$

where $\|\cdot\|_\mu$ stand for the norm in the space $L^2(X, \mu)$. Identity (15) can be seen as an L^2 version of the smoothing property of the semigroup P_t which is used in the strong solution approach to find C^k solutions. Identity (15) is well known and easy to obtain if we know an algebra of functions which is a core for $\mathcal{N}$ (see Section 2.2 for precise references).

Let us now take Eq. (1) with time reversal $t \mapsto T - t$. We obtain the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\mathrm{Tr}[Qu_{xx}] + \langle Ax + F(x), u_x\rangle - H(Q^{1/2}u_x) + f, & t \in [0, T[, x \in D(A) \\ u(0, x) = \varphi(x), \quad x \in X, \end{cases} \tag{16}$$

which can be seen as a perturbation of (12). By applying the formula for variation of constants, the above Eq. (16) can be written in integral form as

$$u(t) = P_t\phi + \int_0^t P_{t-s}\left(f - H\left(Q^{1/2}u_x(s)\right)\right) \mathrm{d}s.$$

Let $W_Q^{1,2}(X, \mu)$ denote the Sobolev space endowed with the norm

$$\|\phi\|_1^2 = \int_X |\phi|^2 \mathrm{d}\mu + \int_X \left|Q^{1/2}\phi_x\right|^2 \mathrm{d}\mu. \tag{17}$$

Now, and this is a key point, identity (15) allows us to use the Banach Fixed Point Theorem and to prove the existence of a unique solution $u : [0, T] \mapsto W_Q^{1,2}(X, \mu)$ for the integral equation (17). Then we identify the solution with the value function V of the stochastic control problem and, by dynamic programming, we construct the optimal feedback control $DH\left(Q^{1/2}V_x\right)$ but only for almost every $(t, x) \in [0, T] \times X$ with respect to the measure *Leb* $\otimes \mu$. Imposing more regularity on the data we can obtain more regular solutions. Equivalently, the original control problem may be approximated by more regular problems converging in an appropriate sense to the initial one (see Section 5).

We would like to emphasize the fact that the operator $D_Q = Q^{1/2}D$ need not to be closable. In fact, D_Q is not closable in our main examples (see Sections 6.1 and 6.2 and also [31]) and gives rise to the unpleasant fact that in general $W_Q^{1,2}(X, \mu) \not\subseteq L^2(X, \mu)$. We deal with this problem in Section 2.3.

The strategy sketched above gives a solution to a large class of Eq. (1) and a large class of the optimal stochastic control problems with rather mild conditions on regularity of the data; the functions $\varphi, f : X \mapsto \mathbb{R}$ are merely square integrable with respect to the measure μ (we will write $\varphi, g \in L^2(X, \mu)$) while $F : X \to X$ and $H : X \to \mathbb{R}$ are Lipschitz continuous. Moreover, if $F(x) \in Q^{1/2}(X)$ then the noise in (5) may be arbitrarily degenerated.

To sum up, we propose a general procedure (obviously, it does not cover all interesting control problems), which provides a well defined solution to (1) identified with the value function and gives the optimal control in a feedback form. In some sense it is an L^2-counterpart of the concept of strong solution and of viscosity solution (which are useful mainly in the case of uniformly continuous data, but see [39,50] for more refined concepts). Moreover, let us mention that the Lipschitz property of F is not essential for our method. The identity (15) may be proved for a much larger class of equations than (11). We made it to keep this paper to a reasonable size and to present the main idea on a relatively simple system. Finally, the case of a locally Lipschitz Hamiltonian is not treated here but will be a subject of forthcoming research (we recall that in the special case $H(p) = |Q^{1/2}p|^2/2$ problem (1) can be solved by applying the Hopf transform, see on this [18,33,25]).

The plan of the paper is the following. In Section 2 we give some notation (Section 2.1), state the main assumptions and results on the uncontrolled problem (11) (Section 2.2) and give some preliminary results (needed later but that may be interesting in themselves) on the gradient operator D_Q (Section 2.3) and on the auxiliary operator $\mathcal{K}$ (Section 2.4). In Section 3 we prove the main results about problem (1) while Section 4 is devoted to the approximation results for the solution of (1) which are needed for the application to the control problem. In Section 5 we show how to apply results of Sections 3 and 4 to the control problem (6) and in Section 6 we apply the above techniques to selected examples.

2. Preliminaries

2.1. Some notation

The following notation will be used troughout the paper. X is a separable Hilbert space with norm $|\cdot|$ and inner product $\langle\cdot,\cdot\rangle$.

We denote by $C_b(X)$ (respectively $UC_b(X)$) the space of all continuous (respectively uniformly continuous) and bounded functions $\phi : X \mapsto \mathbb{R}$. The symbols $C_b(X; X)$ (respectively $UC_b(X; X)$) will mean that such functions take values in X. Similar meanings hold for the spaces $C_b([0, T] \times X)$, $UC_b([0, T] \times X)$ and so on. Moreover $C_b^k(X)$ denotes the space of of functions $\phi : X \to \mathbb{R}$, which are Fréchet differentiable up to order k, $k \geq 1$, such that $\phi, D\phi, \ldots, D^k\phi$ are continuous and bounded, where $D^k\phi$ denotes the k-th Fréchet derivative of ϕ. In the same way we define the space $C_b^k(X, X)$ of X-valued functions with continuous and bounded Fréchet derivatives up to the k-th order.

In some case we will drop the subscript b, writing simply $C(X)$, $UC(X)$ and so on. This will mean that the elements of such spaces may also be unbounded. $C_0^k(\mathbb{R}^n)$ denotes the space of all k-times differentiable, real-valued functions on $\mathbb{R}^n$ with compact support, $k \leq \infty$, $n \geq 1$.

Given a measure μ on X, $L^2(X, \mu)$ stands for the space of all functions $X \mapsto \mathbb{R}$ which are square-integrable and $L^2(X, \mu; X)$ will denote the space of X-valued square-integrable functions. In both cases the norm of the function ϕ will be denoted in the same way:

$$\|\phi\| = \left(\int_X |\phi(x)|^2 \mu(\mathrm{d}x)\right)^{1/2}.$$

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space with the filtration satisfying the usual conditions. We denote by $M_W^2(t, T; X)$ the space of all progressively measurable processes $z : [t, T] \mapsto X$ such that

$$\mathbb{E}\int_t^T |z(s)|^2\,\mathrm{d}s < \infty.$$

The norms of operators acting in various spaces will be denoted by $\|\cdot\|$ with subscripts indicating the spaces explicitly in cases the notation might be ambiguous.

2.2. The uncontrolled problem

We will study first some properties of Eq. (11). The following are standing assumptions for the rest of the paper. The results will be enunciated without recalling these conditions.

Hypothesis 2.1. (A) The operator A generates a strongly continuous semigroup $\left(e^{tA}\right)$ on X and there exist $M \geq 1$, and $\omega \in \mathbb{R}$ such that

$$\left\|e^{tA}\right\| \leq Me^{\omega t}, \quad \forall t \geq 0.$$

(B) The process (W_t) is a standard cylindrical Wiener process on X defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, where $(\mathcal{F}_t)$ is a filtration satisfying the usual conditions. Moreover, the operator $Q = Q^* \geq 0$ is bounded on X.

(C) For every $t > 0$

$$\operatorname{tr}(Q_t) < \infty, \tag{18}$$

where

$$Q_t = \int_0^t \mathrm{e}^{sA} Q \mathrm{e}^{sA^*} \mathrm{d}s. \tag{19}$$

(D) The function $F : X \to X$ is Gateaux differentiable with

$$\sup_{x \in X} \|DF(x)\| < \infty.$$

(E) There exists a nondegenerate invariant measure μ for Eq. (11). Moreover,

$$\int_X |x|^2 \mu(\mathrm{d}x) < \infty.$$

If Hypothesis 2.1 holds then Eq. (11) has a unique solution $(y(\cdot; t, x))$ (see [22, Chapter 7]) which satisfies the integral equation

$$y(s; t, x) = \mathrm{e}^{(s-t)A} x + \int_t^s \mathrm{e}^{(s-r)A} F(y(r; t, x)) \mathrm{d}r + \int_t^s \mathrm{e}^{(s-r)A} Q^{1/2} \mathrm{d}W(r).$$

Moreover, part (E) of Hypothesis 2.1 allows us to extend the transition semigroup (P_t) defined in (13) to a strongly continuous semigroup of contractions on the space $L^2(X, \mu)$ with the generator $\mathcal{N}$ defined in (14) (see for example [23]).

Let P_n be an orthogonal projection in X such that $\dim \operatorname{im}(P_n) = n$ and $\operatorname{im}(P_n) \subset \operatorname{dom}(A^*)$. We define the space

$$\mathcal{F}C_0^2(A^*) = \left\{ \phi \in C_0^2(X) : \phi = f \circ P_n, n \geq 0, f \in C_0^2(\mathbb{R}^n) \right\}.$$

In the notation $f \circ P_n$ above we identify $P_n x$ with the the vector $(\langle x, h_1 \rangle, \ldots, \langle x, h_n \rangle) \in \mathbb{R}^n$, where $h_1, \ldots, h_n$ generate the space $\operatorname{im}(P_n)$.

Lemma 2.2. *For each $\phi \in \mathcal{F}C_0^2(A^*)$ we have $\phi \in \operatorname{dom}(\mathcal{N})$ and*

$$\mathcal{N}\phi(x) = \frac{1}{2} \operatorname{tr}\left(Q D^2 \phi(x) \right) + \left\langle x, A^* D\phi(x) \right\rangle + \langle F(x), D\phi(x) \rangle. \tag{20}$$

Proof. Applying the Ito formula to the process $\phi(y(t, x))$ we find easily that for any $x \in X$

$$\lim_{t \to 0} \frac{P_t \phi(x) - \phi(x)}{t} = \frac{1}{2} \operatorname{tr}\left(Q D^2 \phi(x) \right) + \left\langle x, A^* D\phi(x) \right\rangle + \langle F(x), D\phi(x) \rangle. \tag{21}$$

It follows from the definition of $\mathcal{F}C_0^2(A^*)$ that the function

$$x \to \frac{1}{2} \operatorname{tr}\left(Q D^2 \phi(x) \right) + \left\langle x, A^* D\phi(x) \right\rangle$$

is in $L^2(X, \mu)$. Since $D\phi$ is bounded by definition we obtain from Hypothesis 2.1

$$\int_X \langle F(x), D\phi(x) \rangle^2 \mu(\mathrm{d}x) \leq \int_X c \left(1 + |x|^2 \right) \mu(\mathrm{d}x) < \infty,$$

for a certain $c > 0$. Hence, using Dominated Convergence, we find that the convergence in (21) takes place in $L^2(X, \mu)$ and that (20) holds. Therefore, $\phi \in \mathrm{dom}(\mathcal{N})$ and (20) holds. □

Let $\zeta_t^{x,h}$, $t \geq 0$, $h, x \in X$, denote the solution to the following differential equation (see [22, Chapter 7] for details):

$$\frac{\mathrm{d}\zeta_t^{x,h}}{\mathrm{d}t} = (A + DF(y(t,x)))\zeta_t^{x,h}, \qquad \zeta_0^{x,h} = h. \tag{22}$$

By Hypothesis 2.1 $|\zeta_t^{x,h}| \leq a\mathrm{e}^{\alpha t}|h|$ for some $\alpha, a > 0$ and therefore the solution to (22) defines, for every $t \geq 0$, $x \in X$ and any path $\{y(s,x) : s \leq t\}$, a bounded operator $\zeta_t^x : X \to X$. Moreover, for $\phi \in C_b^1(X)$

$$\langle DP_t\phi(x), h\rangle = \mathbb{E}\left(\left\langle\left(\zeta_t^x\right)^* D\phi(y(t,x)), h\right\rangle\right), \quad h \in X. \tag{23}$$

In particular, if $\phi \in C_b^1(X)$ then $Q^{1/2}DP_t\phi(x)$ is well defined for every $x \in X$.

2.3. The gradient operator D_Q

We define the operator

$$D_Q\phi = Q^{1/2}D\phi, \quad \phi \in \mathcal{F}C_0^2\left(A^*\right),$$

where $D\phi$ denotes the Fréchet derivative of ϕ. For $\phi \in \mathcal{F}C_0^2(A^*)$ we define the norm

$$\|\phi\|_1^2 = \|\phi\|^2 + \left\|D_Q\phi\right\|^2$$

and the completion of $\mathcal{F}C_0^2(A^*)$ with respect to the norm $\|\cdot\|_1$ will be denoted by $W_Q^{1,2}(X,\mu)$. Since we do not assume that D_Q is closable we will recall below for the reader's convenience a standard construction of $W_Q^{1,2}(X,\mu)$ which will be important in the following study of the HJ equation.

The space $W_Q^{1,2}(X,\mu)$ may be identified as a subset of $L^2(X,\mu)\times L^2(X,\mu;X)$ which consists of all pairs

$$(\psi, \Psi) \in L^2(X,\mu) \times L^2(X,\mu;X)$$

such that there exists a sequence $(\phi_n) \subset \mathcal{F}C_0^2(A^*)$ with the property that,

$$\begin{aligned} &\phi_n \to \psi, \quad \text{in } L^2(X,\mu),\\ &D_Q\phi_n \to \Psi, \quad \text{in } L^2(X,\mu;X). \end{aligned}$$

Closability implies that, for any two pairs $(\psi_1, \Psi_1), (\psi_2, \Psi_2) \in W_Q^{1,2}(X,\mu)$ such that $\psi_1 = \psi_2$ in $L^2(X;\mu)$ we have also $\Psi_1 = \Psi_2$, so that $W_Q^{1,2}(X,\mu)$ is naturally embedded in $L^2(X,\mu)$. If D_Q is not closable then we can find a sequence $(\phi_n) \subset \mathcal{F}C_0^2(A^*)$ such that

$$\phi_n \to 0 \quad \text{in } L^2(X,\mu) \quad \text{and} \quad D_Q\phi_n \to \Psi \neq 0, \quad \text{in } L^2(X,\mu;X).$$

Therefore, elements of $W_Q^{1,2}(X,\mu)$ cannot be identified, in general, with functions from $L^2(X,\mu)$ (e.g. the above element $(0, \Psi)$).

We will show that even in the case when D_Q is not closable, it still enjoys some useful properties when applied to the semigroup (P_t). Namely, we will show that D_Q is closable in a weaker sense that we define below. We will show that this weaker definition is satisfied in a wide class of problems, including those satisfying our Hypothesis 2.1.

Definition 2.3. Let $\mathcal{D} \subset \mathrm{dom}(\mathcal{N})$ be a core of $\mathcal{N}$ and assume that $\mathcal{D} \subset C_b^1(X)$. We say that the operator $\left(D_Q, \mathcal{D}\right)$ is closable on $\mathrm{dom}(\mathcal{N})$ if the following condition is satisfied.

Let $(\phi_n) \subset \mathcal{D}$ be such that

$$\phi_n \to 0, \quad \mathcal{N}\phi_n \to 0 \quad \text{in } L^2(X, \mu),$$

and

$$Q^{1/2} D\phi_n \to \psi, \quad \text{in } L^2(X, \mu; X).$$

Then $\psi = 0$.

Let us define an operator $\mathcal{K}$ as follows: given $\phi \in C_b^1(X)\mathcal{K}\phi$ is a function from $[0, T]$ to $C_b^1(X; X)$ given by

$$\mathcal{K}\phi(t) = D_Q P_t \phi.$$

The next proposition is closely related to the similar results in [24], but we present here a completely different proof.

Proposition 2.4. *For every $\phi \in C_b^1(X)$*

$$\int_0^T \left\| D_Q P_t \phi \right\|^2 \mathrm{d}t = \|\phi\|^2 - \|P_T \phi\|^2. \tag{24}$$

Moreover, the operator $\mathcal{K}$ has a unique extension to $\mathrm{dom}(\mathcal{N})$ *and for each $\phi \in$* $\mathrm{dom}(\mathcal{N})$

$$\int_0^T \|\mathcal{K}\phi(t)\|^2 \mathrm{d}t = \|\phi\|^2 - \|P_T \phi\|^2.$$

Proof. Let us recall first the following result (see p. 181 of [52]).

Lemma 2.5. *Assume that $F \in UC_b^2(X)$. Then for every $\phi \in UC_b^2(X)$*

$$\phi\left(y(t, x)\right) = P_t \phi(x) + \int_0^t \left\langle Q^{1/2} D P_{t-s} \phi(y(s, x)), \mathrm{d}W(s) \right\rangle \quad \mathbb{P}\text{- a.e.} \tag{25}$$

Step 1. We will show that (25) holds for any F which is Gateaux differentiable with $l = \sup_x |DF(x)| < \infty$ and any $\phi \in C_b^1(X)$. Indeed, fix $\phi \in UC_b^2(X)$ and let (F_n) be a sequence of mappings $F_n : X \to X$ such that

$$\sup_n \|DF_n\|_\infty \le l,$$

and for all $x \in X$,

$$\lim_{n\to\infty} F_n(x) = F(x), \quad \text{and} \quad \lim_{n\to\infty} DF_n(x) = DF(x), \quad x \in X.$$

Existence of such a sequence is proved for example in [47] and Theorem A.1 in [47] implies that

$$\lim_{n\to\infty}\sup_{t\le T}\mathbb{E}\left|y_n(t,x)-y(t,x)\right|^2=0, \tag{26}$$

where $y_n(\cdot,x)$ is a unique solution of the equation

$$\begin{cases}\mathrm{d}y_n=(Ay_n+F_n(y_n))\,\mathrm{d}t+\sqrt{Q}\mathrm{d}W,\\ y(0,x)=x.\end{cases} \tag{27}$$

Let $P_t^n\phi(x)=\mathbb{E}\phi(y_n(t,x))$ be the corresponding transition semigroup. Then for every $x\in X$

$$\lim_{n\to\infty}P_t^n\phi(x)=P_t\phi(x)\quad\text{and}\quad\lim_{n\to\infty}DP_t^n\phi(x)=DP_t\phi(x) \tag{28}$$

by (22), (23) and (26). We find easily that (28) yields (25) for any F which has uniformly bounded Gateaux derivative and any $\phi\in UC_b^2(X)$.

Assume now that F satisfies Hypothesis 2.1 and $\phi\in C_b^1(X)$. Then, using the same construction as in [47] we can find a sequence $(\phi_n)\subset UC_b^2(X)$, such that for all $x\in X$

$$\lim_{n\to\infty}\phi_n(x)=\phi(x)\quad\text{and}\quad\lim_{n\to\infty}D\phi_n(x)=D\phi(x),$$

and moreover,

$$\|\phi_n\|_\infty\le\|\phi\|_\infty\quad\text{and}\quad\|D\phi_n\|_\infty\le\|D\phi\|_\infty.$$

Then by (23)

$$\lim_{n\to\infty}P_t\phi_n(x)=P_t\phi(x)\quad\text{and}\quad\lim_{n\to\infty}DP_t\phi_n(x)=DP_t\phi(x).$$

This yields (25) for all $\phi\in C_b^1(X)$.

Step 2. Let $\phi\in C_b^1(X)$. Then (25) yields

$$\mathbb{E}\phi^2(y(t,x))=(P_t\phi(x))^2+\int_0^t\mathbb{E}\left|Q^{1/2}DP_{t-s}\phi(y(s,x))\right|^2\mathrm{d}s.$$

Integrating this identity with respect to μ and using the fact that μ is an invariant measure we obtain (24) for all $\phi\in C_b^1(X)$. Note that by (23) we have $P_t:C_b^1(X)\mapsto C_b^1(X)$ which gives that $(I-\mathcal{N})^{-1}C_b^1(X)\subset C_b^1(X)$. Moreover $(I-\mathcal{N})^{-1}C_b^1(X)$ is a core for $\mathcal{N}$ by a standard argument. Hence, for any $\phi\in\mathrm{dom}(\mathcal{N})$ we can find a sequence $(\phi_n)\subset(I-\mathcal{N})^{-1}C_b^1(X)$ such that $\phi_n\to\phi$ in $L^2(X,\mu)$ and (24) implies that $\left(D_QP_t\phi_n\right)$ is a Cauchy sequence in $L^2\left(0,T;L^2(X,\mu;X)\right)$. Therefore, the operator $\mathcal{K}$ can be extended to a linear operator

$$\mathcal{K}:\mathrm{dom}(\mathcal{N})\to L^2\left(0,T;L^2(X,\mu;X)\right),$$

and

$$\int_0^T\|\mathcal{K}\phi(t)\|^2\,\mathrm{d}t=\|\phi\|^2-\|P_T\phi\|^2,\quad\phi\in\mathrm{dom}(\mathcal{N}). \tag{29}$$

In fact the extension could be done to the whole of $L^2(X,\mu)$ but we will do that later. □

Remark 2.6. The crucial fact for the proof of Proposition 2.4 is the Gateaux differentiability of F which is assured by Hypothesis 2.1. This condition can be relaxed in some situations. For example assume that F is Lipschitz, $F(x) \in Q^{1/2}(X)$ and

$$\sup_{x\in X} \frac{|Q^{-1/2}F(x)|}{1+|x|} < \infty.$$

By the result in [48] there exists a set $\mathcal{Z} \subset X$ such that $\nu(\mathcal{Z}) = 0$ for arbitrary Gaussian measure ν on X and F is Gateaux differentiable at each point $x \in X - \mathcal{Z}$. Since the above conditions imply that the law of $y(t, x)$ is absolutely continuous with respect to a Gaussian measure (see [22]) it follows (22) and (23) still hold and then Proposition 2.4 can be proved in the same way. □

Remark 2.7. If $F = 0$ then the operator $\mathcal{N}$ reduces to the Ornstein–Uhlenbeck operator $\mathcal{L}$ and the semigroup (P_t) is called the Ornstein–Uhlenbeck semigroup. In this case the invariant measure for (P_t) is the Gaussian measure $N(0, Q_\infty)$ (recall that Q_t and Q_∞ are defined in (19)) and the concept of closability as well as the smoothing properties of the semigroup (P_t) have a useful control theoretic interpretation in terms of the linearly controlled system

$$y' = Ay + Q^{1/2}z, \qquad y(0) = 0, \tag{30}$$

(see e.g. [22, Appendix B]). In fact (see [31]) the closability is equivalent to the fact that the set

$$\left\{x \in X : Q^{1/2}x \in Q_\infty^{1/2}(X)\right\} \quad \text{is dense in } X.$$

Note that $h \in Q_\infty^{1/2}(X)$ if and only if the system (30) can be driven to h in an infinite time using the square integrable control z.

Moreover $DP_t\phi$ is well defined for $t > 0$ if and only if

$$e^{tA}(X) \subseteq Q_t^{1/2}(X)$$

i.e. every point of X is null controllable in time t (this is also equivalent to the strong Feller property of the semigroup P_t). In this case the singularity of $\|DP_t\phi\|$ at 0^+ goes as the norm of the operator

$$Q_t^{-1/2}e^{tA} = \Gamma(t). \tag{31}$$

Finally $D_Q P_t\phi$ is well defined for $t > 0$ if and only if

$$e^{tA}Q^{1/2}(X) \subseteq Q_t^{1/2}(X) \tag{32}$$

i.e. every point of $Q^{1/2}(X)$ is null controllable in time t. In this case the singularity of $\left\|D_Q P_t\phi\right\|$ at 0^+ goes as the norm of the operator

$$Q_t^{-1/2}e^{tA}Q^{1/2}$$

(which is equal to $\Gamma(t)\, Q^{1/2}$ when the strong Feller property holds). □

Remark 2.8. If D_Q is closable in $L^2(X, \mu)$ then $\mathcal{K}\phi(t) = \overline{D_Q}P_t\phi(t)$ for all $t > 0$ and $\phi \in L^2(X, \mu)$. In this case (24) is easier to obtain and all the machinery to study the HJ equation and the associated control problem is much simpler. This is true in particular when Q is boundedly invertible. Closability follows also, rather straightforwardly, if $\mathcal{N}$ is associated to a nonsymmetric Dirichlet form on $L^2(X, \mu)$, see [44]. In general the question of closability is rather difficult. Let us note that there are interesting control problems for which the operator D_Q is not closable (see Section 6 or also [31]). This fact has been our main motivation for introducing the weaker notion of closability in Definition 2.3. □

2.4. The operator $\mathcal{K}$

We will study here some properties of the operator $\mathcal{K}$ which will be a key tool in proving our main results.

Proposition 2.9. *The operator $\mathcal{K}$ extends to a bounded operator*

$$\mathcal{K} : L^2(X,\mu) \to L^2\left(0,T;L^2(X,\mu)\right)$$

with

$$\|\mathcal{K}\phi\|^2_{L^2(0,T;L^2(X,\mu))} = \|\phi\|^2 - \|P_T\phi\|^2 . \tag{33}$$

Proof. The proof follows immediately from (24). □

The next lemma is crucial for our study of the HJB equation (1).

Lemma 2.10. *For $f \in L^2\left(0,T;L^2(X,\mu)\right)$ let*

$$G_1 f(t) = \int_0^t P_{t-s} f(s)\mathrm{d}s, \quad t \le T,$$

and

$$G_2 f(t) = \int_0^t \mathcal{K}\left(f(s)\right)(t-s)\mathrm{d}s.$$

Then

$$\int_0^T \|G_1 f(t)\|^2 \,\mathrm{d}t \le T^2 \int_0^T \|f(t)\|^2\mathrm{d}t. \tag{34}$$

Moreover, $G_2 f(t) \in L^2(X,\mu;X)$ for almost every $t \in [0,T]$ and

$$\int_0^T \|G_2 f(t)\|^2 \,\mathrm{d}t \le T \int_0^T \|f(t)\|^2\mathrm{d}t. \tag{35}$$

Proof. The first estimate is obvious. We will prove only the second inequality. Assume first that $f \in C_b^1([0,T]\times X)$ and $f(t) \in \mathcal{F}C_0^2(A^*)$ for all $t \ge 0$. Then $D_Q P_{t-s} f(s)$ is well defined for $s \le t$ and so is $D_Q G_1(t)$. Moreover,

$$\int_0^T \|G_2 f(t)\|^2 \,\mathrm{d}t \le \int_0^T \left(\int_0^t \left\|D_Q P_{t-s} f(s)\right\| \mathrm{d}s\right)^2 \mathrm{d}t$$
$$\le \int_0^T t\int_0^t \left\|D_Q P_{t-s} f(s)\right\|^2 \mathrm{d}s\mathrm{d}t \le T\int_0^T\int_s^T \left\|D_Q P_{t-s} f(s)\right\|^2 \mathrm{d}t\mathrm{d}s.$$

Hence by (24)

$$\int_0^T \|G_2(t)\|^2 \,\mathrm{d}t \le T\int_0^T \|f(t)\|^2\mathrm{d}t.$$

If $f \in L^2\left(0,T;L^2(X,\mu)\right)$ is arbitrary, then there exists a sequence $f_n \in C_b^1([0,T]\times X)$, $f_n(t) \in \mathcal{F}C_0^2(A^*)$, which converges to f in $L^2\left(0,T;L^2(X,\mu)\right)$. Repeating the above arguments for

$$G_1^n(t) = \int_0^t P_{t-s} f_n(s)\mathrm{d}s$$

we find that

$$\int_0^T \left\| D_Q\left(G_1^n(t) - G_1^m(t)\right)\right\|^2 \mathrm{d}t \le T\int_0^T \|f_n(t) - f_m(t)\|^2\,\mathrm{d}t.$$

Hence the sequence $D_Q G_1^n$ is convergent in $L^2\left(0,T;L^2(X,\mu)\right)$. Moreover, by the Fubini Theorem

$$\begin{aligned}\int_0^T \left\| D_Q G_1^n(t) - G_2(t)\right\|^2 \mathrm{d}t &= \int_0^T \left\|\int_0^t \left[D_Q P_{t-s} f_n(s)\mathrm{d}s - \mathcal{K}\left(f(s)\right)(t-s)\right]\mathrm{d}s\right\|^2 \mathrm{d}t \\ &\le T\int_0^T \mathrm{d}s \int_s^T \left\| D_Q P_{t-s} f_n(s) - \mathcal{K}\left(f(s)\right)(t-s)\right\|^2 \mathrm{d}t \\ &= T\int_0^T \mathrm{d}s\int_s^T \|\mathcal{K}\left(f_n(s) - f(s)\right)(t-s)\|^2\,\mathrm{d}t\end{aligned}$$

which gives, by Proposition 2.9

$$\begin{aligned}\int_0^T \left\| D_Q G_1^n(t) - G_2(t)\right\|^2 \mathrm{d}t &= T\int_0^T \left[\|f_n(s) - f(s)\|^2 - \|P_{T-s}\left(f_n(s) - f(s)\right)\|^2\right]\mathrm{d}s \\ &\le T\int_0^T \|f_n(s) - f(s)\|^2\,\mathrm{d}s \qquad (36)\end{aligned}$$

so that $D_Q G_1^n$ is convergent in $L^2\left(0,T;L^2(X,\mu)\right)$ to G and (35) holds. □

Remark 2.11. Let $f_n \to f$ in $L^2\left(0,T;L^2(X,\mu)\right)$. Then, by (36), there exists a subsequence $\left(f_{n_k}\right)$ such that for a.e. $s,t \in [0,T]$ and $s \le t$,

$$D_Q P_{t-s} f_{n_k}(s) \to \mathcal{K}\left(f(s)\right)(t-s) \quad \text{in } L^2(X,\mu).$$

This fact will be useful in Section 5. □

Now we use the above to derive a useful approximation result. Let $\varphi \in L^2(X,\mu)$ and $f \in L^2\left(0,T;L^2(X,\mu)\right)$. Consider the Cauchy problem

$$\begin{cases} u'(t) = \mathcal{N}u(t) + f(t) & t \in\ ]0,T] \\ u(0) = \varphi. \end{cases} \qquad (37)$$

Define the mild solution of (37) as

$$u(t) = P_t\varphi + \int_0^t \left[P_{t-s} f(s)\right]\mathrm{d}s. \qquad (38)$$

Then the following holds.

Proposition 2.12. *Let $(\varphi_n) \subset L^2(X,\mu)$ and $(f_n) \subset L^2\left(0,T;L^2(X,\mu)\right)$ be such that*

$$\begin{aligned}&\varphi_n \longrightarrow \varphi \quad \text{in } L^2(X,\mu) \\ &f_n \longrightarrow f \quad \text{in } L^2\left(0,T;L^2(X,\mu)\right).\end{aligned}$$

Then, setting

$$u_n(t) = P_t\varphi_n + \int_0^t \left[P_{t-s} f_n(s)\right] \mathrm{d}s \tag{39}$$

and

$$\tilde{D}_Q u_n(t) = \mathcal{K}\varphi_n(t) + \int_0^t \mathcal{K}\left(f_n\left(s\right)\right)(t-s)\,\mathrm{d}s$$

$$\tilde{D}_Q u(t) = \mathcal{K}\varphi(t) + \int_0^t \mathcal{K}\left(f\left(s\right)\right)(t-s)\,\mathrm{d}s$$

we have

$$u_n \longrightarrow u \quad \text{in } C\left([0,T]; L^2(X,\mu)\right), \tag{40}$$

$$\tilde{D}_Q u_n \longrightarrow \tilde{D}_Q u \quad \text{in } L^2\left(0,T; L^2(X,\mu;X)\right). \tag{41}$$

Proof. We start with the first claim. By subtracting (38) from (39) we get

$$u_n(t) - u(t) = P_t(\varphi_n - \varphi) + \int_0^t P_{t-s}\left(f_n(s) - f(s)\right) \mathrm{d}s$$

so that, by strong continuity of P_t,

$$\|u_n(t) - u(t)\|^2 \le C_T \left[\|\varphi_n - \varphi\|^2 + \int_0^t \|f_n(s) - f(s)\|^2 \,\mathrm{d}s\right]$$

which gives (40), taking the supremum on $[0, T]$. To prove (41) we apply Lemma 2.10. In fact

$$\tilde{D}_Q\left(u_n(t) - u(t)\right) = \mathcal{K}(\varphi_n - \varphi)(t) + \int_0^t \mathcal{K}\left(f_n(s) - f(s)\right)(t-s)\,\mathrm{d}s$$

so that, by (33) and (35)

$$\int_0^T \left\|\tilde{D}_Q u_n(t) - \tilde{D}_Q u(t)\right\|^2 \le \|\varphi_n - \varphi\|^2 + T\int_0^T \|f_n(s) - f(s)\|^2 \,\mathrm{d}s$$

which gives (41). □

The above approximation results substantially tells us that for the mild solutions of Cauchy problems like (37) an operator $\tilde{D}_Q$, that extends D_Q, can be well defined.

3. The HJB equation

In this section we study the existence and uniqueness of solutions to the following HJB equation (where we set $H\left(Q^{1/2}p\right) = H_0(p)$)

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t}(t) = \mathcal{N}u(t) - H\left(D_Q u(t)\right) + f(t), \\ u(0) = \phi \in L^2(X,\mu), \quad t \le T. \end{cases} \tag{42}$$

We assume that the following conditions are satisified.

Hypothesis 3.1. (A) The function H_0 (the Hamiltonian) can be written as $H_0(p) = H\left(Q^{1/2}p\right)$, where $H : X \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant l.
(B) We assume that $f \in L^2\left(0, T : L^2(X, \mu)\right)$ and $\phi \in L^2(X, \mu)$.

Remark 3.2. Note that at the moment the HJB equation (1) is not related to any control problem and therefore the Hamiltonian H_0 need not to be of the special form (8). In fact our existence and uniqueness results will hold under the above assumptions, even if no control problem is associated to (1). □

Using the semigroup (P_t) and the variation of constants formula we can rewrite Eq. (42) in the following integral form

$$u(t) = P_t\phi - \int_0^t P_{t-s}H\left(D_Q u(s)\right) \mathrm{d}s + \int_0^t P_{t-s} f(s)\, \mathrm{d}s, \quad 0 \le t \le T. \tag{43}$$

We will use this integral form (which is often called "mild form") to define a solution and to state our existence and uniqueness result. However, due to the nonclosability of the operator D_Q, an unpleasant problem arises in defining the concept of solution to (42). If D_Q was closable, then it would be natural to define the solution of Eq. (43) (that will be called the mild solution of Eq. (42)) as an element of $L^2\left(0, T; W_Q^{1,2}(X, \mu)\right)$ such that (43) is satisfied for a.e. $t \in [0, T]$ and μ a.e. But here D_Q may be not closable, so elements of $W_Q^{1,2}(X, \mu)$ are not functions in general, but pairs of functions belonging to the product space $L^2(X, \mu) \times L^2(X, \mu; X)$ as recalled in Section 2.2. We will see that, thanks to Proposition 2.9 and Lemma 2.10 this difficulty can be overcome.

The following definition of solution takes into account that we are dealing with pairs of functions.

Definition 3.3. By a solution of Eq. (43) (or a mild solution of Eq. (42)) we mean a pair of functions

$$(u, U) \in L^2\left(0, T; W_Q^{1,2}(X, \mu)\right) \subset L^2\left(0, T; L^2(X, \mu)\right) \times L^2\left(0, T; L^2(X, \mu; X)\right)$$

such that, for a.e. $t \in [0, T]$ and μ a.e.

$$u(t) = P_t\phi + \int_0^t P_{t-s}H\left(U(s)\right) \mathrm{d}s + \int_0^t P_{t-s} f(s)\, \mathrm{d}s, \quad 0 \le t \le T \tag{44}$$

and

$$U(t) = \mathcal{K}(\phi)(t) - \int_0^t \mathcal{K}\left(H(U(s))\right)(t-s)\mathrm{d}s + \int_0^t \mathcal{K}\left(f(s)\right)(t-s)\mathrm{d}s. \tag{45}$$

Remark 3.4. Note that the second Eq. (45) is an obvious consequence of (44) if the operator D_Q is closable and then $U = D_Q u$. □

We now introduce a suitable nonlinear operator $\mathcal{M}$ which will allow us to use the fixed point argument.

For $v \in L^2(0, T; L^2(X, \mu))$ such that $v(t) \in C_b^1(X)t$-a.e. we define the norm $|||\cdot|||$ by the formula

$$\|\!|v|\!\|^2 = \int_0^T \left(\|v(t)\|^2 + \left\| Q^{1/2} Dv(t) \right\|^2 \right) \mathrm{d}t.$$

Next we define the operator $\mathcal{M}_1$ as follows:

$$\mathrm{dom}(\mathcal{M}_1) = \left\{ v \in L^2\left(0, T; L^2(X, \mu)\right) : v(t) \in C_b^1(X) \ t\text{-a.e. and } \|\!|v|\!\| < \infty \right\},$$

and for $v \in \mathrm{dom}(\mathcal{M}_1)$

$$\mathcal{M}_1 v(t) = P_t \phi + \int_0^t P_{t-s} H\left(D_Q v(s)\right) \mathrm{d}s + \int_0^t P_{t-s} f(s)\, \mathrm{d}s, \quad t \le T.$$

Note that by Lemma 2.10 $D_Q \mathcal{M}_1 v \in L^2(0, T; L^2(X, \mu; X))$ is well defined for every $v \in \mathrm{dom}(\mathcal{M}_1)$.

Lemma 3.5. *Assume that Hypotheses 2.1 and 3.1 hold. Then $\mathcal{M}_1$ extends to a Lipschitz mapping $\overline{\mathcal{M}}_1 : L^2(0, T; W_Q^{1,2}(X, \mu)) \to L^2(0, T; L^2(X, \mu))$. Moreover, the mapping $D_Q \mathcal{M}_1$: $\mathrm{dom}(\mathcal{M}_1) \to L^2(0, T; L^2(X, \mu; X))$ also extends to a Lipschitz mapping*

$$\overline{D_Q \mathcal{M}_1} : L^2(0, T; W_Q^{1,2}(X, \mu)) \to L^2(0, T; L^2(X, \mu; X)).$$

Proof. Since, for suitable $b > 0$, $|H(x)| \le b(1 + |x|)$ it follows from Lemma 2.10 that $\mathcal{M}_1 v \in L^2\left(0, T; L^2(X, \mu)\right)$ and $D_Q \mathcal{M}_1 v \in L^2\left(0, T; L^2(X, \mu)\right)$ for every $v \in \mathrm{dom}(\mathcal{M}_1)$. Let $v_1, v_2 \in \mathrm{dom}(\mathcal{M}_1)$. Then

$$\mathcal{M}_1 (v_1 - v_2)(t) = \int_0^t P_{t-s} \left(H\left(D_Q v_1(s)\right) - H\left(D_Q v_2(s)\right) \right) \mathrm{d}s$$

and therefore, since $\|P_t\| = 1$,

$$|\mathcal{M}_1 (v_1 - v_2)(t)| \le l \int_0^t \left| D_Q v_1(t) - D_Q v_2(t) \right| \mathrm{d}t.$$

Hence,

$$\int_0^T \|\mathcal{M}_1 (v_1 - v_2)(t)\|^2 \mathrm{d}t \le l^2 T^2 \int_0^T \left| D_Q v_1(t) - D_Q v_2(t) \right|^2 \mathrm{d}t.$$

It follows that $\mathcal{M}_1$ may be extended to the whole of $L^2\left(0, T; W_Q^{1,2}(X, \mu)\right)$ by continuity and the resulting mapping is Lipschitz with the constant lT. Similarly,

$$D_Q \mathcal{M}_1 (v_1 - v_2)(t) = \int_0^t D_Q P_{t-s} \left(H\left(D_Q v_1(s)\right) - H\left(D_Q v_2(s)\right) \right) \mathrm{d}s$$

and using notation from Lemma 2.10 we obtain

$$\begin{aligned}
&\int_0^T \left\| D_Q \mathcal{M} (v_1 - v_2)(t) \right\|^2 \mathrm{d}t \\
&\quad = \int_0^T \left\| G_2 \left(H\left(D_Q v_1\right) - H\left(D_Q v_2\right) \right)(t) \right\|^2 \mathrm{d}t \\
&\quad \le T \int_0^T \left\| H\left(D_Q v_1(t)\right) - H\left(D_Q v_2(t)\right) \right\|^2 \mathrm{d}t \le l^2 T \int_0^T \left\| D_Q \left(v_1(t) - v_2(t)\right) \right\|^2 \mathrm{d}t,
\end{aligned}$$

and therefore $D_Q\mathcal{M}_1$ extends to a Lipschitz mapping on $L^2(0,T;W_Q^{1,2}(X,\mu))$ with constant lT. □

Remark 3.6. We observe that, in fact, the operators $\overline{\mathcal{M}}_1, \overline{D_Q\mathcal{M}}_1$ are defined on the space $L^2\left(0,T;L^2(X,\mu;X)\right)$ i.e. they depend only on the second component of elements of $L^2\left(0,T;W_Q^{1,2}(X,\mu)\right)$. It is convenient for us to define them on $L^2\left(0,T;W_Q^{1,2}(X,\mu)\right)$ to apply the fixed point argument below. □

Now we define the operator

$$\mathcal{M}: L^2\left(0,T;W_Q^{1,2}(X,\mu)\right)\to L^2\left(0,T;W_Q^{1,2}(X,\mu)\right)$$
$$\mathcal{M}(u,U)=(\overline{\mathcal{M}}_1(u,U),\overline{D_Q\mathcal{M}}_1(u,U)).$$

Using Proposition 2.9 and Lemma 2.10 we find that for a.e. $t\in[0,T]$

$$\overline{\mathcal{M}}_1(u,U)(t)=P_t\phi-\int_0^t P_{t-s}H(U(s))\mathrm{d}s+\int_0^t P_{t-s}f(s)\mathrm{d}s,$$

and

$$\overline{D_Q\mathcal{M}}_1(u,U)(t)=\mathcal{K}(\phi)(t)-\int_0^t\mathcal{K}\left(H(U(s))\right)(t-s)\mathrm{d}s+\int_0^t\mathcal{K}\left(f(s)\right)(t-s)\mathrm{d}s.$$

Theorem 3.7. *Assume that Hypotheses 2.1 and 3.1 hold. Then for every $\phi\in L^2(X,\mu)$ there exists a unique mild solution (u,U) to Eq. (42). Moreover $u\in C\left([0,T];L^2(X,\mu)\right)$ and $U=\tilde{D}_Qu$.*

Proof. We will apply the Fixed Point Theorem to the mapping $\mathcal{M}$ in the space $L^2\left(0,T;W_Q^{1,2}(X,\mu)\right)$ endowed with the norm $|||\cdot|||$ with T sufficiently small. We have

$$|||\mathcal{M}v-\mathcal{M}w|||\le l\sqrt{T(T+1)}|||v_1-v_2|||. \tag{46}$$

Indeed, by Lemma 3.5

$$\int_0^T\left\|\overline{\mathcal{M}}_1v_1(t)-\overline{\mathcal{M}}_1v_2(t)\right\|^2\mathrm{d}t\le l^2T^2|||v_1-v_2|||^2, \tag{47}$$

and

$$\int_0^T\left\|\overline{D_Q\mathcal{M}}_1v_1(t)-\overline{D_Q\mathcal{M}}_1v_2(t)\right\|^2\mathrm{d}t\le Tl^2|||v_1-v_2|||^2, \tag{48}$$

for $v_1,v_2\in L^2\left(0,T;L^2(X,\mu)\right)$. Clearly (47) and (48) yield (46), hence $\mathcal{M}$ is a strict contraction for T sufficiently small. Since the constant in (46) is independent of ϕ, the solution can be continued indefinitely and this concludes the proof of Existence and Uniqueness. Finally, since $H(U)\in L^2\left([0,T];L^2(X,\mu)\right)$ and (P_t) is a C_0-semigroup, we find that $u\in C\left([0,T],L^2(X,\mu)\right)$. □

A stronger result can be proved if D_Q is closable in $L^2(X,\mu)$ in which case $W_Q^{1,2}(X,\mu)$ is continuously embedded in $L^2(X,\mu)$.

Theorem 3.8. *Assume that Hypotheses 2.1 and 3.1 hold. Assume moreover that D_Q is closable. Then there exists a unique mild solution u of (42) in the sense that the couple $(u, D_Q u)$ satisfies Definition 3.3. Moreover u belongs to $L^2\left(0, T; W_Q^{1,2}(X, \mu)\right) \cap C\left([0, T], L^2(X, \mu)\right)$. Finally, if $f \in C_b\left((0, T], L^2(X, \mu)\right)$ then $D_Q u \in C_b\left([\varepsilon, T]; L^2(X, \mu; X)\right)$ for every $\varepsilon > 0$.*

Proof. By Theorem 3.7 there exists a unique solution u of (42) such that $u \in L^2\left(0, T; W_Q^{1,2}(X, \mu)\right)$ and since D_Q is closable, $W_Q^{1,2}(X, \mu) \subset L^2(X, \mu)$ and the first part of the Theorem follows. Assume that $f \in C_b\left((0, T], L^2(X, \mu)\right)$. Then we can repeat the proof of Theorem 3.7 in the space of all $u \in C_b\left((0, T]; L^2(X, \mu)\right)$ such that $D_Q u \in C_b\left([\varepsilon, T]; L^2(X, \mu; X)\right)$ for every $\varepsilon > 0$. This yields easily the desired result. □

We finally give a regularity result.

Proposition 3.9. *Assume that Hypotheses 2.1 and 3.1 hold. Let (u, U) be the mild solution of (42). If $\phi \in C_b^1(X)$ and $f \in C_b^{1,1}([0, T] \times X)$ then $U \in C\left([0, T], L^2(X, \mu; X)\right)$.*

Proof. It is enough to observe that the terms $P_t\phi$ and $\int_0^t P_{t-s} f(s)\, ds$ in (43), thanks to (22) and Lemma 2.10, are such that $D_Q P_t \phi$ and $D_Q \int_0^t P_{t-s} f(s) ds$ belong to $C\left([0, T], L^2(X, \mu; X)\right)$. Then one can apply the fixed point theorem in a space of more regular functions getting the required regularity. □

Remark 3.10. We note that the uniqueness of the solution stated in Theorem 3.7 has to be understood with respect to the reference measure μ. It may happen that there are two different classical solutions that are equal μ-a.e. In the case of HJB equations arising from stochastic control problems, as in Section 5 we can identify (μ-a.e.) the mild solution with the value function. In the case when the value function is continuous (which may be the case under relatively mild assumptions) then we may say (thanks to the nondegeneracy of μ) that the value function is the unique continuous mild solution (in the sense that any other solution is equal to it *at every point of* X). □

4. Approximation of mild solutions

We now show, following the approach of [32], that the mild solution of our equation can be obtained as the limit μ-a.e. of classical solutions.

We start by defining the operator $\mathcal{N}_0$ as follows:

$$\begin{cases} D(\mathcal{N}_0) = \{\eta \in UC_b^2(X) \,:\, \eta_{xx} \in UC_b(X, \mathcal{L}_1(X));\ A^*\eta_x \in UC_b(X); \\ \qquad\qquad x \to \langle F(x), \eta_x \rangle \in UC(X) \cap L^2(X, \mu) \\ \qquad\qquad \text{and} \quad x \to \langle x, A^*\eta_x \rangle \in UC(X) \cap L^2(X, \mu)\} \\ \mathcal{N}_0 \eta = \dfrac{1}{2}\mathrm{Tr}[Q\eta_{xx}] + \langle x, A^*\eta_x \rangle + \langle F(x), \eta_x \rangle. \end{cases} \tag{49}$$

It can be easily seen that $\mathcal{F}C_0^2(A^*) \subseteq D(\mathcal{N}_0)$ so that (see [21]) $\mathcal{N}_0 \subset \mathcal{N}$ and $D(\mathcal{N}_0)$ is dense in $L^2(X, \mu)$. Moreover $D(\mathcal{N}_0)$ is also dense in $UC_b(X)$ in the sense of the so-called $\mathcal{K}$-convergence (the uniform convergence on compact subsets plus uniform boundedness, see [11]). We can now define the concepts of strict and strong solution of Eq. (42).

Definition 4.1. A function $u : [0, T] \times X \to \mathbb{R}$ is a strict solution of Eq. (42) if u has the following regularity properties

$$\begin{cases} u(\cdot, x) \in C^1([0, T]), \quad \forall x \in X \\ u(t) \in D(\mathcal{N}_0) \quad \forall t \in [0, T] \quad \text{and} \quad \sup_{t\in[0,T]} \|u(t)\|_{D(\mathcal{N}_0)} < +\infty \\ u,\ u_t,\ \tilde{D}_Q u, \in C_b([0, T] \times X), \quad \mathcal{N}_0 u \in C([0, T] \times X) \cap L^2(X, \mu) \end{cases}$$

and satisfies (42) in the classical sense with $\tilde{D}_Q$ in place of D_Q.

Note that this definition is slightly different from the one of [32] in that it does not require the boundedness of $\mathcal{N}_0 u$. This comes from the presence of the nonlinear, and possibly unbounded, term F which was assumed to be bounded in [32].

Definition 4.2. A function $u : [0, T] \times X \to \mathbb{R}$ is a strong solution of Eq. (42) if $u \in L^2\left(0, T; W_Q^{1,2}(X, \mu)\right)$ and there exist three sequences $\{u_n\}$, $\{f_n\} \subset L^2\left(0, T; W_Q^{1,2}(X, \mu)\right)$ and $\{\varphi_n\} \subset D(\mathcal{N}_0)$ such that for every $n \in \mathbb{N}$, u_n is the strict solution of the Cauchy problem:

$$\begin{cases} w_t = \mathcal{N}_0 w - H(D_Q w) + f_n \\ w(0) = \varphi_n \end{cases}$$

and moreover, for $n \to +\infty$

$$\begin{aligned} &\varphi_n \longrightarrow \varphi && \text{in } L^2(X, \mu) \\ &f_n \longrightarrow f && \text{in } L^2\left(0, T; L^2(X, \mu)\right) \\ &u_n \longrightarrow u && \text{in } C\left([0, T]; L^2(X, \mu)\right) \\ &\tilde{D}_Q u_n \longrightarrow \tilde{D}_Q u && \text{in } L^2\left(0, T; L^2(X, \mu; X)\right) \end{aligned}$$

Proposition 4.3. *Assume that Hypotheses 2.1 and 3.1 hold. The couple $(u, U) \in L^2\left(0, T; W_Q^{1,2}(X, \mu)\right)$ is a mild solution of Eq. (42) if and only if $U = \tilde{D}_Q u$ and u is a strong solution.*

Proof. Let (u, U) be the mild solution of (42). By the definition of $\tilde{D}_Q u$ in Proposition 2.12 and the Definition of mild solution 3.3 we immediately get $U = \tilde{D}_Q u$. Let $\{\varphi_n\}$, $\{\psi_n\}$ be two sequences such that

$$\begin{aligned} &\varphi_n \in D(\mathcal{N}_0); \qquad \psi_n \in C([0, T]; D(\mathcal{N}_0)) \\ &\varphi_n \overset{n\to+\infty}{\longrightarrow} \varphi \quad \text{in } L^2(X, \mu) \\ &\psi_n \overset{n\to+\infty}{\longrightarrow} -H(\tilde{D}_Q u) + f \quad \text{in } L^2\left(0, T; L^2(X, \mu)\right). \end{aligned}$$

These sequences exist thanks to approximation lemmas proved e.g. in [11,21]. Since we have

$$u(t) = P_t\varphi + \int_0^t \left[P_{t-s}\left(-H(\tilde{D}_Q u(s)) + f(s)\right)\right] \mathrm{d}s,$$

then setting

$$u_n(t, x) = P_t\varphi_n + \int_0^t P_{t-s}\psi_n(s)\mathrm{d}s$$

by Proposition 2.12 we get that

$$u_n \overset{n\to+\infty}{\longrightarrow} u \quad \text{in } C\left([0,T]; L^2(X,\mu)\right)$$
$$\tilde{D}_Q u_n \overset{n\to+\infty}{\longrightarrow} \tilde{D}_Q u \quad \text{in } L^2\left(0,T; L^2(X,\mu;X)\right).$$

Moreover u_n satisfies, in the classical sense, the approximated HJ equation:

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = \mathcal{N} u_n - H(\tilde{D}_Q u_n) + f_n, & t \in]0,T]\; x \in D(A) \\ u(0,x) = \varphi_n(x), & x \in X, \end{cases} \tag{50}$$

where we set

$$f_n = \psi_n - [-H(\tilde{D}_Q u_n)] \overset{n\to+\infty}{\longrightarrow} f \quad \text{in } L^2\left(0,T; L^2(X,\mu)\right).$$

This proves that a mild solution is always strong. Vice versa it is easy to check that a strong solution is always a mild one. In fact, if u is a strong solution and u_n, f_n, φ_n are its approximating sequences as in Definition 4.2 then, by the formula for variation of constants, for every n we have

$$u_n(t) = P_t\varphi_n + \int_0^t P_{t-s}\left[-H(\tilde{D}_Q u_n(s)) + f_n(s)\right] \mathrm{d}s$$

so, setting $\psi_n = -H(\tilde{D}_Q u_n) + f_n$ we get

$$= P_t\varphi_n + \int_0^t P_{t-s}\left[\psi_n(s)\right] \mathrm{d}s$$

where $\varphi_n \in D(\mathcal{N}_0)$, $\psi_n \in L^2\left(0,T; L^2(X,\mu)\right)$ and

$$\varphi_n \overset{n\to+\infty}{\longrightarrow} \varphi \quad \text{in } L^2(X,\mu)$$
$$\psi_n \overset{n\to+\infty}{\longrightarrow} -H(\tilde{D}_Q u) + f \quad \text{in } L^2\left(0,T; L^2(X,\mu)\right).$$

Then we can apply Proposition 2.12 and pass to the limit for $n \to +\infty$ to get the claim. □

Remark 4.4. We observe that the sequences $(\varphi_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ can be always taken with values in $\mathcal{F}C_0^2(A^*)$ i.e. finite dimensional with respect to a fixed orthonormal basis in X. However the approximate solutions $(u_n)_{n\in\mathbb{N}}$ are not in general finite dimensional, except for some special cases (e.g. when $F = 0$, and A, Q are diagonal operators with respect to the same orthonormal basis in X). Of course these cases could be interesting from the point of view of numerical approximations, this happens e.g. in some fluid dynamics models (see e.g. [18–20]). □

Remark 4.5. In the case when D_Q is closable then, using the same arguments of Theorem 3.8 above we can prove that

$$\tilde{D}_Q u_n \overset{n\to+\infty}{\longrightarrow} \tilde{D}_Q u \quad \text{in } C\left([\varepsilon,T]; L^2(X,\mu;X)\right)$$

for every $\varepsilon > 0$. □

Remark 4.6. Using results of Section 2.2 one can prove also the following approximation result similar to the ones of this section. If $\phi_n \in C_b^1(X)$ and $\lim_{n\to\infty} \phi_n = \phi$ in $L^2(X,\mu)$ then

$$\lim_{n\to\infty} \int_0^T \left(\|u_n(t) - u(t)\|^2 + \left\| D_Q (u_n(t) - u(t)) \right\|^2 \right) \mathrm{d}t = 0.$$

The same results also holds if we approximate f by $f_n \in C_b^1([0,T] \times X)$. □

5. Dynamic programming

Consider a stochastic controlled system governed by the state equation

$$y(s) = \mathrm{e}^{(s-t)A} x + \int_t^s \mathrm{e}^{(s-r)A} \left[Q^{1/2} F(y(r)) + Q^{1/2} h_1(z(r)) \right] \mathrm{d}r + \int_t^s \mathrm{e}^{(r-t)A} Q^{1/2} \mathrm{d}W(r), \quad s \geq t \geq 0 \tag{51}$$

where $x \in X$ which is a separable Hilbert space, A, Q, F, W satisfy Hypothesis 2.1, the function $h_1 : X \mapsto X$ is measurable and $z \in M_W^2(t,T;X)$. Eq. (51) can be regarded as the mild form of the stochastic differential equation

$$\begin{cases} \mathrm{d}y(s) = \left[Ay(s) + Q^{1/2} F(y(s)) + Q^{\frac{1}{2}} h_1(z(s)) \right] \mathrm{d}s + Q^{1/2} \mathrm{d}W(s), & t \leq s \leq T \\ y(t) = x, \quad x \in X. \end{cases} \tag{52}$$

The following Proposition is proved in [33] and, in a special case, in [8], (see also [22, Ch 7.1]).

Proposition 5.1. *Let $h_1 : X \mapsto X$ be continuous and sublinear. Then, for all $z \in M_W^2(t,T;X)$, Eq. (51) has a unique solution $y(\cdot,t,x,z) \in M_W^2(t,T;X)$. Moreover, if for some $\beta > 0$,*

$$\int_0^T t^{-\beta} \left\| \mathrm{e}^{tA} Q^{1/2} \right\|_{HS}^2 \mathrm{d}t < +\infty,$$

then the solution $y(\cdot,t,x,z)$ is continuous with probability one.

We now consider the following abstract optimal control problem in the so-called relaxed setting (see e.g. [51]). Given $0 \leq t \leq T < \infty$ we denote by $\overline{\mathcal{A}}_{t,T}$ the set of admissible (relaxed) controls. The set consists of:

- probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$,
- cyilindrical Brownian motions W, on $[t,T]$.
- measurable processes $z \in M_W^2(t,T;X)$ with $\sup_{s\in[t,T]} |z(s)| \leq R$ for a given constant $R > 0$ possibly infinite.

We will use the notation $(\Omega, \mathcal{F}, \mathbb{P}, W, z) \in \overline{\mathcal{A}}_{t,T}$. When no ambiguity arises we will leave aside the probability space (regarding it as fixed) and consider admissible controls simply as processes $z \in \mathcal{A}_{t,T} := M_W^2(t,T;X)$ with $\sup_{s\in[t,T]} |z(s)| \leq R$.

Let now $x \in X$ and $(\Omega, \mathcal{F}, \mathbb{P}, W, z) \in \overline{\mathcal{A}}_{t,T}$. We try to minimize the cost functional

$$J(t,x;z) = \mathbb{E} \left\{ \int_t^T [f(y(s;t,x,z)) + \frac{1}{2} h_2(z(s))] \mathrm{d}s + \varphi(y(T;t,x,z)) \right\} \tag{53}$$

over all (relaxed) controls $z \in \mathcal{A}_{t,T}$.

Here $f, \varphi : X \to \mathbb{R}$ satisfy Hypothesis 3.1, $h_2 : X \to \mathbb{R}$ is measurable and bounded from below and $y(\cdot; t, x, z)$ is the mild solution of Eq. (51). The value function of this problem is defined as

$$V(t,x) = \inf\left\{J(t,x;z) : z \in \mathcal{A}_{t,T}\right\}. \tag{54}$$

The corresponding Hamilton–Jacobi equation reads as follows

$$\begin{cases} -\dfrac{\partial v}{\partial t} = \mathcal{N} v - H(D_Q v) + f(x), \quad t > 0, \quad x \in D(A) \\ v(T,x) = \varphi(x), \quad x \in X, \end{cases} \tag{55}$$

where the Hamiltonian H is given by

$$H(p) = \sup_{z \in X} \left\{-\langle h_1(z), p\rangle - h_2(z)\right\}. \tag{56}$$

To apply our results we need to assume that Hypotheses 2.1 and 3.1-(B) hold and moreover

Hypothesis 5.2. (i) $h_1 : X \mapsto X$ is continuous and either (a) bounded or (b) sublinear and there exists $R > 0$ such that $|z(s)| \le R$ for each $t \le s \le T$ and $z \in \mathcal{A}_{t,T}$.
(ii) $h_2 : X \to \mathbb{R}$ is measurable and bounded below.

Remark 5.3. Hypothesis 5.2 says, in particular, that h_1 and h_2 are such that the Hamiltonian function $H : X \to \mathbb{R}$ defined by (56) is Lipschitz continuous, so also Hypothesis 3.1-(A) is satisfied. □

We now show how to apply our results on HJB equations to obtain a verification theorem and existence of optimal feedbacks for the above optimal control problem. We will need some technical lemmas that guarantee non triviality.

Lemma 5.4. *Assume that Hypotheses 2.1 and 5.2 hold and let*

$$\rho_z = \exp\left(\int_0^T \langle h_1(z(r)), \mathrm{d}W(r)\rangle - \frac{1}{2}\int_0^T |h_1(z(r))|^2\,\mathrm{d}r\right).$$

Then $\mathbb{E}^x \rho_z = 1$ *for a.e.* x *where* $\mathbb{E}^x$ *is the expected value with respect to the law of the process* $y(\cdot, 0, x)$. *Moreover, there exists a set* $\mathcal{Z} \subset X$ *such that* $\mu(X - \mathcal{Z}) = 0$ *and*

$$\sup_{x \in \mathcal{Z}} \mathbb{E}^x \rho_z^2 < \infty.$$

Finally, the laws of the processes $y(\cdot, 0, x)$ *and* $y(\cdot, 0, x, z)$ *are equivalent.*

Proof. Standard and omitted. □

Lemma 5.5. *Assume that Hypotheses 2.1 and 5.2 hold and that* $w \in L^2\left(0,T; L^2(X,\mu)\right)$ *(or* $L^2\left(0,T; L^2(X,\mu;X)\right)$*). Then the map*

$$(s,x) \mapsto \mathbb{E} w\left(s, y(s;t,x,z)\right)$$

belongs to $L^1\left((t,T) \times X, Leb \otimes \mu\right)$

Proof. If $z = 0$ then $y(\cdot; t, x, z) = y(\cdot; t, x)$ is a solution to (11) and so μ is its stationary measure. Therefore,

$$\begin{aligned}
&\int_t^T \int_H |\mathbb{E}w(s, y(s; t, x))| \mu(\mathrm{d}x)\, \mathrm{d}s \\
&\quad \le \int_t^T \int_H \mathbb{E}|w(s, y(s; t, x))| \mu(\mathrm{d}x)\, \mathrm{d}s = \int_t^T \int_H P_{s-t}|w(s, \cdot)|(x)\mu(\mathrm{d}x)\, \mathrm{d}s \\
&\quad = \int_t^T \int_H |w(s, x)| \mu(\mathrm{d}x)\, \mathrm{d}s \le \mathbb{C}_{\mathbb{T}} \int_t^{\mathbb{T}} \int_{\mathbb{H}} |w(s, x)|^2 \mu(\mathrm{d}x)\, \mathrm{d}s < +\infty.
\end{aligned}$$

Invoking Lemma 5.4 we find that

$$\begin{aligned}
&\int_t^T \int_H |\mathbb{E}w(s, y(s; t, x, z))| \mu(\mathrm{d}x)\, \mathrm{d}s \\
&\quad \le \int_t^T \int_H \mathbb{E}|w(s, y(s; t, x, z))| \mu(\mathrm{d}x)\, \mathrm{d}s = \int_t^T \int_H \mathbb{E}|\rho_z w(s, y(s; t, x))| \mu(\mathrm{d}x)\, \mathrm{d}s \\
&\quad \le \int_t^T \int_H \left(\mathbb{E}|\rho_z|^2\, \mathbb{E}|w(s, y(s; t, x))|^2\right)^{1/2} \mu(\mathrm{d}x)\, \mathrm{d}s \\
&\quad \le C_{T,z} \left(\int_t^T \int_H \mathbb{E}|w(s, y(s; t, x))|^2 \mu(\mathrm{d}x)\, \mathrm{d}s\right)^{1/2} \\
&\quad = C_{T,z} \left(\int_t^T \int_H P_{s-t}|w(s, \cdot)|^2(x)\mu(\mathrm{d}x)\, \mathrm{d}s\right)^{1/2} \\
&\quad = C_{T,z} \left(\int_t^T \int_H |w(s, \cdot)|^2(x)\mu(\mathrm{d}x)\, \mathrm{d}s\right)^{1/2} < +\infty.
\end{aligned}$$

and the claim follows. □

Lemma 5.6. *Assume that Hypothesis 2.1, 3.1 and 5.2 hold. Let $(u, U) \in L^2\left(0, T; W_Q^{1,2}(X, \mu)\right)$ be the mild solution of (55). Then, for every $t \in [0, T]$, $x \in X$ and $z \in \mathcal{A}_{t,T}$, the following identity holds*

$$\begin{aligned}
&v(t, x) + \int_t^T \left\{H(\tilde{D}_Q v(s, y(s))) + \langle h_1(z(s)), \tilde{D}_Q v(s, y(s))\rangle + h_2(z(s))\right\} \mathrm{d}s \\
&\quad = \mathbb{E}\left\{\int_t^T [f(y(s)) + h_2(z(s))]\mathrm{d}s + \varphi(y(T))\right\} = J(t, x, z) \qquad (57)
\end{aligned}$$

where $y(s) \overset{def}{=} y(s; t, x, z)$ is the mild solution of (51).

Proof. Let $(\varphi_n)_{n\in\mathbb{N}}$, $(\psi_n)_{n\in\mathbb{N}}$ be suitable approximating sequences as in Section 4. Then we set

$$u_n(t, x) = P_t\varphi_n + \int_0^t P_{t-s}\psi_n(s)\mathrm{d}s$$

Then we know that u_n satisfies, in the classical sense, the approximated Hamilton–Jacobi equation:

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = \mathcal{N} u_n - H(\tilde{D}_Q u_n) + f_n, & t \in]0,T]\ x \in D(A) \\ u(0,x) = \varphi_n(x), & x \in X, \end{cases} \tag{58}$$

where we set

$$f_n(t,x) = \psi_n(x) + H(\tilde{D}_Q u_n) \overset{n\to+\infty}{\longrightarrow} f \quad \text{in } L^2\left(0,T; L^2(X,\mu;X)\right)$$

(if D_Q is closable then the convergence is in $C\left([\varepsilon,T]; L^2(X,\mu;X)\right)$ for every $\varepsilon > 0$ and we may put D_Q instead of $\tilde{D}_Q$). Let $v_n(s,x) = u_n(T-s,x)$. By using Ito's formula as in [33] we obtain

$$\mathrm{d}v_n(s,y(s)) = \left[\frac{\partial v_n}{\partial s}(s,y(s)) + \frac{1}{2}\mathrm{Tr}\ Q v_{nxx}(s,y(s))\right]\mathrm{d}s + \left\langle \mathrm{d}y(s), \frac{\partial v_n}{\partial x}(s,y(s))\right\rangle. \tag{59}$$

Then use (52) and (58), integrate on $[t,T]$ and take the expectation to obtain

$$\begin{aligned} &\mathbb{E}\varphi_n(y(T)) - v_n(t,x) \\ &\quad = \mathbb{E}\int_t^T \left[\langle \tilde{D}_Q v_n(s,y(s)), h_1(z(s))\rangle + H(\tilde{D}_Q u_n(s,y(s))) - f_n(T-s,y(s))\right]\mathrm{d}s. \end{aligned} \tag{60}$$

Now we pass to the limit for $n \to +\infty$ in (60) by using (4.2) and the two Lemmas 5.4 and 5.5 above. It follows that

$$\begin{aligned} &\mathbb{E}\varphi(y(T)) - v(t,x) \\ &\quad = \mathbb{E}\int_t^T \left[\langle D_Q v(s,y(s)), h_1(z(s))\rangle + H(D_Q v(s,y(s))) - f(y(s))\right]\mathrm{d}s \end{aligned}$$

which gives (57) by rearranging the terms. □

Theorem 5.7. *Assume that Hypothesis 2.1, 3.1 and 5.2 hold. Assume also that H is differentiable. Then problem (55) has a unique mild solution v which coincides with the value function V defined in (54). Moreover, for any $(t,x) \in [0,T] \times X$, there exists a unique optimal control for problem (53) in the relaxed sense. Furthermore, the optimal relaxed control z^* is related to the corresponding optimal state y^* by the feedback formula*

$$z^*(s) = DH(\tilde{D}_Q V(s, y^*(s))). \tag{61}$$

Proof. First we remark that, by (56) for every $s \in [t,T]$ and $z \in M_W^2(t,T;X)$ the following inequality holds

$$H(\tilde{D}_Q v(s,y(s))) - \langle z(s), \tilde{D}_Q v(s,y(s))\rangle + h_2(z(s)) \geq 0 \tag{62}$$

so that by (57) it follows that $v(t,x) \leq V(t,x)$ on $[0,T] \times X$. To prove the reverse inequality, let us first recall that, by the regularity of h_2, the minimum of (62) is attained if and only if, for almost every $(t,x,\omega) \in [0,T] \times X \times \Omega$,

$$z(t) = DH(\tilde{D}_Q v(t,y(t)))$$

(see e.g. [26, Section I.8]). We then consider the closed loop equation (with $T \geq s \geq t \geq 0$)

$$y(s) = e^{(s-t)A}x + \int_t^s e^{(s-r)A}\left[Q^{\frac{1}{2}}F(y(r)) + Q^{\frac{1}{2}}DH(\tilde{D}_Q v(s, y(s)))\right]dr + W_A(t, s). \tag{63}$$

This equation has a solution $y^*(s)$ (see e.g. [22, Ch. 8]). At this point, taking

$$z^*(s) = DH(\tilde{D}_Q v(s, y^*(s; t, x))) \tag{64}$$

we have the equality in (62) and so by (57) $v(t, x) \geq V(t, x)$ on $[0, T] \times X$. Moreover, the choice (64) provides the optimal control at (t, x). Finally, the feedback formula (61) follows from (64) and from the equality $v = V$. □

6. Examples

6.1. Stochastic controlled delay equations

Let us consider a simple controlled stochastic differential equation with a delay $r > 0$:

$$\begin{cases} dx(t) = (a_0x(t) + a_1x(t - r) + bz_0(t))\,dt + b\,dW_0(t), \\ x(0) = x_0, \quad x(\theta) = x_1(\theta), \quad \theta \in [-r, 0). \end{cases} \tag{65}$$

This kind of equation is used e.g. in advertising models (see [45]) and can be studied as a stochastic controlled equation in $\mathbb{R}$ (see e.g. [41] or, more recently, [49]). We use here the setting introduced in [12] by rewriting the equation as a controlled stochastic evolution equation in the space $X = \mathbb{R} \times L^2(-r, 0; \mathbb{R})$ as follows. Consider the linear operator on X:

$$D(A) = \left\{\left(\begin{cases} x_0 \\ x_1(\cdot) \end{cases}\right) \in \mathbb{R} \times W^{1,2}(-r, 0; \mathbb{R})\right\}$$

$$A\begin{pmatrix} x_0 \\ x_1(\cdot) \end{pmatrix} = \begin{pmatrix} a_0x_0 + a_1x_1(-r) \\ x_1'(\cdot) \end{pmatrix}.$$

Then A generates a strongly continuous semigroup $S(t)$ on X and, for $x = (x_0, x_1(\cdot)) \in X$, $S(t)\,x$ can be written in term of the solution of the linear deterministic delay equation

$$\begin{cases} \dot{y}(t) = a_0y(t) + a_1y(t - r), \\ y(0) = x_0, \quad y(\theta) = x_1(\theta), \quad \theta \in [-r, 0), \end{cases} \tag{66}$$

as follows:

$$S(t)\,x = \begin{pmatrix} y(t) \\ y(t + \cdot) \end{pmatrix} \in X, \quad t \geq 0,$$

(see [12]). Then, set

$$z = \begin{pmatrix} z_0 \\ z_1(\cdot) \end{pmatrix} \in X$$

$$W = \begin{pmatrix} W_0 \\ W_1 \end{pmatrix} \in X$$

where z_1 is a fictitious control belonging to $L^2(-r, 0; \mathbb{R})$ and W_1 is a cylindrical white noise in $L^2(-r, 0; \mathbb{R})$, and define $Q : X \mapsto X$ as

$$Q\begin{pmatrix} x_0 \\ x_1(\cdot) \end{pmatrix} = \begin{pmatrix} b^2 x_0 \\ 0 \end{pmatrix}.$$

Then the controlled stochastic delay equation (65) can be rewritten as the unique mild solution of a linear evolution equation

$$\begin{cases} \mathrm{d}Y = \left[AY + Q^{1/2}z\right]\mathrm{d}t + Q^{1/2}\mathrm{d}W, \\ X(0) = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \in H. \end{cases} \tag{67}$$

We assume that

$$a_0 < 1, \quad a_0 < -a_1 < \sqrt{\gamma^2 + a_0^2}, \tag{68}$$

where $\gamma \in (0, \pi)$ and $\gamma \coth \gamma = a_0$. Under this condition equation (67) has a unique invariant measure μ which is nondegenerate (see [23, Chapter 10]).

Let $D_Q = Q^{1/2}D$ be an operator in $L^2(X, \mu)$ with $\mathrm{dom}\left(D_Q\right) = C_b^1(H)$. It is shown in [31] that the operator D_Q is not closable on $L^2(X, \mu)$. This fact shows that it is important to treat cases where the operator D_Q is not closable. Moreover it can be easily seen that Hypothesis 2.1 holds true in this case so that here D_Q is closable in the weak sense introduced in Definition 2.3, so our theory can be applied.

Now consider the problem of minimizing the functional (setting $x = (x_0, x_1)$)

$$J_0(t, x; z_0) = \mathbb{E}\left\{\left[\int_t^T f_0\left(x\left(s; t, x, z_0\right)\right) + h_0\left(z_0\left(s\right)\right)\right]\mathrm{d}s + \varphi_0(x(T; t, x, z_0))\right\}$$

$z_0 \in M_W^2(t, T; \mathbb{R})$ with $\sup_{s\in[t,T]} |z_0(s)| \le R$ for a given constant $R > 0$. The above functional can be rewritten as follows. Set

$$f\left(x_0, x_1\right) = \left(f_0\left(x_0\right), 0\right); \qquad h\left(z_0, z_1\right) = \left(h_0\left(z_0\right), 0\right);$$
$$\varphi\left(x_0, x_1\right) = \left(\varphi_0\left(x_0\right), 0\right)$$

so

$$J_0(t, x; z_0) = J(t, x; z) = \mathbb{E}\left\{\int_t^T \left[f\left(Y\left(s; t, x, z\right)\right) + h\left(z\left(s\right)\right)\right]\mathrm{d}s + \varphi(Y(T; t, x, z))\right\}.$$

The value function of this problem is defined as

$$V(t, x) = \inf\left\{J(t, x; z) : z \in M_W^2(t, T; X),\ \sup_{s\in[t,T]} |z(s)| \le R\right\} \tag{69}$$

and the HJ equation is exactly (55) with the Hamiltonian H_0 given by

$$H_0\left(p\right) = \sup_{z\in X}\left\{-\langle z, p\rangle_X - h\left(z\right)\right\} = \sup_{z\in\mathbb{R}}\left\{-\langle z_0, p_0\rangle_{\mathbb{R}} - h_0\left(z_0\right)\right\}.$$

Then all the results of Sections 3–5 hold true, and we can find the optimal feedback.

Remark 6.1. We observe that here, for simplicity of presentation, we considered a simple one dimensional case of controlled stochastic delay equations. In fact in our framework we can treat more general cases like semilinear d-dimensional equations of the following type

$$\begin{cases} \mathrm{d}x(t) = \Bigg[a_0 x(t) + \sum_{i=1}^{N} a_i x(t+\theta_i) + F_0\left(x(t), x(t+\theta_1), \ldots, x(t+\theta_n)\right) \\ \qquad\qquad + b z_0(t)\Bigg]\mathrm{d}t + b\mathrm{d}W_0(t), \\ x(0) = x_0, \quad x(\theta) = x_1(\theta), \quad \theta \in [-r, 0). \end{cases} \tag{70}$$

where the map F_0 needs to satisfy suitable assumptions to have existence of a nontrivial invariant measure for the system, see [23, Section 10.3] on this (for example the case when F_0 is bounded fits in our theory). Finally we could also treat in the same way a control problem where the costs f_0 and ϕ_0 depend also on the history of the state x.

6.2. *Control of stochastic PDE's of first order*

We will consider a controlled stochastic differential equation

$$\mathrm{d}y(t,\zeta) = \left(\frac{\partial y}{\partial \zeta}(t,\zeta) + F_0(y(t,\cdot),\zeta) + b(y(t,\zeta))z(t,\zeta)\right)\mathrm{d}t + b(y(t,\zeta))\mathrm{d}W(t), \\ \zeta \geq 0, \tag{71}$$

where b is a bounded continuous function, W is a one dimensional Wiener process and

$$F_0(y(t,\cdot),\zeta) = b(y(t,\zeta))\int_0^{\zeta} b(y(t,r))\mathrm{d}r.$$

This equation is important in financial modelling, see [46]. It provides a description of time evolution of the forward rates under the nonarbitrage assumption. We will study this equation in the following abstract framework. Let $H^\kappa = L^2\left((0,\infty), \rho_\kappa(\zeta)\mathrm{d}\zeta\right)$, where $\rho_\kappa(\zeta) = \mathrm{e}^{-\kappa\zeta}$ with $\kappa > 0$. In particular $H^0 = L^2(\mathbb{R})$. The scalar product and the norm in H^κ will be denoted by $\langle\cdot,\cdot\rangle_\kappa$ and $|\cdot|_\kappa$ respectively. Let

$$A = \frac{\partial}{\partial\zeta}, \quad \mathrm{dom}(A) = H^1_\kappa(0,\infty).$$

Then

$$\mathrm{e}^{tA}x(\zeta) = x(t+\zeta), \quad t,\zeta \geq 0,$$

and it is easy to check that

$$\left\|\mathrm{e}^{tA}\right\|_{H^\kappa \to H^\kappa} \leq \mathrm{e}^{-\kappa t}.$$

We will assume that

$$B : H^\kappa \to H^\kappa, \quad B(x)(\zeta) = b(x(\zeta))$$

is a Lipschitz mapping and the mapping $F : H^\kappa \to H^\kappa$ defined by

$$F(x)(\zeta) = b(x(\zeta))\int_0^{\zeta} b(x(r))\mathrm{d}r,$$

is a Lipschitz mapping as well. Then Eq. (71) may be rewritten as an abstract equation

$$\mathrm{d}y(t) = (Ay(t) + F(y(t)) + Bz(t))\,\mathrm{d}t + B(y(t))\mathrm{d}W(t), \tag{72}$$

where $z(t) \in H^\kappa$ is a control. We need also to consider an uncontrolled equation

$$\mathrm{d}y(t) = (Ay(t) + F(y(t)))\,\mathrm{d}t + B(y(t))\mathrm{d}W(t). \tag{73}$$

The proof of the next lemma is similar to the proof provided in [29] and is thus omitted.

Lemma 6.2. *Assume that*

$$\|b\|_\infty + |b|_\kappa \le c,$$

with $c > 0$ small enough. Then there exists a nondegenerate invariant measure for Eq. (73).

Given the above lemma we can apply the theory of the HJ equation developed in the previous section to study the optimal control problem for Eq. (72). Note that, as for the previous example, in this case $\left(D_Q, \mathcal{D}\right)$ is not closable, see [31] for details. □

Remark 6.3. Using the same framework as in the case of the Musiela equation, we can consider the optimal control of first order equations arising in economic theory (see e.g. [5]) and in the theory of population dynamics (see e.g. [3,38]). □

6.3. Second order SPDE in the whole space

Let $H^\kappa = L^2(\mathbb{R}, \rho_\kappa(\zeta)\mathrm{d}\zeta)$, where $\rho_\kappa(\zeta) = \mathrm{e}^{-\kappa|\zeta|}$ with $\kappa > 0$. In particular $H^0 = L^2(\mathbb{R})$. The scalar product and the norm in H^κ will be denoted by $\langle\cdot,\cdot\rangle_\kappa$ and $|\cdot|_\kappa$ respectively. Fix $m > 0$ and let $A^{(0)} = \Delta - mI$, where Δ is the Laplacian in H^0 and let $S^{(0)}(t)$ denote the semigroup on H^0 generated by $A^{(0)}$. The semigroup $\left(S^{(0)}(t)\right)$ is selfadjoint on H^0 and

$$\left\|S^{(0)}(t)\right\| \le \mathrm{e}^{-mt}. \tag{74}$$

By the results in [23, Section 9.4.1] $\left(S^{(0)}(t)\right)$ can be uniquely extended to a C_0-semigroup $\left(S^{(\kappa)}(t)\right)$ on H^κ with the generator denoted by $A^{(\kappa)}$. Moreover,

$$\left\|S^{(\kappa)}(t)\right\| \le \mathrm{e}^{\left(\frac{1}{2}\kappa^2 - m\right)t}, \quad t \ge 0. \tag{75}$$

We will consider the equation

$$\mathrm{d}y = \left(A^{(\kappa)}y + JF(y)\right)\mathrm{d}t + J\mathrm{d}W, \tag{76}$$

where W is a standard cylindrical Wiener process on $H^{(0)}$ and $J : H^{(0)} \to H^{(\kappa)}$ is an imbedding: $Jx = x$. Moreover, we assume that the Lipschitz mapping $F : H^0 \to H^0$ is bounded.

It was proved in [23] that for any $\kappa > 0$ and $m > 0$ the solution (76) is well defined in H^κ and it admits an invariant measure $\mu = N(0, Q_\infty)$. Moreover, $\ker(Q_\infty) = \{0\}$ for any $\kappa > 0$ and $m > 0$. Then by the recent results in [13] there exists a nondegenerate invariant measure μ^F for y which has a density with respect to μ.

Let us consider a controlled equation

$$\mathrm{d}y(t) = (Ay(t) + JF(y(t)) - Jz(t))\,\mathrm{d}t + J\mathrm{d}W(t),$$

where z is a control taking values in $L^2(\mathbb{R})$. It may be shown that the transition semigroup of this process is never strongly Feller, hence the theory of HJB equations developed in [7–9,32, 33] does not apply in this case. We can apply however all the results of the previous sections to obtain a unique optimal feedback control for the process y.

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References

[1] N.U. Ahmed, Optimal control of ∞-dimensional stochastic systems via generalized solutions of HJB equations, Discuss. Math. Differ. Incl. Control Optim. 21 (1) (2001) 97–126.

[2] N.U. Ahmed, Generalized solutions of HJB equations applied to stochastic control on Hilbert space, Nonlinear Anal. 54 (3) (2003) 495–523.

[3] S. Anita, Analysis and control of age-dependent population dynamics, in: Mathematical Modelling: Theory and Applications, Kluwer Academic Publishers, Dordrecht, 2000.

[4] V. Barbu, G. Da Prato, Hamilton–Jacobi equations in Hilbert spaces, in: Research Notes in Mathematics, Pitman, Boston, 1983.

[5] E. Barucci, F. Gozzi, On capital accumulation in a vintage model, Res. Economics 52 (1998) 159–188.

[6] V. Borkar, T. Govindan, Optimal control of semilinear stochastic evolution equations, Nonlinear Anal. 23 (1) (1994) 15–35.

[7] P. Cannarsa, G. Da Prato, Direct solution of a second order Hamilton–Jacobi equation in Hilbert spaces, in: G. Da Prato, L. Tubaro (Eds.), Stochastic Partial Differential Equations and Applications, in: Pitman Research Notes in Mathematics, vol. 268, 1992, pp. 72–85.

[8] P. Cannarsa, G. Da Prato, Second order Hamilton–Jacobi equations in infinite dimensions, SIAM J. Control Optim. 29 (2) (1991) 474–492.

[9] S. Cerrai, Optimal control problems for stochastic reaction-diffusion systems with non-Lipschitz coefficients, SIAM J. Control Optim. 39 (2001) 1779–1816.

[10] S. Cerrai, Stationary Hamilton–Jacobi equations in Hilbert spaces and applications to a stochastic optimal control problem, SIAM J. Control Optim. 40 (2001) 824–852.

[11] S. Cerrai, F. Gozzi, Strong solutions of Cauchy problems associated to weakly continuous semigroups, Differential Integral Equations 8 (3) (1995) 465–486.

[12] A. Chojnowska-Michalik, Representation theorem for general stochastic delay equations, Bull. Acad. Pol. Sci. Ser. Sci. Math. 26 (7) (1978) 634–641.

[13] A. Chojnowska-Michalik, Transition semigroups for stochastic semilinear equations on Hilbert spaces, Dissertationes Math. 396 (2001) 59 pages.

[14] V.S. Borkar, R.T. Chari, S.K. Mitter, Stochastic quantization of field theory in finite and infinite volume, J. Funct. Anal. 81 (1) (1988) 184–206.

[15] P.L. Chow, J.L. Menaldi, Infinite dimensional Hamilton–Jacobi–Bellman equations in Gauss–Sobolev spaces, Nonlinear Anal. 29 (4) (1997) 415–426.

[16] M.G. Crandall, H. Ishii, P.L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. (New Series) A.M.S. 27 (1) (1992) 1–67.

[17] G. Da Prato, Some results on Bellman equation in Hilbert spaces, SIAM J. Control Optim. 23 (1985) 61–71.

[18] G. Da Prato, A. Debussche, Control of the stochastic Burgers model of turbulence, SIAM J. Control Optim. 37 (4) (1999) 1123–1149.

[19] G. Da Prato, A. Debussche, Differentiability of the transition semigroup of the stochastic Burgers equation, and application to the corresponding Hamilton–Jacobi equation, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 9 (4) (1998) 267–277.

[20] G. Da Prato, A. Debussche, Dynamic programming for the stochastic Navier–Stokes equations, Math. Model. Numer. Anal. 34 (2) (2000) 459–475. (Special issue for R. Temam's 60th birthday).

[21] G. Da Prato, J. Zabczyk, Regular densities of invariant measures in Hilbert spaces, J. Funct. Anal. 130 (2) (1995) 427–449.

[22] G. Da Prato, J. Zabczyk, Stochastic equations in infinite dimensions, in: Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge (UK), 1992.

[23] G. Da Prato, J. Zabczyk, Ergodicity for Infinite Dimensional Systems, in: London Mathematical Society Lecture Note Series, vol. 229, Cambridge University Press, 1996.

[24] G. Da Prato, J. Zabczyk, Second Order Partial Differential Equations in Hilbert Spaces, in: London Mathematical Society Lecture Note Series, vol. 293, Cambridge University Press, 2002.

[25] G. Da Prato, J. Zabczyk, Differentiability of the Feynman–Kac semigroup and a control application, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 8 (3) (1997) 183–188.

[26] W.H. Fleming, H.M. Soner, Controlled Markov Processes and Viscosity Solutions, Springer-Verlag, Berlin, New-York, 1993.

[27] M. Fuhrman, G. Tessitore, Nonlinear Kolmogorov equations in infinite dimensional spaces: the backward stochastic differential equations approach and applications to optimal control, Ann. Probab. 30 (3) (2002) 1397–1465.

[28] M. Fuhrman, G. Tessitore, The Bismut–Elworthy formula for Backward SDEs and applications to nonlinear Kolmogorov equations and control in infinite dimensional spaces, Stoch. Stoch. Rep. 74 (1–2) (2002) 429–464.

[29] B. Goldys, M. Musiela, Lognormality of rates and term structure models, Stoch. Anal. Appl. 18 (2000) 375–396.

[30] B. Goldys, B. Maslowski, Ergodic control of semilinear stochastic equations and Hamilton–Jacobi equations, J. Math. Anal. Appl. 234 (2) (1999) 592–631.

[31] B. Goldys, F. Gozzi, J.M.A.M. Van Neerven, On closability of directional gradients, Potential Anal. 18 (2003) 289–310.

[32] F. Gozzi, Regularity of solutions of a second order Hamilton–Jacobi equation and application to a control problem, Comm. Partial Differential Equations 20 (5 & 6) (1995) 775–826.

[33] F. Gozzi, Global regular solutions of second order Hamilton–Jacobi equations in Hilbert spaces with locally Lipschitz nonlinearities, J. Math. Anal. Appl. 198 (1996) 399–443.

[34] F. Gozzi, E. Rouy, Regular solutions of second order stationary Hamilton–Jacobi equation, J. Differential Equations 130 (1) (1996) 201–234.

[35] F. Gozzi, E. Rouy, A. Swiech, Second order Hamilton–Jacobi equation in Hilbert spaces and stochastic boundary control, SIAM J. Control Optim. 38 (2) (2000) 400–430.

[36] F. Gozzi, A. Swiech, Hamilton–Jacobi–Bellman equations for the optimal control of the Duncan–Mortensen–Zakai equation, J. Funct. Anal. 172 (2) (2000) 466–510.

[37] T. Havarneanu, Existence for the dynamic programming equation of control diffusion processes in Hilbert space, Nonlinear Anal. 9 (1985) 619–629.

[38] M. Iannelli, Mathematical problems in the description of age structured populations, in: Mathematics in Biology and Medicine (Bari, 1983), 19–32, in: Lecture Notes in Biomath., vol. 57, Springer, Berlin, 1985.

[39] H. Ishii, On uniqueness and existence of viscosity solutions of fully nonlinear second-order elliptic PDE's, Comm. Pure Appl. Math. 42 (1989) 15–45.

[40] H. Ishii, Viscosity solutions of nonlinear second-order partial differential equations in Hilbert spaces, Comm. Partial Differential Equations 18 (1993) 601–651.

[41] K. Ito, M. Nisio, On stationary solutions of a stochastic differential equation, J. Math. Kyoto Univ. 4 (1) (1964) 1–75.

[42] M. Kocan, A. Swiech, Second order unbounded parabolic equations in separated form, Studia Math. 115 (1995) 291–310.

[43] P.-L. Lions, Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions. Part I: The case of bounded stochastic evolution, Acta Math. 161 (1988) 243–278;
G. Da Prato, L. Tubaro (Eds.), Part II: Optimal Control of Zakai's equation, in: Lecture Notes in Mathematics, vol. 1390, Springer-Verlag, Berlin, 1989, pp. 147–170; Part III: Uniqueness of viscosity solutions for general second order equations, J. Funct. Anal. 86 (1989) 1–18.

[44] Z.M. Ma, M. Röckner, Introduction to the Theory of (Non Symmetric) Dirichlet Forms, Springer-Verlag, 1992.

[45] C. Marinelli, On stochastic modelling and optimal control in advertising. Ph.D. Thesis, Columbia University, 2004.

[46] M. Musiela, Stochastic PDEs and term structure models, in: Journées Internationales de Finance, IGR-AFFI, La Baule, 1993.

[47] S. Peszat, J. Zabczyk, Strong Feller property and irreducibility for diffusions on Hilbert Spaces, Ann. Probab. 23 (1995) 157–172.

[48] R.R. Phelps, Gaussian null sets and differentiability of Lipschitz maps on Banach spaces, Pacific. J. Math. 77 (1978) 523–531.

[49] M. Scheutzow, Qualitative behavior of stochastic delay equations with a bounded memory, Stochastics 12 (1984) 41–80.
[50] A. Swiech, Unbounded second order partial differential equations in infinite dimensional Hilbert spaces, Comm. Partial Differential Equations 19 (1994) 1999–2036.
[51] J. Yong, X.Y. Zhou, Stochastic Control: Hamiltonian Systems and HJB Equations, Springer-Verlag, 1999.
[52] J. Zabczyk, Parabolic equations in Hilbert spaces, in: N. Krylov, M. Röckner, J. Zabczyk (Eds.), Stochastic PDE's and Kolmogorov Equations in Infinite Dimensions, in: LNM, vol. 1715, Springer, 1999.