Global Existence and Gradient Estimates for the Quasilinear Parabolic Equations of $m$-Laplacian Type with a Nonlinear Convection Term

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Received December 28, 1998; revised April 19, 1999

In this paper, we derive precise estimates for $u(t)$ including smoothing effects near $t=0$ and decay as $t\to\infty$, as well as global existence of the solutions $u(t)$ to the initial-boundary value problem in a bounded domain in $\mathbb{R}^n$ for the quasilinear parabolic equation of the $m$-Laplacian type with a nonlinear convection term $b(u)\nabla u$. For the initial data $u_0$ we only assume $u_0 \in L^q(\Omega)$, $1 < q < \infty$. © 2000 Academic Press

Key Words: quasilinear parabolic equation; gradient estimate; global existence; convection.

1. INTRODUCTION

In this paper we are concerned with global solutions to the initial-boundary value problem for nonlinear parabolic equations:

$$u_t - \text{div} \left\{ \sigma(\|u\|^2) \nabla u \right\} + b(u) \cdot \nabla u = 0, \quad x \in \Omega, \quad t > 0, \quad (1.1)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega; \quad u(x, t) = 0, \quad x \in \partial \Omega, \quad t > 0, \quad (1.2)$$
where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with a smooth, say, $C^2$ boundary $\partial \Omega$, $\sigma(|Vu|^2)$ is a function like $\sigma(|Vu|^2) = |Vu|^m$, $m \geq 0$, and $b(u)$ is a nonlinear vector field such that

$$|b(u)| \leq k_0 |u|^{\beta}$$  \hspace{1cm} (1.3)

with some $\beta \geq 0$ and $k_0 > 0$.

When $b(u) \equiv 0$, our equation is one of the most typical nonlinear parabolic equations and has been investigated from various points of view. Concerning gradient estimates, we know, for example, that if $u_0 \in L^1(\Omega)$, the problem (1.1)–(1.2) admits a unique solution $u(t)$ in the class

$$L^m_{loc}((0, \infty); W^{1,m+2}(\Omega)) \cap W^{1,2}_{loc}((0, \infty); L^2(\Omega)) \cap C([0, \infty); L^1(\Omega))$$

satisfying

$$\|\nabla u(t)\|_{m+2} \leq C_0(\|u_0\|_1)(1+t)^{-1/m} (1+t^{-\mu}), \hspace{1cm} t > 0, \hspace{1cm} (1.4)$$

where $\mu = (1+\lambda)/(m+2)$ and $\lambda = N/(mN + m + 2)$. (Cf. Alikakos and Rostamian [2], Nakao [10, 11].) The estimate (1.4) follows from a multiplier technique. Further, under a certain geometrical condition on $\partial \Omega$, we can prove

$$\|\nabla u(t)\|_{\infty} \leq C_0(1+t)^{-1/m} (1+t^{-\xi}), \hspace{1cm} t > 0, \hspace{1cm} (1.5)$$

with $\xi = (N + 2\mu)/(2m + 4 + mN)$ (cf. Alikakos and Rostamian [2, 3]).

We note that the estimates (1.4)–(1.5) show certain smoothing effects near $t = 0$ and decay property as $t \to \infty$.

The object of this paper is to derive precise gradient estimates as well as global existence of solutions for the perturbed problem (1.1)–(1.2) with initial data $u_0 \in L^q(\Omega), \ q \geq 1$, which generalizes (1.5). We note that the principal term $-\text{div}\{\sigma(|Vu|^2) \nabla u\}$ generates nonlinear semigroup in any $L^q(\Omega)$, while the term $b(u) \cdot Vu$ generates a nonlinear semi-group only in $L^1(\Omega)$. So, it seems to be difficult to apply nonlinear semi-group theory to our problem. Even if we could apply a nonlinear semi-group theory to our problem we would know the existence of a weaker solution and at most the estimate

$$\|u(t)\|_q \leq C_1^{-1} \|u_0\|_q,$$

which is not sufficient for our purpose.

Physically, the term $b(u) \cdot Vu$ describes an effect of convection with a velocity field $b(u)$. Some linear parabolic equation or nonlinear equations of porous medium type with such a perturbation have been investigated by several authors (cf. Gilding and Peletier [8], Nakao [13], Escobedo,
Vazquez, and Zuazua [6], Escobedo and Zuazua [7], Zuazua [21], Okamoto and Oharu [18], etc. and some of techniques seem to be useful for our problem too. But, there seems to be very little results on m-Laplacian type diffusion-convection equations.

Quite recently, in [4] we have treated the problem (1.1)-(1.2) with \( b(u) \cdot \nabla u \) replaced by a stronger perturbation \( g(\nabla u) \) with \( |g(\nabla u)| \leq C |\nabla u|^\beta \), \( \beta > m \). We have proved in [4] some global existence theorems and some estimates for \( \|\nabla u(t)\|_\infty \) under the assumptions that the mean curvature \( H(x) \) of \( \partial \Omega \) is nonpositive and \( \|\nabla u_0\|_p \) is small for certain \( p_0 \geq m + 2 \). Such a geometric condition was introduced by Serrin [19] for the study of the quasilinear elliptic problems of the mean curvature type and employed also in [2, 5, 15, 17], etc. Our perturbation \( b(u) \cdot \nabla u \) is weaker than \( g(\nabla u) \), but, we prove here a global existence, uniqueness and gradient estimates of solutions with weaker initial data \( u_0 \# L^q, q \geq 1 \). It should be noted that we make no smallness and differentiability condition on \( u_0 \) nor any geometrical assumption on the boundary \( \partial \Omega \). Our result seems to be new even for the non-perturbed case \( b(u) \equiv 0 \).

To derive precise estimates for \( \nabla u(t) \), in particular, \( \|\nabla u(t)\|_\infty \) we must treat the perturbation term \( b \cdot \nabla u \) very carefully, which is the main task of this paper. Also we must treat carefully a boundary integral, which is not needed if we make the geometrical condition on \( \partial \Omega \).

2. PRELIMINARIES AND STATEMENT OF RESULTS

The function spaces we use are all familiar and the definition of them are omitted. But, we note that \( \| \cdot \|_p \) and \( \| \cdot \|_{1,p} \), \( 1 \leq p \leq \infty \) denote \( L^p(\Omega) \) and \( W^{1,p}(\Omega) \) norms respectively. We often drop the letter \( \Omega \) in these notations.

Let us state our precise assumption on \( \sigma(\cdot) \) and \( b(u) \).

Hyp. A. \( \sigma(\cdot) \) is a continuous function on \( R^+ = [0, \infty) \) and satisfies the conditions:

\[
k_0 |x|^m \leq \sigma(x^2) \leq k_1 |x|^m \tag{2.1}
\]

and

\[
(\sigma(|\xi|^2) \xi - \sigma(|\eta|^2) \eta, \xi - \eta) \geq 0 \tag{2.2}
\]

for \( \forall \xi, \eta \in R^N \).
Hyp. B. \( b(u) = (b_1(u), ..., b_N(u)) \) is an \( \mathbb{R}^N \)-valued function on \( \mathbb{R}^1 \), satisfying
\[
|b(u)| \leq k_1 |u|^{\beta}
\] (2.3)
for some \( \beta \geq 0 \).

**Definition 1.** We say a measurable function \( u(x, t) \) on \( \Omega \times \mathbb{R}^+ \) to be a solution of the problem (1.1) iff \( u \in C(R^+; \mathcal{H}(\Omega)) \cap L^\infty((0, \infty); L^\infty(\Omega)), \sigma(|\nabla u|^2) \nabla u \in L^1_{loc}((0, \infty); L^1(\Omega)), u(0) = u_0 \) and the equality
\[
\int_0^\infty \int_{\Omega} \{ -u(t) \varphi', t^\sigma(|\nabla u|^2) \nabla u \nabla \varphi + B(u) \nabla \varphi \} \, dx \, dt = 0
\] (2.4)
holds for all \( \varphi \in C^1_0((0, \infty); \mathcal{C}^1_0(\Omega)) \), where we set
\[
B(u) = \int_0^u b(s) \, ds.
\]

Our first result reads as follows.

**Theorem 1.** Let \( u_0 \in L^q, q \geq 1 \). Then, under Hyp. A and B, the problem (1.1)–(1.2) admits a unique solution \( u(t) \) in the class
\[
L^\infty((0, \infty); W^{1, m+2}_0) \cap W^{1, 2}_{loc}((0, \infty); L^2) \cap C(R^+; L^1) \cap L^\infty(R^+; L^q),
\] (2.5)
satisfying
\[
\|u(t)\|_{m+2} \leq C_0 (1 + t)^{-\frac{1}{m+2}} (1 + t^{-\lambda}), \quad t > 0,
\] (2.6)
and
\[
\|u(t)\|_{\infty} \leq C_0 (1 + t)^{-\frac{1}{m+2}} (1 + t^{-\lambda}), \quad t > 0,
\] (2.7)
where we set
\[
\lambda = N/(mN + q(m + 2)), \quad \sigma = (2\beta - m - mqN)\lambda, \quad \mu = \frac{1 + 2(\sigma - 1) + (2 - q)\lambda}{m+2},
\] (2.8) (2.9) (2.10)
Remark. When \( m = 0 \) the term \((1 + t)^{-m}\) should be replaced by \( e^{-\lambda_0 t} \) with some \( \lambda_0 > 0 \). This applies to other statements below.

To state our main result concerning the estimates for \( \|\nabla u(t)\|_\infty \), we need a regularity assumption.

Hyp. \( \tilde{A} \). \( \sigma(\cdot) \) belongs to \( C(R^+) \cap C'(0, \infty) \).

Remark. Under Hyp. A and Hyp. \( \tilde{A} \) we see

\[
\sigma'(v^2) \geq 0, \quad v \neq 0. \tag{2.11}
\]

Under the following the main result would be a little improved.

Hyp. C. When \( N \geq 2 \), \( \partial \Omega \) is of \( C^2 \)-class and the mean curvature \( H(x) \) of \( \partial \Omega \) at \( x \in \partial \Omega \) with respect the outward normal is nonpositive.

**Theorem 2.** Under the hypotheses Hyp. A, B, and \( \tilde{A} \), the solution \( u(t) \) in Theorem 1 further belongs to \( L^p_{\text{loc}}((0, \infty); W^{1,p}_0) \) for any \( p > m + 2 \), and satisfies

\[
\|\nabla u(t)\|_p \leq C_p(1 + t)^{-\theta} (1 + t^{-\xi_p}), \quad t > 0, \tag{2.12}
\]

for all \( p \geq m + 2 \), where \( C_p \) is a constant depending on \( \|u_0\|_q \) and \( p \), where

\[
v = \min \left\{ \frac{1}{m}, \frac{\beta}{m^2} \right\}. \tag{2.13}
\]

and

\[
\xi_p = \frac{1}{p + m - p\theta} \left\{ \frac{(p + m)(1 - \theta)\mu}{m + 2} + \max \left\{ \theta, \frac{(p + m) \hat{a} - (p + m - pm) \ p^{-1}}{1} \right\} \right\}, \tag{2.14}
\]

with

\[
\theta = \frac{p + m}{2} \left( \frac{1}{m + 2} \frac{1}{p} \frac{1}{N} \frac{1}{2m + 4} \right)^{-1} \tag{2.15}
\]

and

\[
\hat{a} = \max \{ a, \mu \}.
\]

Under the additional assumption Hyp. C the above result holds with \( \hat{a} \) replaced by \( a \).
Here, we note that
\[
\lim_{p \to \infty} \xi_p = \frac{2 \mu + N \max\{1, \tilde{\alpha}\}}{mN + 2m + 4} = \xi_{\infty}. \tag{2.16}
\]

Our main result is the following.

**Theorem 3.** Under the same hypotheses as in Theorem 2, the solutions \(u(t)\) in Theorem 1 belong in fact to \(L^\text{loc}_{\infty}(0, \infty; W^{1,0})\) and satisfy the estimates:

\[
\|\nabla u(t)\|_{\infty} \leq Ct^{-\tilde{\alpha}+}, \quad 0 < t \leq 1, \quad \text{if } \tilde{\alpha} \leq 1,
\]
and

\[
\|\nabla u(t)\|_{\infty} \leq C_s t^{-\tilde{\alpha}+}, \quad 0 < t \leq 1, \quad \text{if } \tilde{\alpha} > 1,
\]

where \(\varepsilon\) is an arbitrarily small positive number and \(C_s\) is a constant depending on \(\|u_0\|_q\) and \(\tilde{\alpha}\).

Further, we have

\[
\|\nabla u(t)\|_{\infty} \leq C(1 + t)^{-\tilde{\gamma}}, \quad 1 \leq t,
\]

with

\[
\tilde{\gamma} = \min\left\{1, \frac{2\beta - m}{m} \right\}.
\]

If we make the further assumption Hyp. C, we can take \(\varepsilon\) for \(\tilde{\alpha}\) in the definition of \(\xi_{\infty}\).

For the proofs of Theorems we use the following Lemmas.

**Lemma 1 (Gagliardo–Nirenberg).** Let \(\beta \geq 0, N > p \geq 1, \beta + 1 \leq q\) and \(1 \leq r \leq q \leq (\beta + 1)Np/(N - p)\). Then for \(u\) such that \(|u|^r u \in W^1_r(\Omega)\), we have

\[
\|u\|_q \leq C^{1/(\beta + 1)} \|u\|^{1-q}_r \|u\|^{\beta}_r \|u\|^{(\beta+1)}_{1,p} \tag{2.20}
\]

with \(\theta = (\beta + 1)(r^{-1} - q^{-1})/(N^{-1} - p^{-1} + (\beta + 1) r^{-1})\), where \(C\) is a constant independent of \(q, r, \beta\) and \(\theta\) if \(N \neq p\), and a constant depending on \(q/(\beta + 1)\) if \(N = p\).
Lemma 2. Let $y(t)$ be a nonnegative differentiable function on $(0, T]$ satisfying
$$y'(t) + A t^\theta - 1 y_1 + \theta (t) \leq B t^{-k} y(t) + C t^{-\delta}$$
with $A, \theta > 0, \lambda \theta \geq 1, B, C \geq 0, k \leq 1$. Then, we have
$$y(t) \leq A^{-1/\theta} (2 \lambda + 2 BT^{1-k}) t^{-\lambda} + 2 C (\lambda + BT^{1-k})^{-1} t^{-\delta}, \quad 0 < t \leq T.$$
For a proof of Lemma 2 see Ohara [16]. The following is also useful.

Lemma 3. Let $y(t)$ be a differentiable function on $[1, \infty)$ such that
$$y'(t) + A t^\mu y_1 + \mu (t) \leq B t^{-k} \quad 1 \leq t,$$
for some $A, B, \mu > 0, k \geq 0$. Then
$$y(t) \leq C t^{-\gamma} \quad t \geq 1$$
with some $C > 0$ independent of $y$ and $\gamma = \min \{(1 + \mu)/\theta, (\mu + k)/(1 + \theta)\}$.

The proof of Lemma 3 is elementary and omitted.

To derive the estimates for $\|u(t)\|_\infty$ and $\|\nabla u(t)\|_\infty$, we employ Moser’s technique, and for this we prepare the following lemma.

Lemma 4. Let $p_1 \geq 1$ and define $p_n$ inductively by
$$p_n = R p_{n-1} - m$$
with $R > 1, m > 0$. Further, we set
$$\theta_n = NR (1 - p_{n-1}^{-1} p_n^{-1}) (N(R - 1) + r)^{-1}$$
and
$$\beta_n = (p_n + m) \theta_n^{-1} - p_n,$$
for $n = 2, 3, \ldots$. where $r > 0$. Finally, for given $\lambda_1 \geq 0$, we define $\{\lambda_n\}$ by
$$\lambda_n = (1 + \lambda_{n-1}(\beta_n - m)) \beta_n^{-1},$$
for $n = 2, 3, \ldots$. Then, we have
$$\lim_{n \to \infty} \lambda_n = \frac{p_1 \lambda_1 r + N}{p_1 r + m N}$$
Proof. The defining equation of $\lambda_n$ can be rewritten as

$$\lambda_n - \frac{1}{m} = \frac{\beta_n - m}{\beta_n} \left( \lambda_{n-1} - \frac{1}{m} \right), \quad n = 2, 3, \ldots$$

Here, setting $w_n = p_n r + mN$, we see by a direct calculation

$$\frac{\beta_n - m}{\beta_n} = \frac{p_{n-1} w_n}{p_n w_{n-1}}.$$ 

Therefore,

$$\lambda_n - \frac{1}{m} = \frac{w_n p_1}{p_n w_1} \left( \hat{\lambda}_1 - \frac{1}{m} \right).$$

From this, we have that

$$\lim_{n \to \infty} \lambda_n = \frac{p_1 r}{w_1} \left( \hat{\lambda}_1 - \frac{1}{m} \right) + \frac{1}{m} = \frac{p_1 \hat{\lambda}_1 r + N}{p_1 r + mN}.$$ 

This completes the proof of Lemma 4. \[\square\]

3. PROOF OF THEOREM 1

Let $\sigma_\varepsilon(x^2) = \sigma(x^2 + \varepsilon)$, $\varepsilon > 0$, and consider the approximate problem

$$u_{\varepsilon} - \text{div} \left\{ \sigma_\varepsilon(|\nabla u|) \nabla u \right\} + b u \nabla u = 0, \quad x \in \Omega, \quad t > 0, \quad (3.1)$$

$$u(x, 0) = u_{0, \varepsilon}(x), \quad x \in \Omega; \quad u(x, t) = 0, \quad x \in \partial \Omega, \quad t > 0, \quad (3.2)$$

where $u_{0, \varepsilon}$ is a function in $C_0^3(\Omega)$ such that $\lim_{\varepsilon \to 0} u_{0, \varepsilon} = u_0$ in $L^q$.

The problem (3.1)-(3.2) is a standard quasilinear parabolic equation and admits a unique smooth solution $u_{\varepsilon}(t)$ on $[0, \infty)$ for each $\varepsilon$. (See Ladyzenskaya, Solonnikov, and Uraltseva [9]). We shall derive various estimates for $u_{\varepsilon}(t)$.

**Proposition 1.** For $u = u_{\varepsilon}(t)$, we have

$$\|u(t)\| \leq \|u_{0, \varepsilon}\|, \quad 0 \leq t < \infty,$$ 

and

$$\|u(t)\| \leq C(\|u_0\|) \frac{t}{t-1}, \quad 0 < t \leq 1.$$
where we set

\[ \lambda = N/(mN + q(m + 2)) \]

and \( C(\|u_0\|_q) \) denote a constant depending on \( \|u_0\|_q \), but, independent of \( \epsilon \).

**Proof.** The estimate (3.3) is standard. Indeed, roughly speaking, this follows by multiplying the equation by \( |u|^\gamma - 2 u \) and integrating by parts, in which we use the fact

\[ \int_{\Omega} \mathbf{b}(u) \cdot \nabla u^\gamma dx = 0. \]

To prove (3.4), we again use the same technique with \( q \) replaced by \( p \geq 2 \) to get

\[ \frac{1}{p} \frac{d}{dt} \|u(t)\|_p^p + C \|u(\gamma + m)/(m + 2)\|_{1, m + 2} \leq 0 \]  

(3.5)

for some \( C > 0 \), where we sometimes write \( \alpha' \) for \( |u|^\gamma - 1 u \), \( \gamma > 0 \). This situation is the same as in the equation without perturbation and the result is known (cf. Veron [20]). For convenience of the readers, however, we sketch the proof.

We take \( p_1 = q \) and \( p_n = (m + 2)p_{n-1} - m, \ n = 2, 3, \ldots \) Then, by Gagliardo–Nirenberg inequality,

\[ \|u\|_{p_n} \leq C(m^2 + 1)(p_n + m) \|u\|^{1 - \theta_n} \|u(\gamma + m)/(m + 2)\|_{1, m + 2}^{(m + 2)\theta_n}(p_n + m) \]  

(3.7)

with

\[ \theta_n = (m + 2)N(1 - p_{n-1}p_n^{-1})/(m + 2 + N(m + 1)). \]

It follows from (3.5) with \( p = p_n \) and (3.7) that

\[ \frac{d}{dt} \|u(t)\|_{p_n} + C(m + 2)\theta_n p_n^{-1} - m \|u\|_{p_n} \|u\|_{p_n}^{1 + \theta_n} \leq 0 \]  

(3.9)

where we set

\[ \theta_n = (p_n + m)\theta_n^{1 - p_n}. \]

We claim from (3.3) and (3.9) that there exist a bounded sequence \( \{\eta_n\} \) and a convergent sequence \( \{\lambda_n\} \) such that

\[ \|u(t)\|_{p_n} \leq \eta_n {t}^{-\lambda_n}, \quad 0 < t \leq 1. \]  

(3.10)
Indeed, by induction we can prove (3.10) with
\[ \hat{\lambda}_n = (1 + \hat{\lambda}_{n-1} (\beta_n - m)/\beta_n), \quad \text{and} \quad \eta_n = \eta_{n-1} (C^{m/\beta_n} \hat{\lambda}_n p_n^{m+1} \sqrt[\beta_n]{m}) . \]
(3.11)

Applying Lemma 4 to our \( \{ \hat{\lambda}_n \} \), we obtain \( \lim_{n \to \infty} \hat{\lambda}_n = \hat{\lambda} \). It is not difficult to show that \( \{ \eta_n \} \) is bounded. \[ \blacksquare \]

**Proposition 2.** For \( u = u_j(t) \), we have
\[ \| u(t) \|_\infty \leq C (1 + t)^{-1/m}, \quad t \geq 1. \] (3.12)

**Proof.** The proof is again standard. Taking \( p = 2 \) in (3.5), we have
\[ \frac{d}{dt} \| u(t) \|_2^2 + C \| \nabla u(t) \|_m^2 \leq 0 \] (3.13)
and
\[ \frac{d}{dt} \| u(t) \|_2^2 + C \| u(t) \|_m^2 \leq 0 \] (3.14)
which implies
\[ \| u(t) \|_2 \leq C (1 + t)^{-1/m}, \quad t \geq 1. \] (3.15)

Now, setting \( \tau = \log(1 + t), t \geq 1 \), and \( W(\tau) = (1 + t)^{1/m} u(t) \), and substituting this into (3.5), we obtain
\[ \frac{1}{p} \frac{d}{dt} \| W(\tau) \|_p^p + C \| \nabla W(\tau) \|_m^{p+2} \| W(\tau) \|_m^2 \leq \frac{1}{m} \| W(\tau) \|_p^p, \quad \tau \geq \log 2. \] (3.16)

This implies (see Alikakos [1]) that
\[ \sup_{t \geq 1} \| u(t) \|_\infty (1 + t)^{1/m} = \sup_{\tau \geq \log 2} \| W(\tau) \|_\infty \]
\[ \leq C \max \{ 1, \sup_{\tau \geq \log 2} \| W(\tau) \|_2, \| W(\log 2) \|_\infty \} \]
\[ = C \max \{ 1, \sup_{t \geq 1} \| u(t)(1 + t)^{1/m} \|_2, \| u(1) \|_\infty \} < \infty. \]
This is (3.13). \[ \blacksquare \]

Let us proceed to the estimation of \( \nabla u_j(t) \).
PROPOSITION 3.3. For \( u = u_t(x) \), we have
\[
\|\nabla u(t)\|_{m+2} \leq C_0 t^{-\mu}, \quad 0 < t \leq 1,
\]  
(3.17)
where we denote by \( C_0 \) constants depending on \( \|u_0\|_q \) and we recall \( \mu = \{1 + 2(x-1)^+ + (2-q)^+ \lambda\}/(m+2) \) and \( \lambda = \hat{\lambda}(2\beta - m - mg/N)^+ \).

Proof. We employ a technique used in Nakao [11]. Multiplying (3.1) by \( u_t \), we see
\[
\frac{1}{2} \frac{d}{dt} \Gamma_\lambda(|\nabla u|^2) + \|u_t(t)\|_2^2 = -\int_\Omega b(u) \cdot \nabla u_t \, dx
\]  
(3.18)
where
\[
\Gamma_\lambda(|\nabla u|^2) = \int_0^{\|\nabla u\|^2} \sigma_\lambda(s) \, ds \, dx.
\]
Hence, by Hyp. B,
\[
\int_\Omega |b(u)| \, |\nabla u| \, |u_t| \, dx \leq \frac{1}{2} \|u_t(t)\|_2^2 + \frac{k_1}{2} \int_\Omega |u|^{2p} \, |\nabla u|^2 \, dx.
\]  
(3.19)
Thus we have from (3.18)-(3.19) that
\[
\frac{d}{dt} \Gamma_\lambda(|\nabla u|^2) + \|u_t(t)\|_2^2 \leq C \int_\Omega |u|^{2p} \, |\nabla u|^2 \, dx.
\]  
(3.20)
Next, multiplying (3.1) by \( u \), we have
\[
\Gamma_\lambda(|\nabla u|^2) \leq C \int_\Omega \sigma_\lambda(|\nabla u|^2) \, |\nabla u|^2 \, dx
\]  
\[
= -\int_\Omega u_t u \, dx \leq C \|u_t\|_2 \|u\|_2.
\]  
(3.21)
Thus, it follows from (3.20) and (3.21) that
\[
\Gamma_\lambda(|\nabla u|^2) \leq C \left( C \int_\Omega |u|^{2p} \, |\nabla u|^2 \, dx - \frac{d}{dt} \Gamma_\lambda \right)^{1/2} \|u(t)\|_2,
\]
which is equivalent to
\[
\frac{d}{dt} \Gamma_\lambda(|\nabla u|^2) + C \|u(t)\|_2^2 \Gamma_\lambda^2(|\nabla u|^2) \leq C \int_\Omega |u|^{2p} \, |\nabla u|^2 \, dx.
\]  
(3.22)
Let us estimate the right hand side of (3.22). We see
\[
\int |u|^{2\beta} |\nabla u|^{2} \, dx \leqslant \|u(t)\|_{\infty}^{2} \int \int |u|^{2\beta-a} |\nabla u|^{2} \, dx
\leqslant \|u(t)\|_{\infty}^{2} \left( \int |u|^{(2\beta-a)(m+2)/m} \, dx \right)^{m(m+2)/m} \|\nabla u\|_{m+2}^{2},
\] (3.23)
for any \(a\) with \(0 < a < \beta\).

First, we consider the case \(2\beta > m+m/N\). We take
\[a = (2\beta - m - mq/N)^+.
\]
Then, by Gagliardo–Nirenberg inequality, we have
\[
\left( \int |u|^{(2\beta-a)(m+2)/m} \, dx \right)^{m(m+2)/m} \leqslant \|u(t)\|_{p_{0}}^{(2\beta-a)(1-\theta)} \|\nabla u\|_{m+2}^{m},
\] (3.24)
where we should take
\[
\theta = p_{0}^{-1} - m(2\beta-a)^{-1} (m+2)^{-1},
\]
and
\[p_{0} = (2\beta - m-a) N/m.
\]
We note that \(p_{0} = q\) if \(2\beta \geqslant m+mq/N\), and \(p_{0} = (2\beta - m) N/m \geqslant 1\) if \(m+N/m \leqslant 2\beta \leqslant m+Nq/m\). Then, we obtain from (3.24) that
\[
\left( \int |u|^{(2\beta-a)(m+2)/m} \, dx \right)^{m(m+2)/m} \|u(t)\|_{\infty}^{a} \leqslant C_{0} t^{-\theta} \|\nabla u(t)\|_{m+2}^{m},
\]
and hence, from (3.23), that
\[
\int |u|^{2\beta} |\nabla u|^{2} \, dx \leqslant ct^{-\theta} \Gamma_{a}^{2}\|\nabla u\|^{2}.
\] (3.25)
Thus we obtain from (3.22) that
\[
\frac{d}{dt} \Gamma_{a}(t) + C t^{(2-\theta)+} \Gamma_{a}^{2}(t) \leqslant C t^{-\theta} \Gamma_{a}(t)
\]
where we set for simplicity \(\Gamma_{a}(t) = \Gamma_{a}(\|\nabla u(t)\|^{2})\).
By Young’s inequality, we have
\[ \frac{d}{dt} \Gamma_a(t) + C t^{2(2-q)^*} \Gamma_a^2(t) \leq C t^{-2s - 2(2-q)^*}, \quad 0 < t \leq 1. \] (3.26)

Applying Lemma 2 to (3.26), we obtain
\[ \Gamma_a(t) \leq C_0 t^{-(1 + 2(2-q)^* + (2-q)^*)}, \quad 0 < t \leq 1, \]
which implies (3.17).

Secondly we consider the case $2 \beta < m + m/N$. If $m \leq 2 \beta < m + m/N$, we easily see
\[ \int_\Omega |u|^{2 \beta} |Vu|^2 \, dx \leq C \|u(t)\|^{2 \beta(1 - \theta)} \|Vu(t)\|^{2 \beta + 2} \leq C_0 \|Vu(t)\|^{m+2}, \quad 0 < t \leq 1 \]
where
\[ \theta = \left(1 - \frac{m}{2 \beta(m + 2)}\right) \left(\frac{1}{N} - \frac{1}{m + 2} + 1 \right)^{-1} \left(\frac{m}{2 \beta}\right). \]

If $2 \beta \leq m$, we see
\[ \int_\Omega |u|^{2 \beta} |Vu|^2 \, dx \leq C \|Vu(t)\|^{2 \beta + 2} \leq C \|Vu(t)\|^{m+2} + 1. \] (3.25)

Thus, we obtain, instead of (3.26),
\[ \frac{d}{dt} \Gamma_a(t) + C t^{2(2-q)^*} \Gamma_a^2(t) \leq C_0 (\Gamma_a(t) + 1), \quad 0 < t \leq 1, \] (3.26)’
which yields again (3.17) in the case under consideration.

Next, we shall show the decay estimate of $\|Vu(t)\|_{m+2}$ as $t \to \infty$.

**Proposition 4.** For $u = u_*(x, t)$, we have
\[ \|Vu(t)\|_{m+2} \leq C(1 + t)^{-v}, \quad t \geq 1. \] (3.27)
with $v = \min\{1/m, \beta/m^2\}$.

**Proof.** We return to the inequality (3.22). Since
\[ \int_\Omega |u|^{2 \beta} |Vu|^2 \, dx \leq C \|Vu\|^{m+2} \|u(t)\|^{2 \beta(m+2)}, \]

and
\[\|u(t)\|_2 \leq C \|\nabla u(t)\|_{m+2} \leq C T^3(t)^{3/(m+2)},\]
application of Young's inequality to (3.22) yields
\[
\frac{d}{dt} \Gamma(t) + C(\Gamma(t))^{2(m+2)/(m+2)} \leq C \|u(t)\|_{2(m+1)/m}^{2(m+1)/m} \leq C(1+t)^{-2(m+1)/m}, \quad t \geq 1. \tag{3.28}
\]
Thus applying Lemma 3 to (3.28), we obtain
\[\Gamma(t) \leq C(1+t)^{-m/2}, \quad t \geq 1.\]
which implies
\[\|\nabla u(t)\|_{m+2} \leq C(1+t)^{-\gamma}, \quad t \geq 1.\]
To prove the convergence of \(u_t\), we further need the following Propositions.

**Proposition 5.** For \(u = u_0(x, t)\), we have
\[\int_t^T \|u(s)\|_2^2 ds \leq C_0(T) t^{-\gamma}, \quad 0 < t \leq T, \tag{3.29}\]
for any \(T > 0\), where \(\gamma = \mu(m+2) + (m+1)\).

**Proof.** It follows from (3.20) and (3.25)((3.25)' that
\[
\int_t^T \|u(s)\|_2^2 ds \leq \Gamma(t) + C \int_t^T |u|^{2m} |\nabla u|^2 dx
\leq C t^{-\mu(m+2)} + C \int_t^T s^{-\gamma}(\Gamma(s) + 1) ds
\leq C(T) t^{-\gamma}, \quad 0 < t \leq T.
\]

**Proposition 6.** For \(\varepsilon_1, \varepsilon_2 > 0\), we have
\[\|u_{\varepsilon_1}(t) - u_{\varepsilon_2}(t)\|_1 \leq \|u_{\varepsilon_0, \varepsilon_1} - u_{\varepsilon_0, \varepsilon_2}\|_1, \quad 0 < t. \tag{3.30}\]

**Proof.** For \(\delta > 0\), we take a function \(\rho_d(v) \in C^1(\mathbb{R})\) such that \(\rho_d(v) = 1\)
if \(v \geq \delta\), \(\rho_d(v) = -1\) if \(v \leq -\delta\), \(\rho_d(0) = 0\) and \(0 \leq \rho_d'(s) \leq 2s^{-1}\) if \(|s| \leq \delta\).
Multiplying the difference of the equations for \( u_1 \) and \( u_2 \) by \( \rho_\delta(u_1 - u_2) \), we have

\[
\frac{d}{dt} \int_\Omega \rho_\delta(x) \, dx + \int_\Omega \left( (B(u_1) - B(u_2)) \cdot \rho_\delta' \nabla (u_1 - u_2) \right) \, dx \\
+ \int_\Omega \{ \sigma_{\epsilon}(\nabla u_1) \} \nabla u_1 - \sigma_{\epsilon}(\nabla u_2) \} \nabla u_2 \} \rho_\delta' \nabla (u_1 - u_2) \, dx = 0.
\]

Here, by the monotonicity of \( \sigma_{\epsilon}(\cdot) \) and the facts;

\[
\lim_{\delta \to 0} \int_\Omega (B(u_1) - B(u_2)) \rho_\delta' \nabla (u_1 - u_2) \, dx = 0
\]

and

\[
\lim_{\delta \to 0} \int_{|u_1 - u_2|<\delta} (B(u_1) - B(u_2)) \rho_\delta' \nabla (u_1 - u_2) \, dx = 0
\]

we obtain

\[
\frac{d}{dt} \int_\Omega |u_1(t) - u_2(t)| \, dx \leq 0
\]

which implies (3.30).

Now, we have finished the preparation of the proof of Theorem 1.

Completion of the proof of Theorem 1. Let \( u_\epsilon(t) \) be the approximate solutions of (3.1)–(3.2). By Proposition 3.1–3.6, we see that there exists a subsequence of \( \{ u_\epsilon(t) \} \) (again denoted by \( \{ u_\epsilon(t) \} \)), such that as \( \epsilon \to 0 \),

\[
\begin{align*}
&u_\epsilon(\cdot) \to u(\cdot) \quad \text{strongly in } C(\mathbb{R}^+; L^1), \\
u_\epsilon(\cdot) \to u(\cdot) \quad \text{weakly}^* \text{ in } \mathcal{L}^\infty((0, \infty); L^\infty) \cap L^\infty(\mathbb{R}^+; L^\infty) \\
&\cap L^\infty_{\text{loc}}((0, \infty); W^{1,m+2}_0)
\end{align*}
\]

and

\[
\frac{\partial}{\partial t} u_\epsilon(t) \to \frac{\partial}{\partial t} u(t) \quad \text{weakly in } L^2_{\text{loc}}((0, \infty); L^2).
\]
In particular, we have
\[ u_e(\cdot) \to u(\cdot) \quad \text{in} \quad C_{loc}(0, \infty); L^2. \]

Since \( A(u_e) = -\text{div}\{\sigma_e(\nabla u_e)\nabla u_e\} \) is bounded in \( W^{-1, \infty(\cdot \cdot + 1)} \equiv (W_0^{1, m+2})^* \), we see further that
\[ A(\cdot) \to \chi \quad \text{weakly* in} \quad L^\infty_{loc}(0, \infty); (W_0^{1, m+2})^* \]
for some \( \chi \in L^\infty_{loc}(0, \infty); (W_0^{1, m+2})^* \).

We first note that from the Eq. (2.5) with \( u \) and \( \sigma(\nabla u) \) replaced by \( u_e \) and \( \sigma_e(\nabla u) \), respectively, we can easily show
\[
\int_0^T \langle u_e \phi - B(u) \nabla \phi \rangle dx \, dt + \langle \chi, \phi \rangle = 0. \tag{3.31}
\]

It remains to show \( \chi = -\text{div}\{\sigma(\nabla u)^2\} \nabla u \), i.e.,
\[
\langle \chi, \phi \rangle_{\delta, T} = \lim_{\varepsilon \to 0} \int_0^T \langle A(u_e), \phi \rangle dt = \lim_{\varepsilon \to 0} \int_0^T \sigma_e(\nabla u_e)^2 \nabla u_e \nabla \phi \, dx \, dt
\]
\[
= \int_{\Omega} \sigma(\nabla u)^2 \nabla u \nabla \phi \, dx \, dt = \langle A(u), \phi \rangle_{\delta, T}
\]
for any \( T > \delta > 0 \) and \( \phi \in C^1([\delta, T]; C^0(\Omega)) \). This is proved by a standard monotonicity argument, i.e., Minty’s trick. Indeed, we first note that (3.31) is valid for any
\[
\bar{\varphi} \in W_0^{1,2}(0, \infty); W_0^{1, m+2}(\Omega)
\]
\[
\cap L^\infty_{loc}(0, \infty); W_0^{1, m+2}(\Omega) \cap L^\infty(\Omega)
\]
and using this we can prove that
\[
\|u(T)\|_\delta^2 - \|u(\delta)\|_\delta^2 - \int_\delta^T \langle \chi, u \rangle_{\delta, t} = 0, \quad (3.32)
\]
for any \( T > \delta > 0 \), where
\[
\langle w^*, w \rangle_{\delta, T} = \int_{\delta}^T \langle w^*, w \rangle_{W_0^{1, m+2}; W_0^{1, m+2}} \, dt
\]
By the monotonicity of \( A \), we see that
\[
\langle A(u_e) - A(f), u_e - f \rangle_{\delta, T} \\
= \int_{\delta}^{T} \int_{\Omega} (\sigma_{e}(\nabla u_e) \nabla u_e - \sigma_{e}(\nabla f) \nabla f) \, dx \, dt \\
\geq 0.
\] (3.33)
for any \( f \in L^\infty([\delta, T]; W_{0}^{1,m+2}(\Omega) \cap L^\infty(\Omega)). \)

On the other hand, by (3.1)–(3.2),
\[
\lim_{t \to 0} \langle A(u_e) - A(f), u_e - f \rangle_{\delta, T} \\
= \lim_{t \to 0} \left( \int_{\delta}^{T} \int_{\Omega} u_e(\phi - u_e) \, dx \, dt \\
+ \int_{\delta}^{T} \int_{\Omega} B(u_e) \nabla u_e \nabla \phi \, dx \, dt - \langle A_{j}(\phi), u_e - f \rangle_{\delta, T} \right) \\
= -\int_{\delta}^{T} \int_{\Omega} u_e(\phi - u_e) \, dx \, dt + \int_{\delta}^{T} \int_{\Omega} B(u) \nabla u \nabla \phi \, dx \, dt - \langle A(\phi), u_e - f \rangle_{\delta, T} \\
= \langle \chi - A(f), u_e - f \rangle_{\delta, T}
\] (3.34)
where we have used (3.31) and (3.32) at the last step.

It follows from (3.33) and (3.34) that
\[
\langle \chi - A(f), u - f \rangle_{\delta, T} \geq 0,
\] (3.35)
for any \( f \in L^\infty([\delta, T]; W_{0}^{1,m+2}(\Omega) \cap L^\infty(\Omega)). \) which implies, by Minty's trick,
\( \chi = A(u) \).

Finally, we must show the uniqueness. For this, let \( u, v \) be two possible solutions. Then, by the same argument as in Proposition 3.6, we have
\[
\|u(t) - v(t)\|_1 \leq \|u(\delta) - v(\delta)\|_1
\] (3.36)
for any \( t > \delta > 0. \) Taking \( \delta \to 0 \), we conclude that \( u(t) \equiv v(t) \) on \( \mathbb{R}^+ \). The proof of Theorem 1 is now complete.

4. ESTIMATE OF \( \|\nabla u(t)\|_p, \ p > m + 2 \)

In this section, we proceed to the estimation of \( \|\nabla u(t)\|_p \) under the hypotheses Hyp. A, \( \tilde{A}, \) B and prove Theorem 2. By the proof of Theorem 1,
it suffices to derive the estimate (2.11) for an assumed smooth solution $u(t)$. We use notations

$$[D^2u]^2 = \sum_{i,j=1}^{N} u_{i,j}^2, \quad u_{i,j} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad u_i = \frac{\partial u}{\partial x_i}. $$

Multiplying the Eq. (3.1) by $-\text{div} \{ |\nabla u|^{p-2} \nabla u \}$, $p \geq m + 2$, and integrating by parts, we have

$$\frac{1}{p} \frac{d}{dt} |\nabla u|^p + \int_{\Omega} \text{div} \{ \sigma \nabla u \} \text{div} \{ |\nabla u|^{p-2} \nabla u \} \, dx$$

$$= -\int_{\Omega} b \cdot \nabla u \text{div}(|\nabla u|^{p-2} \nabla u) \, dx. \quad (4.1)$$

Here, using further integration by parts (see [2, 5, 15]),

$$\int_{\Omega} \text{div} \{ \sigma \nabla u \} \text{div} \{ |\nabla u|^{p-2} \nabla u \} \, dx$$

$$= \int_{\Omega} |\nabla u|^{p-2} (\sigma_{ij} + 2\sigma u_i u_j + u_k u_{k,i}) \, dx$$

$$+ \frac{p-2}{4} \int_{\Omega} \sigma |\nabla u|^{p-4} |\nabla (|\nabla u|^2)|^2 \, dx - (N-1) \int_{\partial \Omega} \sigma |\nabla u|^p H(x) \, dS$$

$$\geq k_0 \left\{ \int_{\Omega} |\nabla u|^{p+m-2} |D^2u|^2 \, dx + \frac{p+m-4}{4} \int_{\Omega} |\nabla u|^{p+m-4} |\nabla (|\nabla u|^2)|^2 \, dx \right\}$$

$$- C(N-1) \int_{\partial \Omega} |\nabla u|^{p+m} \, dS \right\}. \quad (4.2)$$

Next, we see for $p \geq m + 2$,

$$\int_{\Omega} b(u) \cdot \nabla u \text{div}(|\nabla u|^{p-2} \nabla u) \, dx$$

$$\leq k_0 \int_{\Omega} |\nabla u|^{p+m-2} |D^2u|^2 \, dx + C p^2 \int_{\Omega} |u|^{2p} |\nabla u|^{p-m} \, dx. \quad (4.3)$$
It follows from (4.1), (4.2), and (4.3) that
\[
\frac{1}{p} \frac{d}{dt} \| \nabla u(t) \|_p^p + \frac{k_0}{2} \int_\Omega |\nabla u|^{p+m-2} |D^2 u|^2 \, dx + \frac{k_0(p-2)}{4} \int_\Omega |\nabla u|^{p+m-4} |V(|\nabla u|^2)|^2 \, dx \\
\leq C \int_\Omega |\nabla u|^{p+m} \, ds + C p^2 \int_\Omega |u|^{2p} |\nabla u|^{p-m} \, dx \tag{4.4}
\]
and further,
\[
\frac{1}{p} \frac{d}{dt} \| \nabla u(t) \|_p^p + \frac{C_1}{p} \| \nabla u \|_{p(m/2)}^2 \leq \int_\Omega |u|^{2p} |\nabla u|^{p-m} \, dx + C p \int_\Omega |\nabla u|^{p+m} \, dx + C \int_\partial \Omega |\nabla u|^{p+m} \, ds. \tag{4.5}
\]

The first term of the right hand side of (4.5) is estimated as in (3.25). Indeed, when \( a \equiv (2\beta - m - mq/N)^+ > 0 \), we see
\[
\int_\Omega |u|^{2p} |\nabla u|^{p-m} \, dx \leq \| u \|_{2(p+m)/m} \| |\nabla u|^{p-m} \|_p \leq \| u(t) \|_\infty^2 \| u \|_{2q/(q-\theta)}^{2q/(q-\theta)} |\nabla u|_p^p
\]
with \( \theta = m/(2\beta - a) \). Hence, we have
\[
\int_\Omega |u|^{2p} |\nabla u|^{p-m} \, dx \leq C_a t^{-\theta} \| \nabla u \|_p^p, \quad 0 < t \leq 1, \tag{4.6}
\]
with \( \alpha = \tilde{\alpha}(2\beta - m - mq/N)^+ \). It is easy to see that in fact, (4.6) holds if \( 2\beta \geq m \).

When \( 2\beta \leq m \), we have, instead of (4.6),
\[
\int_\Omega |u|^{2p} |\nabla u|^{p-m} \, dx \leq C(1 + \| \nabla u \|_p^p), \quad 0 < t \leq 1. \tag{4.6'}
\]

To estimate the third term in (4.5), the boundary integral, we use a standard trace theorem and Gagliardo–Nirenberg inequality;
\[
\int_\partial \Omega |\nabla u|^{p+m} \, ds \leq C_1 \| \nabla u \|_{p(m/2)}^2 \|_{H^{1/2}} \leq \| \nabla u \|_{(p+m)/2} \| \nabla u \|_{H^{1/2}} \| \nabla u \|_{H^1} \leq \frac{1}{4} C_1 \| \nabla u \|_{(p+m)/2}^2 + C p \int_\Omega |\nabla u|^{p+m} \, dx. \tag{4.7}
\]
To estimate the second term of the right hand side of (4.5) or (4.7) we make the following device as in [14, 17]:

\[ \|u\|_{p+m} \leq h_{m}^{\theta_{1}} \|u\|_{p}^{\theta_{2}} \|u\|_{(p+m)/2}^{2 \theta_{3}}, \]

where we have chosen

\[ \theta_{1} = \frac{m}{3m+4}, \quad \theta_{2} = \frac{2m(m+2)}{(p+m)(3m+4)}, \quad \text{and} \quad \theta_{3} = \frac{2p(m+2)}{(p+m)(3m+4)} > 0. \]

From (4.5) through (4.8) and the estimate (3.17) we arrive at the basic inequality

\[ \frac{1}{p} \frac{d}{dt} \|u(t)\|_{p} + \frac{C_{1}}{p} \|u\|_{(p+m)/2}^{2} \leq C_{p}^{3} (t^{-\tilde{s}} \|u\|_{p}^{1} + 1) \]

where we recall \( \tilde{s} = \max\{z, mu\} \).

Now, by Lemma 1, we have

\[ \|u\|_{p} \leq C^{1/p} \|u\|_{m+2}^{1-\theta} \|u\|_{(p+m)/2}^{2 \theta(m+2)} \]

with

\[ \theta = \frac{p+m}{2} \left( \frac{1}{m+2} - \frac{1}{p} \right) \left( \frac{1}{2} - \frac{1}{m} \right) \left( \frac{p+m}{2m+4} \right)^{-1}. \]

Thus we have from (4.9) that

\[ \frac{d}{dt} \|u(t)\|_{p} + \frac{CC_{1}}{p} \|u\|_{m+2}^{1-\theta} \|u\|_{(p+m)/2}^{2 \theta} \leq C_{p}^{3} (t^{-\tilde{s}} \|u\|_{p}^{1} + 1), \]

and further, by Young's inequality and (3.17), we have

\[ \frac{d}{dt} \|u(t)\|_{p} + \frac{CC_{1}}{p} \|u\|_{m+2}^{1-\theta} \|u\|_{(p+m)/2}^{2 \theta} \leq C_{p}^{3} t^{-\tilde{s}}, \quad 0 < t \leq 1, \]

with some \( C_{p} > 0 \) and

\[ \tilde{s}_{p} = \frac{(p+m)(\tilde{s} + p(1-\theta))}{(p+m-p\theta)}. \]
Moreover, applying Lemma 2 to (4.10), we have
\[
\|\nabla u(t)\|_p \leq C_p t^{-\xi_p} \tag{4.11}
\]
for some constant \(C_p > 0\), where
\[
\xi_p = \frac{(p + m)(1 - \theta)\mu}{(p + m - p\theta)} + \frac{1}{p + m - p\theta} \max\{\theta, ((p + m)(\bar{\alpha} - 1) + p\theta)/p\}.
\]

If we make an additional assumption Hyp. C we know (cf. Engler, Kawohl, and Luckhaus [4])
\[
\|\nabla u\|_2^2 - (N-1) \oint_{\partial \Omega} v^2 H(x) \, dx \geq \lambda_0 \|v\|_{\hat{H}_1}^2,
\]
with some \(\lambda_0 > 0\). Therefore we can drop the second and third terms in (4.5) and consequently (4.9) holds with \(\bar{\alpha}\) replaced by \(\alpha\).

Next, we shall show the estimate for \(t \geq 1\). For \(t \geq 1\) we have
\[
\|\nabla u\|_{p + m} \leq \|\nabla u\|_{m+2}^{1-\theta} \|\nabla u\|_{1,2}^{(p + m)/2} \leq \frac{1}{4} \|\nabla u\|_{p + m} \|\nabla u\|_{m+2}^{p + m}
\]
with a certain \(0 < \theta < 1\) and hence, instead of (4.10),
\[
\frac{d}{dt} \|\nabla u(t)\|_p^p + C_1 p^{-1} \|\nabla u\|_{1,2}^{(p + m)/2} \leq C_1 (1 + t)^{-2\beta/m} \|u(t)\|_p^{p - m} + cp(1 + t)^{-(p + m)/m}. \tag{4.12}
\]
From (4.12) we see
\[
\frac{d}{dt} \|\nabla u(t)\|_p^p + \tilde{C}_1 \|\nabla u(t)\|_{p + m} \leq C_\rho \{(1 + t)^{-(p + m)\beta/m^2} + (1 + t)^{-(p + m)/m}\} \tag{4.13}
\]
which implies by Lemma 3,
\[
\|\nabla u(t)\|_p \leq C_\rho (1 + t)^{-\gamma}
\]
with \(\gamma = \min\{m^{-1}, \beta m^{-2}\} \).
In this final section, we shall derive the estimates for $\|\nabla u(t)\|_\infty$ and prove Theorem 3. To prove the estimate for $0 < t \leq 1$, we return to the inequality (4.9).

Let $p_1 \geq m + 2$ and we define a sequence $\{p_n\}$ by

$$p_n = 2p_{n-1} - m.$$  \hspace{1cm} (5.1)

Then, by Lemma 1, we have

$$\|\nabla u\|_{p_n} \leq C^{2(p_n + m)} \|\nabla u\|_{p_{n-1}}^{1-\theta_n} \|\nabla u\|_{p_n + m}^{2(\theta_n + m)}$$  \hspace{1cm} (5.2)

with

$$\theta_n = N(1 - m/p_n)/(N + 2).$$

It follows from (4.6) and (5.2) that

$$\frac{d}{dt} \|\nabla u(t)\|_{p_n} + C_n C^{2(p_n + m)} \|\nabla u(t)\|_{p_{n-1}}^{1-\theta_n} \|\nabla u(t)\|_{p_n + m}^{\theta_n}$$

$$\leq C p_n^{\frac{1}{\theta_n}} \int_Q |u|^{2\theta} |\nabla u|^{n-m} \, dx \leq C p_n^{\frac{1}{\theta_n}} t^{-\frac{\alpha}{\theta_n}} \|\nabla u(t)\|_{p_n}^{\beta_n + 1},$$  \hspace{1cm} (5.3)

where we recall

$$\alpha = \lambda(2\beta - m - mq/N)^{+} \text{ and } \beta = \max\{\alpha, \beta_n\}.$$  

We claim that there exist a bounded sequence $\{\eta_n\}$ and a convergent sequence $\{\xi_n\}$ such that

$$\|\nabla u(t)\|_{p_n} \leq \eta_n t^{-\xi_n}, \quad 0 < t \leq 1.$$  \hspace{1cm} (5.4)

This is true for $n = 1$ if we set $\xi_1 = \xi_{p_1}$ (see (2.12)). Suppose that (5.4) is valid for $n - 1$ with some $\eta_{n-1}$ and $\xi_{n-1}$. Then, we have from (5.3) that

$$\frac{d}{dt} \|\nabla u(t)\|_{p_n} + C_{n-1} C^{2(p_n + m)(1-1/\theta_n)} \|\nabla u(t)\|_{p_{n-1}}^{1-\theta_n} \|\nabla u(t)\|_{p_n + m}^{\theta_n}$$

$$\leq C (p_n^{\frac{1}{\theta_n}} t^{-\frac{\alpha}{\theta_n}} \|\nabla u(t)\|_{p_n}^{\beta_n + 1})$$  \hspace{1cm} (5.5)

with

$$\beta_n = (p_n + m) \beta_n^{-1} - p_n.$$
By use of Young's inequality, we have from (5.5) that
\[ \frac{d}{dt} \| \nabla u(t) \|_p + A_n t^{-(p_n+m)(1-\theta_n)/2} \| \nabla u(t) \|_p \leq C_n \| u(t) \|_p + A_n t^{-(p_n+m)(1-\theta_n)/(p_n+m-p_n \theta_n)} t^{-\delta_n} + C p^3, \quad 0 < t \leq 1, \] (5.6)

where
\[ A_n = \frac{1}{2} C_1 C^{p_n/m} (p_n+m)(1-\theta_n), \]
\[ C_n = C^{(3(p_n+m)+p_n \theta_n)/(p_n+m-p_n \theta_n)} \]
and
\[ \delta_n = \frac{1}{p_n+m-p_n \theta_n} \left\{ (p_n+m) \bar{z} + p_n (p_n+m)(1-\theta_n) \xi_{n-1} \right\}. \]

Here we note that we may assume \( \eta_n \geq 2 \) and hence the first term of the right hand side of (5.6) is dominant to the second, so the second term \( cp^3 \) can be in fact dropped.

Applying Lemma 2 to (5.6) (without \( cp^3 \)), we obtain (5.4) with
\[ \eta_n = \{ (2A_n)^{-p_n/\theta_n} (1 + (p_n+m)(\theta_n^{-1} - 1) \xi_{n-1})^{p_n/\theta_n} \]
\[ + 2C_n \{ 1 + (p_n+m)(\theta_n^{-1} - 1) \xi_{n-1} \} - 1 \}^{p_n/m} (p_n+m)(1-\theta_n)/(p_n+m-p_n \theta_n) \}
\[ \frac{1}{p_n+m-p_n \theta_n} \left\{ (p_n+m)(1-\theta_n) \xi_{n-1} + \theta_n \right\} \}
\[ + \max\{ 0, (p_n+m)(\bar{z}-1)/p_n \} \} \] (5.7)

and
\[ \xi_n = \frac{(p_n+m)(1-\theta_n) \xi_{n-1}}{p_n+m-p_n \theta_n} \]
\[ + \max\left\{ \frac{\theta_n}{p_n+m-p_n \theta_n}, \frac{\bar{z}(p_n+m)}{p_n(p_n+m-p_n \theta_n)} - \frac{1}{p_n} \right\} \]
\[ = \frac{1}{p_n+m-p_n \theta_n} \left\{ (p_n+m)(1-\theta_n) \xi_{n-1} + \theta_n \right\} \]
\[ + \max\{ 0, (p_n+m)(\bar{z}-1)/p_n \} \} \] (5.8)

Once \( \xi_n \) is known to be bounded, we see from (5.7) that
\[ \eta_n \leq (K p^n)^{1/p_n} \eta_n^{p_n/m}(1-\theta_n)/(p_n+m-p_n \theta_n) \]
for some constants \( \gamma > 0, K > 0 \). Since \( (p_n+m)(1-\theta_n)/(p_n+m-p_n \theta_n) < 1 \) we can easily show that \( \eta_n \) is bounded.
Let us consider the convergency of $\xi_n$.

Case 1 ($\tilde{\xi} \leq 1$). In this case, from (5.8),
$$
\xi_n = \frac{1}{(p_n + m)(1 - \theta_n) \xi_{n-1} + \theta_n}/(p_n + m - p_n \theta_n).
$$

We take $p_1 = m + 2$ and $\xi_1 = \mu$. Then, we can apply Lemma 2 to get
$$
\lim_{n \to \infty} \xi_n = (2 \xi_1 + N)/(2p_1 + mN) = (2\mu + N)/(2m + 4 + mN),
$$
which is the desired result.

Case 2 ($\tilde{\xi} > 1$). In this case,
$$
\xi_n = \frac{1}{p_n + m - p_n \theta_n}\{(p_n + m)(1 - \theta_n) \xi_{n-1} + \theta_n + (p_n + m)(\tilde{\xi} - 1)/p_n\}.
$$

Let us take $p_1 \geq m + 2$ and $A$ such that
$$
A > \frac{1}{m} \left\{ 1 + \frac{(\tilde{\xi} - 1)(N + 2)}{N} \right\}. \quad (5.9)
$$

Then,
$$
A - \tilde{\xi}_n = \frac{(p_n + m)(1 - \theta_n)}{p_n + m - p_n \theta_n} (A - \tilde{\xi}_{n-1}) + K_n,
$$
where we see
$$
K_n = \left( 1 - \frac{(p_n + m)(1 - \theta_n)}{p_n + m - p_n \theta_n} \right) A - \frac{(p_n + m) \tilde{\xi}}{p_n(\tilde{\xi} + 1 - mN)}/p_n + \frac{1}{p_n}
$$
$$
= \frac{m\theta_n}{p_n + m - p_n \theta_n} \left\{ A - \frac{1}{m} \left\{ 1 + \frac{(\tilde{\xi} - 1)(N + 2)}{N} \right\} \right\}
$$
$$
\geq \frac{m\theta_n}{p_n + m - p_n \theta_n} \left\{ A - \frac{1}{m} \left\{ 1 + \frac{(\tilde{\xi} - 1)(N + 2)}{N} \right\} \right\} \geq 0
$$
for sufficiently large $p_1$.

Thus,
$$
A - \tilde{\xi}_n \geq \frac{(p_n + m)(1 - \theta_n)}{p_n + m - p_n \theta_n} (A - \tilde{\xi}_{n-1}). \quad (5.10)
$$
Here, we note that (see (2.14))
\[
\tilde{\xi}_1 = \frac{(p_1 + m)(1 - \theta_1) \mu}{(p_1 + m - p_1 \theta_1)(m + 2)} + \frac{1}{p_1(p_1 + m - p_1 \theta_1)} \max\{\theta_1, (p_1 + m) \tilde{z} - (p_1 + m - p_1 \theta_1)\}
\]
(\theta_1 \text{ is defined by (2.15) with } p = p_1) and
\[
\lim_{p_1 \to \infty} \tilde{\xi}_1 = \frac{2\mu + N\tilde{\xi}}{mN + 2(m + 2)}.
\]

Taking larger \(A\), if necessary, with \(A - \xi_1 > 0\) we observe from (5.10) that \(A - \xi_n \geq 0, n = 1, 2, \ldots\) and (see the proof of Lemma 4))
\[
A - \xi_n \geq \frac{p_1w_n}{\mu} (A - \xi_1)
\]
where
\[
w_n = 2p_n + mN.
\]
We have from this,
\[
\limsup_{n \to \infty} \xi_n \leq \frac{2p_1\xi_1 + mAN}{2p_1 + mN} = I(p_1)
\]
and here,
\[
\lim_{p_1 \to \infty} I(p_1) = \lim_{n_1 \to \infty} \tilde{\xi}_1 = \frac{2\mu + N\tilde{\xi}}{mN + 2(m + 2)}
\]
(see (2.15)).

We conclude that for any \(\varepsilon > 0\), there exists \(p_1 > m + 2\) such that
\[
\limsup_{n \to \infty} \xi_n \leq \frac{2\mu + N\tilde{\xi}}{mN + 2m + 4 + \varepsilon},
\]
which together with (5.4) yields
\[
\|\nabla u(t)\|_{\infty} \leq C_s t^{-\left(2\mu + N\tilde{\xi}\right)(mN + 2m + 4) - \varepsilon}, \quad 0 < t \leq 1.
\]
The estimation of \(\|\nabla u(t)\|_{\infty}\) for \(0 < t \leq 1\) is now completed.
Finally, we show the estimate of \( \|\nabla u(t)\|_\infty \) for \( t \geq 1 \). To do so we return to the inequality (4.5). We set

\[
\psi(\tau) = (1 + t)^\psi u(t), \quad \tau = \log(1 + t)
\]

and

\[
\psi = \min \left\{ m^{-1}, \frac{2\beta - m}{m} \right\}.
\]

Then (4.5) becomes (see (4.7) and (4.8))

\[
\frac{d}{dt} \|\nabla \psi(\tau)\|_p^p + C_1 p^{-1} \|\nabla \psi^{p + m}\|_{1,2}^2
\leq \hat{C} p \|\nabla \psi(\tau)\|_p^p + C \psi^{1 + (1 + t)^{m + 1} - 2\hat{C}} \|\nabla \psi(\tau)\|_p^{p - m}
\]

(5.12)

and, by the choice of \( \hat{C} \), we have

\[
\frac{d}{dt} \|\nabla \psi(\tau)\|_p^p + C_1 p^{-1} \|\nabla \psi^{p + m}\|_{1,2}^2
\leq \hat{C} p \|\nabla \psi(\tau)\|_p^p + C \psi^{1 + (1 + t)^{m + 1}} - 2\hat{C} \|\nabla \psi(\tau)\|_p^{p - m}
\]

(5.13)

Applying Lemma 3.1 in [12] to (5.13), we have

\[
\|\nabla \psi(\tau)\|_\infty \leq C \max \left\{ 1, \sup_{\tau > \log 2} \|\nabla \psi(\tau)\|_{m+2}, \|\nabla \psi(\log 2)\|_\infty \right\} < \infty
\]

which implies

\[
\|\nabla u(t)\|_\infty \leq C (1 + t)^{-\psi}, \quad t \geq 1.
\]

REFERENCES