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The Binet formula, sums and representations of generalized Fibonacci p-numbers

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Abstract

In this paper, we consider the generalized Fibonacci p-numbers and then we give the generalized Binet formula, sums, combinatorial representations and generating function of the generalized Fibonacci p-numbers. Also, using matrix methods, we derive an explicit formula for the sums of the generalized Fibonacci p-numbers.

1. Introduction

We consider a generalization of well-known Fibonacci numbers, which are called Fibonacci p-numbers. The Fibonacci p-numbers $F_p(n)$ are defined by the following equation for $n > p+1$

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \tag{1}$$

with initial conditions

$$F_p(1) = F_p(2) = \cdots = F_p(p) = F_p(p+1) = 1.$$

If we take $p = 1$, then the sequence of Fibonacci p-numbers, $\{F_p(n)\}$, is reduced to the well-known Fibonacci sequence $\{F_n\}$.

The Fibonacci p-numbers and their properties have been studied by some authors (for more details see [1,4–6,8,13–26,29]).

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In 1843, Binet gave a formula which is called "Binet formula" for the usual Fibonacci numbers F_n by using the roots of the characteristic equation $x^2 - x - 1 = 0 : \alpha = \frac{1+\sqrt{5}}{2},\ \beta = \frac{1-\sqrt{5}}{2}$

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

where α is called Golden Proportion, $\alpha = \frac{1+\sqrt{5}}{2}$ (for details see [7,30,28]). In [12], Levesque gave a Binet formula for the Fibonacci sequence by using a generating function. In [2], the authors considered an $n \times n$ companion matrix and its nth power, then gave the combinatorial representation of the sequence generated by the nth power the matrix. Further in [25], the authors derived analytical formulas for the Fibonacci p-numbers and then showed these formulas are similar to the Binet formulas for the classical Fibonacci numbers. Also, in [11], the authors gave the generalized Binet formulas and the combinatorial representations for the generalized order-k Fibonacci [3] and Lucas [27] numbers. In [10], the authors defined the generalized order-k Pell numbers and gave the Binet formula for the generalized Pell sequence. For the common generalization of the generalized order-k Fibonacci and Pell numbers, and its generating matrix, sums and combinatorial representation, we refer readers to [9].

In this paper, we consider the generalized Fibonacci p-numbers and give the generalized Binet formula, combinatorial representations and sums of the generalized Fibonacci p-numbers by using the matrix method.

The generating matrix for the generalized Fibonacci p-numbers is given by Stakhov [23] as follows: Let Q_p be the following $(p+1) \times (p+1)$ companion matrix :

$$Q_p = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \dots & \ddots & \dots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix} \tag{2}$$

and the nth power of the matrix Q_p is

$$Q_p^n = \begin{bmatrix} F_p(n+1) & F_p(n-p+1) & \dots & F_p(n-1) & F_p(n) \\ F_p(n) & F_p(n-p) & \dots & F_p(n-2) & F_p(n-1) \\ \vdots & \vdots & & \vdots & \vdots \\ F_p(n-p+2) & F_p(n-2p+2) & \dots & F_p(n-p) & F_p(n-p+1) \\ F_p(n-p+1) & F_p(n-2p+1) & \dots & F_p(n-p-1) & F_p(n-p) \end{bmatrix}. \tag{3}$$

The matrix Q_p is said to be a generalized Fibonacci p-matrix.

2. The generalized Binet formula

In this section, we give the generalized Binet formula for the generalized Fibonacci p-numbers. We start with the following results.

Lemma 1. *Let* $a_p = \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}$. *Then* $a_p > a_{p+1}$ *for* $p > 1$.

Proof. Since $2p^3 - 2p - 1 > 0$ and $p > 1$, $\left(p^2+2p+1\right)\left(p^2-1\right) > p^4$. Thus, $\left(\frac{p^2-1}{p^2}\right) > \left(\frac{p}{p+1}\right)^2$. Therefore, for $p > 1$, $\left(\frac{p^2-1}{p^2}\right)^{p-1} > \left(\frac{p}{p+1}\right)^2$ and so$\left(\left(\frac{p-1}{p^2}\right) \times \left(\frac{p+1}{p}\right)\right)^{p-1} > \left(\frac{p}{p+1}\right)^2$. Then we have $\left(\frac{p-1}{p^2}\right)^{p-1} > \left(\frac{p}{p+1}\right)^{p+1}$. So the proof is easily seen. □

Lemma 2. *The characteristic equation of the Fibonacci p-numbers* $x^p - x^{p-1} - 1 = 0$ *does not have multiple roots for* $p > 1$.

Proof. Let $f(z) = z^p - z^{p-1} - 1$. Suppose that α is a multiple root of $f(z) = 0$. Note that $\alpha \neq 0$ and $\alpha \neq 1$. Since α is a multiple root, $f(\alpha) = \alpha^p - \alpha^{p-1} - 1 = 0$ and $f'(\alpha) = p\alpha^{p-1} - (p-1)\alpha^{p-2} = 0$. Then

$$f'(\alpha) = \alpha^{p-2}(p\alpha - (p-1)) = 0.$$

Thus $\alpha = \frac{p-1}{p}$, and hence

$$\begin{aligned} 0 &= f(\alpha) = -\alpha^p + \alpha^{p-1} + 1 = \alpha^{p-1}(1-\alpha) + 1 \\ &= \left(\frac{p-1}{p}\right)^{p-1}\left(1 - \frac{p-1}{p}\right) + 1 = \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1} + 1 \\ &= a_p + 1. \end{aligned}$$

Since, by Lemma 1, $a_2 = \frac{1}{4} < 1$ and $a_p > a_{p+1}$ for $p > 1$, $a_p \neq 1$, which is a contradiction. Therefore, the equation $f(z) = 0$ does not have multiple roots. □

We suppose that $f(\lambda)$ is the characteristic polynomial of the generalized Fibonacci p-matrix Q_p. Then, $f(\lambda) = \lambda^{p+1} - \lambda^p - 1$, which is a well-known fact from the companion matrices. Let $\lambda_1, \lambda_2, \ldots, \lambda_{p+1}$ be the eigenvalues of the matrix Q_p. Then, by Lemma 2, we know that $\lambda_1, \lambda_2, \ldots, \lambda_{p+1}$ are distinct. Let Λ be a $(p+1) \times (p+1)$ Vandermonde matrix as follows:

$$\Lambda = \begin{bmatrix} \lambda_1^p & \lambda_1^{p-1} & \ldots & \lambda_1 & 1 \\ \lambda_2^p & \lambda_2^{p-1} & \ldots & \lambda_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_{p+1}^p & \lambda_{p+1}^{p-1} & \ldots & \lambda_{p+1} & 1 \end{bmatrix}.$$

We denote Λ^{T} by V. Let

$$d_k^i = \begin{bmatrix} \lambda_1^{n+p+1-i} \\ \lambda_2^{n+p+1-i} \\ \vdots \\ \lambda_{p+1}^{n+p+1-i} \end{bmatrix}$$

and $V_j^{(i)}$ be a $(p+1) \times (p+1)$ matrix obtained from V by replacing the jth column of V by d_k^i.

Then we can give the generalized Binet formula for the generalized Fibonacci p-numbers with the following theorem.

Theorem 3. *Let $F_p(n)$ be the nth generalized Fibonacci p-number; then*

$$q_{ij} = \frac{\det\left(V_j^{(i)}\right)}{\det(V)}$$

where $Q_p^n = \left[q_{ij}\right]$ and $q_{ij} = F_p(n+j-i-p)$ for $j \geq 2$ and $q_{i,1} = F_p(n+2-i)$ for $j = 1$.

Proof. Since the eigenvalues of the matrix Q_p are distinct, the matrix Q_p is diagonalizable. It is easy to show that $Q_pV = VD$, where $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{p+1})$. Since the Vandermonde matrix V is invertible, $V^{-1}Q_pV = D$. Hence, the matrix Q_p is similar to the diagonal matrix D. So we have the matrix equation $Q_p^nV = VD^n$. Since $Q_p^n = \left[q_{ij}\right]$, we have the following linear system of equations:

$$\begin{aligned}
&q_{i1}\lambda_1^p + q_{i2}\lambda_1^{p-1} + \cdots + q_{i,p+1} = \lambda_1^{p+n+1-i}\\
&q_{i1}\lambda_2^p + q_{i2}\lambda_2^{p-1} + \cdots + q_{i,p+1} = \lambda_2^{p+n+1-i}\\
&\vdots\\
&q_{i1}\lambda_{p+1}^p + q_{i2}\lambda_{p+1}^{p-1} + \cdots + q_{i,p+1} = \lambda_{p+1}^{p+n+1-i}.
\end{aligned}$$

Thus, for each $j = 1, 2, \ldots, p+1$, we obtain

$$q_{ij} = \frac{\det\left(V_j^{(i)}\right)}{\det(V)}.$$

So the proof is complete. □

Thus, we give the Binet formula for the nth Fibonacci p-number $F_p(n)$ by the following corollary.

Corollary 4. *Let $F_p(n)$ be the nth Fibonacci p-number. Then*

$$F_p(n) = \frac{\det\left(V_1^{(2)}\right)}{\det(V)} = \frac{\det\left(V_{p+1}^{(1)}\right)}{\det(V)}.$$

Proof. The conclusion is immediate result of Theorem 3 by taking $i = 2,\ j = 1$ or $i = 1, j = p+1$. □

The following lemma can be obtained from [2].

Lemma 5. *Let the matrix $Q_p^n = \left[q_{ij}\right]$ be as in (3). Then*

$$q_{ij} = \sum_{(m_1,\ldots,m_{p+1})} \frac{m_j + m_{j+1} + \cdots + m_{p+1}}{m_1 + m_2 + \cdots + m_{p+1}} \times \binom{m_1 + m_2 + \cdots + m_{p+1}}{m_1, m_2, \ldots, m_{p+1}}$$

where the summation is over nonnegative integers satisfying $m_1 + 2m_2 + \cdots + (p+1)m_{p+1} = n - i + j$, and defined to be 1 if $n = i - j$.

Then we have the following corollaries.

Corollary 6. *Let $F_p(n)$ be the generalized Fibonacci p-number. Then*

$$F_p(n) = \sum_{(m_1,\ldots,m_{p+1})} \frac{m_{p+1}}{m_1+m_2+\cdots+m_{p+1}} \times \binom{m_1+m_2+\cdots+m_{p+1}}{m_1,m_2,\ldots,m_{p+1}}$$

where the summation is over nonnegative integers satisfying $m_1 + 2m_2 + \cdots + (p+1)m_{p+1} = n + p$.

Proof. In Lemma 5, when $i = 1$ and $j = p + 1$, then the conclusion can be directly seen from (3). □

Corollary 7. *Let $F_p(n)$ be the generalized Fibonacci p-number. Then*

$$F_p(n) = \sum_{(m_1,\ldots,m_{p+1})} \binom{m_1+m_2+\cdots+m_{p+1}}{m_1,m_2,\ldots,m_{p+1}}$$

where the summation is over nonnegative integers satisfying $m_1 + 2m_2 + \cdots + (p+1)m_{p+1} = n - 1$.

Proof. In Lemma 5, if we take $i = 2$ and $j = 1$, then we have the corollary from (3). □

We consider the generating function of the generalized Fibonacci p-numbers. We give the following lemma.

Lemma 8. *Let $F_p(n)$ be the nth generalized Fibonacci number, then for $n > 1$*

$$x^n = F_p(n-p+1)x^p + \sum_{j=1}^{p} F_p\,(n-p+1-j)\,x^{j-1}.$$

Proof. We suppose that $n = p + 1$; then by the definition of the Fibonacci p-numbers

$$x^{p+1} = F_p(2)x^p + F_p(1) = x^p + 1.$$

Now we suppose that the equation holds for any integer $n, n > p + 1$. Then we show that the equation holds for $n+1$. Thus, from our assumption and the characteristic equation the Fibonacci p-numbers,

$$\begin{aligned} x^{n+1} = x^n x &= \left(F_p(n-p+1)x^p + \sum_{j=1}^{p} F_p\,(n-p+1-j)\,x^{j-1} \right) x \\ &= F_p(n-p+1)\left(x^p+1\right) + \sum_{j=1}^{p} F_p\,(n-p+1-j)\,x^{j} \\ &= F_p(n-p+1)x^p + F_p(n-p+1) + F_p\,(n-2p+1)\,x^p \\ &\quad + F_p(n-2p+2)x^{p-1} + \cdots + F_p(n-2p+1)x^2 + F_p(n-p)x \\ &= \left[F_p(n-p+1) + F_p(n-2p+1)\right]x^p + F_p(n-2p+2)x^{p-1} \\ &\quad + F_p\,(n-2p+3)\,x^{p-2} + \cdots + F_p(n-p)x + F_p(n-p+1). \end{aligned} \quad (4)$$

Using the definition of the generalized Fibonacci p-numbers, we have

$$F_p(n-p+1)+F_p(n-2p+1)=F_p\,(n-p+2)\,.$$

Therefore, we can write the Eq. (4) as follows

$$\begin{aligned} x^{n+1} &= F_p(n-p+2)x^p+F_p\,(n-2p+2)\,x^{p-1}\\ &\quad +F_p\,(n-2p+3)\,x^{p-2}+\cdots+F_p(n-p)x+F_p(n-p+1)\\ &= F_p(n-p+2)x^p+\sum_{j=1}^{p}F_p\,(n-p+2-j)\,x^{j-1} \end{aligned} \tag{5}$$

which is what was desired. □

Now we give the generating function of the generalized Fibonacci p-numbers:
Let

$$G_p(x)=F_p(1)+F_p(2)x+F_p(3)x^2+\cdots+F_p(n+1)x^n+\cdots\,.$$

Then

$$G_p(x)-xG_p(x)-x^{p+1}G_p\,(x)=\left(1-x-x^{p+1}\right)G_p(x).$$

By the Eq. (5), we have $\left(1-x-x^{p+1}\right)G_p\,(x)=F_p(1)=1$. Thus

$$G_p(x)=\left(1-x-x^{p+1}\right)^{-1}$$

for $0\le x+x^{p+1}<1$.

Let $f_p(x)=x+x^{p+1}$. Then, for $0\le f_p(x)<1$, we have the following lemma.

Lemma 9. *For positive integers t and n, the coefficient of x^n in $\left(f_p(x)\right)^t$ is*

$$\sum_{j=0}^{t}\binom{t}{j},\quad \frac{n}{p+1}\le t\le n$$

where the integers j satisfy $pj+t=n$.

Proof. From the above results, we write

$$\left(f_p(x)\right)^t=\left(x+x^{p+1}\right)^t=x^t\left(1+x^p\right)^t=x^t\sum_{j=0}^{t}\binom{t}{j}x^{pj}.$$

In the above equation, we consider the coefficient of x^n. For positive integers t and j such that $pj+t=n$ and $j\le t$, the coefficients of x^n are

$$\sum_{j=0}^{t}\binom{t}{j},\quad \frac{n}{p+1}\le t\le n.$$

So we have the required conclusion. □

Now we can give a representation for the generalized Fibonacci p-numbers by the following theorem.

Theorem 10. *Let $F_p(n)$ be the nth generalized Fibonacci p-number. Then, for positive integers t and n,*

$$F_p(n+1) = \sum_{\frac{n}{p+1} \le t \le n} \sum_{j=0}^{t} \binom{t}{j}$$

where the integers j satisfy $pj + t = n$.

Proof. Since

$$\begin{aligned} G_p(x) &= F_p(1) + F_p(2)\,x + F_p(3)x^2 + \cdots + F_p(n+1)\,x^n + \cdots \\ &= \frac{1}{1 - x - x^{p+1}} \end{aligned}$$

and $f_p(x) = x + x^{p+1}$, the coefficient of x^n is the $(n+1)$th generalized Fibonacci p-number, $F_p(n+1)$ in $G_p(x)$. Thus

$$\begin{aligned} G_p(x) &= \frac{1}{1 - x - x^{p+1}} \\ &= \frac{1}{1 - f_p(x)} \\ &= 1 + f_p(x) + \left(f_p(x)\right)^2 + \cdots + \left(f_p(x)\right)^n + \cdots \\ &= 1 + x\left(1 + x^p\right) + x^2 \sum_{j=0}^{2} \binom{2}{j} x^{pj} + \cdots + x^n \sum_{j=0}^{n} \binom{n}{j} x^{pj} + \cdots . \end{aligned}$$

As we need the coefficient of x^n, we only consider the first $n+1$ terms on the right-side. Thus by Lemma 9, the proof is complete. □

Now we give an exponential representation for the generalized Fibonacci p-numbers.

$$\begin{aligned} \ln G_p(x) &= \ln\left[1 - \left(x + x^{p+1}\right)\right]^{-1} \\ &= -\ln\left[1 - \left(x + x^{p+1}\right)\right] \\ &= -\left[-\left(x + x^{p+1}\right) - \frac{1}{2}\left(x + x^{p+1}\right)^2 - \cdots - \frac{1}{n}\left(x + x^{p+1}\right)^n - \cdots\right] \\ &= x\left[\left(1 + x^p\right) + \frac{1}{2}\left(1 + x^p\right)^2 + \cdots + \frac{1}{n}\left(1 + x^p\right)^n + \cdots\right] \\ &= x \sum_{n=0}^{\infty} \frac{1}{n}\left(1 + x^p\right)^n . \end{aligned}$$

Thus,

$$G_p(x) = \exp\left(x \sum_{n=0}^{\infty} \frac{1}{n}\left(1 + x^p\right)^n\right).$$

3. Sums of the generalized Fibonacci p-numbers by matrix methods

In this section, we define a $(p+2) \times (p+2)$ matrix T, and then we show that the sums of the generalized Fibonacci p-numbers can be obtained from the nth power of the matrix T.

Definition 11. For $p \geq 1$, let $T = (t_{ij})$ denote the $(p+2) \times (p+2)$ matrix by$t_{11} = t_{21} = t_{22} = t_{2,p+2} = 1$, $t_{i+1,i} = 1$ for $2 \leq i \leq p+1$ and 0 otherwise.

Clearly, by the definition of the matrix Q_p,

$$T = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & & 0 & 1 & 0 \end{bmatrix} \quad \text{or} \quad T = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & & & \\ 0 & & Q_p & \\ \vdots & & & \\ 0 & & & \end{bmatrix} \tag{6}$$

where the $(p+1) \times (p+1)$ matrix Q_p given by (2).

Let S_n denote the sums of the generalized Fibonacci p-numbers from 1 to n, that is:

$$S_n = \sum_{i=1}^{n} F_p(i). \tag{7}$$

Now we define a $(p+2) \times (p+2)$ matrix C_n as follows

$$C_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ S_n & & & \\ S_{n-1} & & Q_p^n & \\ \vdots & & & \\ S_{n-p} & & & \end{bmatrix} \tag{8}$$

where Q_p^n given by (3).

Then we have the following theorem.

Theorem 12. *Let the $(p+2) \times (p+2)$ matrices T and C_n be as in (6) and (8), respectively. Then, for $n \geq 1$:*

$$C_n = T^n.$$

Proof. We will use the induction method to prove that $C_n = T^n$. If $n = 1$, then, by the definition of the matrix C_n and generalized Fibonacci p-numbers, we have

$$C_1 = T.$$

Now we suppose that the equation holds for n. Then we show that the equation holds for $n+1$. Thus,

$$T^{n+1} = T^n.T$$

and by our assumption,

$$T^{n+1} = C_n T.$$

Since $S_{n+1} = S_n + F_p(n+1)$ and using the definition of the generalized Fibonacci numbers, we can derive the following matrix recurrence relation

$$C_n T = C_{n+1}.$$

So the proof is complete. □

We define two $(p+2) \times (p+2)$ matrices. First, we define the matrix R as follows:

$$R = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & \lambda_1^p & \lambda_2^p & \dots & \lambda_{p+1}^p \\ -1 & \lambda_1^{p-1} & \lambda_2^{p-1} & \dots & \lambda_{p+1}^{p-1} \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & \lambda_1 & \lambda_2 & \dots & \lambda_{p+1} \\ -1 & 1 & 1 & \dots & 1 \end{bmatrix} \tag{9}$$

and the diagonal matrix D_1 as follows:

$$D_1 = \begin{bmatrix} 1 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{p+1} \end{bmatrix} \tag{10}$$

where the λ_i's are the eigenvalues of the matrix Q_p for $1 \le i \le p+1$.

We give the following theorem for the computing the sums of the generalized Fibonacci p-numbers 1 from to n by using a matrix method.

Theorem 13. *Let the sums of the generalized Fibonacci numbers S_n be as in (7). Then*

$$S_n = F_p(n+p+1) - 1.$$

Proof. If we compute the det R by the Laplace expansion of determinant with respect to the first row, then we obtain that $\det R = \det V$, where the Vandermonde matrix V is as in Theorem 3. Therefore, we can easily find the eigenvalues of the matrix R. Since the characteristic equation of the matrix R is $\left(x^p - x^{p-1} - 1\right) \times (x-1)$ and by Lemma 2, the eigenvalues of the matrix R are $1, \lambda_1, \dots, \lambda_{p+1}$ and distinct. So the matrix R is diagonalizable. We can easily prove that $TR = RD_1$, where the matrices T, R and D_1 are as in (6), (9) and (10), respectively. Then we have

$$T^n R = R D_1^n. \tag{11}$$

Since $T^n = C_n$, we write that $C_n R = R D_1^n$. We know that $S_n = (C_n)_{2,1}$. By a matrix multiplication,

$$S_n - \left(\sum_{i=0}^{p} F_p(n+1-i) \right) = -1. \tag{12}$$

By the definition of the generalized Fibonacci p-numbers, we know that $\sum_{i=0}^{p} F_p(n+1-i) = F_p(n+p+1)$. Then we write the Eq. (12) as follows:

$$S_n - F_p(n+p+1) = -1.$$

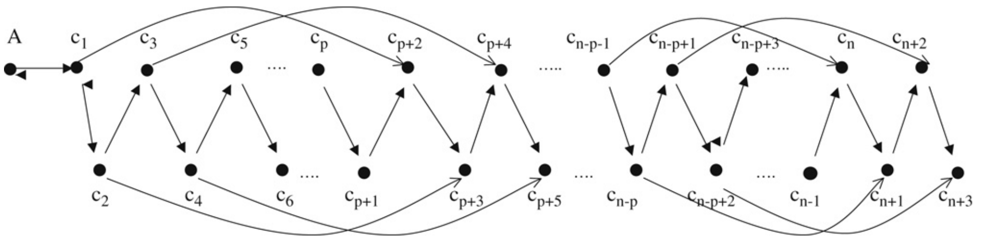

Fig. 1.

Thus,

$$S_n = \sum_{i=1}^{n} F_p(i) = F_p(n + p + 1) - 1.$$

So the proof is complete. □

In [30], the author presents an enumeration problem for the paths from A to c_n, and then shows that the number of paths from A to c_n are equal to the nth usual Fibonacci number. Now, we are interested in a problem of paths. The problem is as in Fig. 1.

It is seen that the number of path from A to $c_1, c_2, \ldots c_{p+1}$ is 1. Also, we know that the initial conditions of the generalized Fibonacci p-numbers, that is, $F_p(1), F_p(2), \ldots, F_p(p+1)$, are 1. Now we consider the case $n > p + 1$. The number of the path from A to c_{p+2} is 2. By the induction method, one can see that the number of the path from A to c_n is the nth generalized Fibonacci p-number.

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