JOURNAL OF APPROXIMATION THEORY 33, 199–213 (1981)

Approximation by Finite Rank Operators

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Communicated by Oved Shisha

Received April 21, 1980

1. INTRODUCTION

There has been much recent interest in the problem of approximating in the space of bounded linear operators $\mathscr{L}(X, Y)$ from one normed linear space X into another Y by certain subsets $\mathscr{M}$ of $\mathscr{L}(X, Y)$. In particular, the case when $\mathscr{M} = \mathscr{K}(X, Y)$, the compact operators, has received considerable attention (see, e.g., [7, 10–17]). A strong impetus in developing a reasonable theory in this case has come from the fact that (for certain spaces X and Y) $\mathscr{K}(X, Y)$ is an "M-ideal" in $\mathscr{L}(X, Y)$ so that one can apply the powerful and elegant M-ideal theory (as developed by Alfsen and Effros [1]) to get substantial information about this problem. (Precise definitions are given below.) Of special importance is the question of the *existence* of best approximations. (A subset M of the normed space Z is called *proximinal* in Z if each $z \in Z$ has a nearest point in M.) From the general M-ideal theory, one obtains immediately that whenever $\mathscr{K}(X, Y)$ is an M-ideal in $\mathscr{L}(X, Y)$, then $\mathscr{K}(X, Y)$ is proximinal in $\mathscr{L}(X, Y)$ (see, e.g., [1]). In general, however, $\mathscr{K}(X, Y)$ is *not* an M-ideal in $\mathscr{L}(X, Y)$ and $\mathscr{K}(X, Y)$ may or may not be proximinal in $\mathscr{L}(X, Y)$. More precisely, there are examples of spaces X, Y such that $\mathscr{K}(X, Y)$ is not an M-ideal in $\mathscr{L}(X, Y)$, but $\mathscr{K}(X, Y)$ is proximinal

* The work of this author was performed while he was a guest at the Institut für Mathematik der Universität Bonn for a period of two weeks in February–March 1979. He is indebted to the Sonderforschungsbereich 72 for its generous financial support during that time.

0021-9045/81/110199–15$02.00/0

in $\mathscr{L}(X, Y)$ (see, e.g., [12, 15, 17]). On the other hand, there is a Hilbert space X and a separable strictly convex Banach space Y such that $\mathscr{K}(X, Y)$ is not proximinal in $\mathscr{L}(X, Y)$ (Holmes and Kripke [9]). In spite of what is known, there are still many nagging open problems connected with the proximinality of $\mathscr{K}(X, Y)$ in $\mathscr{L}(X, Y)$ (see, e.g., [15]).

An interesting related problem concerns the question of when the set of "rank N" operators $\mathscr{K}_N(X, Y)$ is proximinal in $\mathscr{L}(X, Y)$ (or in $\mathscr{K}(X, Y)$). (An operator in $\mathscr{L}(X, Y)$ is said to have *rank* N if its range is contained in an N dimensional subspace of Y.) This problem has obvious practical ramifications as well (e.g., in the theory of integral equations). Unfortunately, the algebraic structure of $\mathscr{K}_N(X, Y)$ is not as nice as that of $\mathscr{K}(X, Y)$. Indeed, whereas $\mathscr{K}(X, Y)$ is a (linear) subspace, $\mathscr{K}_N(X, Y)$ is not even convex. At the present time, we are aware of only relatively few results concerning the proximinality of $\mathscr{K}_N(X, Y)$ in $\mathscr{L}(X, Y)$ for certain special cases X and Y (see, e.g., [7, 14]).

This paper represents a further contribution to a solution of the problem: When is $\mathscr{K}_N(X, Y)$ proximinal in $\mathscr{L}(X, Y)$ or in $\mathscr{K}(X, Y)$? The main result of Section 2 (Theorem 2.2) states that $\mathscr{K}_N(X, Y^*)$ is proximinal in $\mathscr{L}(X, Y^*)$ for *any* normed spaces X and Y. More generally, $\mathscr{K}_N(X, Y)$ is proximinal in $\mathscr{L}(X, Y)$ whenever Y is norm-one complemented in a dual space (Corollary 2.6). As corollaries, we obtain two results of Fakhoury [7, Remark 2.3 (1) and Corollary 2.8] as well as the fact that when Y is an abstract L-space, then $\mathscr{K}_N(X, Y)$ is proximinal in $\mathscr{L}(X, Y)$ (Corollary 2.7). In Section 3, we are concerned with the case when $Y = C_0(S)$, the continuous functions "vanishing at infinity" on a locally compact Hausdorff space S. Theorem 3.2 is a generalization of the result of Fakhoury [7] which states: "If X^* is strictly convex, then $\mathscr{K}_N(X, C_0(S))$ is proximinal in $\mathscr{K}(X, C_0(S))$." When S has the discrete topology, then (Theorem 3.4), $\mathscr{K}(X, C_0(S))$ is proximinal in $\mathscr{K}(X, C_0(S))$ for any space X. It is not known to us whether $\mathscr{K}(X, C_0(S))$ can be replaced by $\mathscr{L}(X, C_0(S))$ in Theorem 3.2 or 3.4. However, Theorem 3.5 (resp. Theorem 3.10) states that if X is uniformly smooth (resp. $X = c_0$), then $\mathscr{K}_N(X, c_0)$ is proximinal in $\mathscr{L}(X, c_0)$. In Section 4, we consider approximating by compact operators. For example, in Theorem 4.1, we give a list of several approximative properties that the set $\mathscr{K}(X, C_0(S))$ in $\mathscr{L}(X, C_0(S))$ possesses *provided* S has the discrete topology. In this case, $\mathscr{K}(X, C_0(S))$ is an M-ideal and hence is proximinal. However, there is substantially more that can be said. For example, we give an explicit formula for a homogeneous Lipschitz continuous selection for the metric projection onto $\mathscr{K}(X, C_0(S))$. (Before this, only the *existence*—*non*constructive—of a continuous homogeneous selection was known. See [10].) In Section 5, we collect a few miscellaneous facts and state some open problems.

We conclude the introduction with some basic notation and terminology.

(All undefined notation or terminology is standard and can be found, e.g., in [6].) If X and Y are (real) normed linear spaces, then $\mathscr{L}(X, Y)$ denotes the normed linear space of all bounded linear operators T from X into Y endowed with the norm $\|T\| = \sup\{\|Tx\| \mid x \in X, \|x\| \leqslant 1\}$. $\mathscr{K}(X, Y)$ is the subset of all *compact operators* in $\mathscr{L}(X, Y)$. That is, $T \in \mathscr{K}(X, Y)$ iff T maps the unit ball in X into a relatively compact subset of Y. For any natural number N, the set of *rank N operators* is the subset $\mathscr{K}_N(X, Y)$ of all operators T in $\mathscr{L}(X, Y)$ with the property that the range of T is at most N dimensional. If S is any locally compact Hausdorff space, $C_0(S)$ will denote the set of all real-valued continuous functions f on S "*vanishing at infinity*" (i.e., $\{s \in S \mid |f(s)| \geqslant \varepsilon\}$ is compact for each $\varepsilon > 0$) and endowed with the supremum norm. If S is compact, then $C_0(S) = C(S)$, the continuous functions on S. If S is any set with the discrete topology, we often write $c_0(S)$ for $C_0(S)$. If M is a subset of a normed space Z and z is in Z, an element y in M is called a *best approximation* to z from M if $\|x - y\| = d(x, M)$, where $d(x, M) = \inf\{\|x - y\| \mid y \in M\}$. M is called *proximinal* in Z if each $z \in Z$ has a best approximation in M. The set of all best approximations in M to z is denoted by $P_M(z)$. The set-valued mapping $P_M: Z \to 2^M$ thus defined is called the *metric projection* onto M. A closed subspace M of Z is called an *M-ideal* if there is a linear projection Q from Z^* onto $M^\perp$ such that $\|z^*\| = \|Qz^*\| + \|z^* - Qz^*\|$ for every $z^* \in Z^*$. If $z \in Z$, then $\hat{z}$ will denote the image of z under the natural embedding of Z into its second dual Z^{**}. That is, $\hat{z}(z^*) = z^*(z)$, $z^* \in Z$. Further, $\hat{Z}$ will denote the set $\{\hat{z} \mid z \in Z\}$.

Throughout this paper, unless explicitly stated otherwise, X and Y will denote arbitrary (real) normed linear spaces, N any given natural number, and S an arbitrary locally compact Hausdorff space.

2. When the Range Space Is a Dual Space

In this section we will consider the case when Y is a dual space or, more generally, Y is norm-one complemented in a dual space.

The following lemma isolates a simple but useful fact that will be needed more than once in the sequel. (Here X and Y are arbitrary.)

2.1. Lemma. *If $F \in \mathscr{K}_N(X, Y)$, then there exist N vectors $y_i \in Y$ and N functionals $x_i^* \in X^*$ such that*

(i) $\|y_i\| = 1 \qquad (i = 1, 2, ..., N)$;

(ii) $\|x_i^*\| \leqslant \|F\| \qquad (i = 1, 2, ..., N)$;

(iii) $Fx = \sum_{i=1}^N x_i^*(x)\, y_i, \qquad x \in X.$

Conversely, if F is defined by (iii) *for some given sets* $\{y_1, y_2, ..., y_N\}$ *in* Y *and* $\{x_1^*, x_2^*, ..., x_N^*\}$ *in* X^*, *then* $F \in \mathscr{K}_N(X, Y)$.

Proof. Let V be an N dimensional subspace of Y which contains the range of F. By Auerbach's lemma [3], there is a basis $\{y_1, y_2, ..., y_N\}$ of V and linear functionals $\{y_1^*, y_2^*, ..., y_N^*\}$ in V^* such that $\|y_i\| = \|y_i^*\| = 1$ $(i = 1, 2, ..., N)$, and $v = \sum_{i=1}^N y_i^*(v) y_i$, $v \in V$. By the Hahn–Banach theorem, we may assume $y_i^* \in Y^*$ $(i = 1, 2, ..., N)$. In particular,

$$Fx = \sum_{i=1}^N y_i^*(Fx) y_i, \qquad x \in X.$$

Let $x_i^* = y_i^* \circ F$. Then $x_i^* \in X^*$ and

$$\|x_i^*\| \leqslant \|y_i^*\| \|F\| \leqslant \|F\| \qquad (i = 1, 2, ..., N).$$

The converse is trivial. ∎

2.2. THEOREM. $\mathscr{K}_N(X, Y^*)$ *is proximinal in* $\mathscr{L}(X, Y^*)$.

Proof. We will actually prove the stronger statement that $\mathscr{K}_N(X, Y^*)$ is "boundedly weak*-operator compact," i.e., any bounded net in $\mathscr{K}_N(X, Y^*)$ has a subnet which converges in the weak*-operator topology to an element of $\mathscr{K}_N(X, Y^*)$. Let (F_δ) be a bounded net in $\mathscr{K}_N(X, Y^*)$, say, $\|F_\delta\| \leqslant c$ for all δ. By Lemma 2.1, there exist a set $\{y_{1\delta}^*, y_{2\delta}^*, ..., y_{n\delta}^*\}$ in Y^* with $\|y_{i\delta}^*\| = 1$ and a set $\{x_{1\delta}^*, x_{2\delta}^*, ..., x_{n\delta}^*\}$ in X^* with $\|x_{i\delta}^*\| \leqslant \|F_\delta\| \leqslant c$ such that

$$F_\delta x = \sum_{i=1}^N x_{i\delta}^*(x) y_{i\delta}^*, \qquad x \in X.$$

Since all the functionals involved are bounded, it follows that by passing to a subnet we may assume that

$$x_{i\delta}^* \xrightarrow{w^*} x_i^* \qquad \text{and} \qquad y_{i\delta}^* \xrightarrow{w^*} y_i^*$$

$(i = 1, 2, ..., N)$ for some $x_i^* \in X^*$ and $y_i^* \in Y^*$. (Here w^* denotes the weak* topology.) Hence for each $x \in X$ and $y \in Y$,

$$(F_\delta x)(y) = \sum_{i=1}^N x_{i\delta}^*(x) y_{i\delta}^*(y) \to \sum_{i=1}^N x_i^*(x) y_i^*(y).$$

Defining F_0 on X by $F_0 x = \sum_{i=1}^N x_i^*(x) y_i^*$, it follows by Lemma 2.1 that $F_0 \in \mathscr{K}_N(X, Y^*)$ and that $F_\delta \to F_0$ in the weak*-operator topology. Thus $\mathscr{K}_N(X, Y^*)$ is boundedly weak*-operator compact as claimed.

Now let $T \in \mathscr{L}(X, Y^*)$ and let (F_n) be a minimizing sequence in $\mathscr{K}_N(X, Y^*)$:

$$\|T - F_n\| \to d(T, \mathscr{K}_N(X, Y^*)).$$

Since (F_n) is bounded, the first part of the proof shows that there is a subnet (F_δ) which converges, in the weak*-operator topology, to some $F_0 \in \mathscr{K}_N(X, Y^*)$. Further,

$$\|T - F_0\| \leqslant \liminf \|T - F_\delta\| = d(T, \mathscr{K}_N(X, Y^*))$$

implies that F_0 is a best approximation to T. ∎

2.3. *Remarks.* (1) Fakhoury [7] had proved a special case of Theorem 2.2 when he showed that the "representable operators" in $\mathscr{L}(L_1, Y^*)$ have best approximations in $\mathscr{K}_N(L_1, Y^*)$.

(2) As noted in the proof of Theorem 2.2, we actually proved the stronger statement that $\mathscr{K}_N = \mathscr{K}_N(X, Y^*)$ is "boundedly weak*-operator compact." In particular, by a result of [5], not only is $\mathscr{K}_N$ proximinal but the metric projection $P_{\mathscr{K}_N}$ is norm-to-weak*-operator upper semicontinuous.

(3) Theorem 2.1 is false in general if $\mathscr{K}_N(X, Y^*)$ is replaced by the compact operators $\mathscr{K}(X, Y^*)$. (See, e.g., the example of Holmes and Kripke [9].)

A (linear) subspace Y of the normed linear space Z is said to be *norm-one complemented* in Z provided there is a bounded linear mapping P from Z onto Y with $P^2 = P$ and $\|P\| = 1$.

The next result is a useful device for asserting the proximinality of $\mathscr{K}_N(X, Y)$ in $\mathscr{L}(X, Y)$ when it is known that $\mathscr{K}_N(X, Z)$ is proximinal in $\mathscr{L}(X, Z)$ for a certain superspace Z which contains Y.

2.5. THEOREM. *If Y is norm-one complemented in a space Z and $\mathscr{K}_N(X, Z)$ is proximinal in $\mathscr{L}(X, Z)$, then $\mathscr{K}_N(X, Y)$ is proximinal in $\mathscr{L}(X, Y)$.*

Proof. Let P be a norm-one projection of Z onto Y. Let $T \in \mathscr{L}(X, Y)$. Since $\mathscr{K}_N(X, Y) \subset \mathscr{K}_N(X, Z)$, it follows that

$$d(T, \mathscr{K}_N(X, Z)) \leqslant d(T, \mathscr{K}_N(X, Y)).$$

Since $\mathscr{L}(X, Y) \subset \mathscr{L}(X, Z)$, it follows by hypothesis that T has a best approximation $\tilde{F} \in \mathscr{K}_N(X, Z)$. Let $F = P\tilde{F}$. Then $F \in \mathscr{K}_N(X, Y)$ and

$$\begin{aligned} \|T - F\| &= \|PT - P\tilde{F}\| = \|P(T - \tilde{F})\| \leqslant \|T - \tilde{F}\| \\ &= d(T, \mathscr{K}_N(X, Z)) \leqslant d(T, \mathscr{K}_N(X, Y)). \end{aligned}$$

Thus F is a best approximation to T from $\mathscr{K}_N(X, Y)$. ∎

2.6. COROLLARY. *If Y is norm-one complemented in a dual space (e.g., if Y is a dual space or if $\hat{Y}$ is norm-one complemented in Y^{**}), then $\mathscr{K}_{\chi}(X, Y)$ is proximinal in $\mathscr{L}(X, Y)$.*

Proof. Theorems 2.2 and 2.5. ∎

2.7. COROLLARY. *If Y is an abstract L-space, then $\mathscr{K}_{\chi}(X, Y)$ is proximinal in $\mathscr{L}(X, Y)$.*

Proof. Every abstract L-space is isometric to a space of type $L_1(\mu)$ for some measure μ (see, e.g., [18]). Also, $L_1(\mu)$ is norm-one complemented in its second dual (see, e.g., [18]). Now apply Corollary 2.6. ∎

2.8. COROLLARY (Fakhoury [7]). *Let S be an extremally disconnected compact Hausdorff space. Then $\mathscr{K}_{\chi}(X, C(S))$ is proximinal in $\mathscr{L}(X, C(S))$.*

Proof. We use the fact (see, e.g., [18]) that $C(S)$ is norm-one complemented in $l_{\infty}(S)$, and then apply Corollary 2.6. ∎

3. WHEN THE RANGE SPACE IS $C_0(S)$

In this section we will be concerned with the case when $Y = C_0(S)$. It is convenient to first have some notation. Let $l_{\infty}(S, X)$ denote the space of all norm bounded functions $f\colon S \to X$ equipped with the supremum norm $\|f\| = \sup\{\|f(s)\| \mid s \in S\}$. If τ denotes either the norm $(\|\cdot\|)$ or weak* (w^*) topology on a dual space X^*, let $C(S(X^*, \tau))$ denote the subspace of $l_{\infty}(S, X^*)$ of all τ-continuous functions $f\colon S \to (X^*, \tau)$.

Further, let

$$C_{w^*}(S, X^*) = \{f \in C(S, (X^*, w^*)) \mid \hat{x} \circ f \in C_0(S), x \in X\},$$

$C_0(S, X^*) = \{f \in C(S, X^*, \|\cdot\|)) \mid \{s \in S \mid \|f(s)\| \geqslant \varepsilon\}$ is compact for every $\varepsilon > 0\}$, and, if V is a subspace of X^*, let

$$C_0(S, V) = \{f \in C_0(S, X^*) \mid f(S) \subset V\}.$$

Note that $C_0(S, V) \subset C_0(S, X^*) \subset C_{w^*}(S, X^*) \subset C(S, (X^*, w^*))$. Further, if S is compact, then $C(S, (X^*, w^*)) = C_{w^*}(S, X^*)$ and $C(S, (X^*, \|\cdot\|)) = C_0(S, X^*)$.

The following representation theorem is essential for our purposes.

3.1. THEOREM. *The space $\mathscr{L}(X, C_0(S))$ is isometrically isomorphic to $C_{w^*}(S, X^*)$ via the mapping $T \in \mathscr{L}(X, C_0(S)) \to \tilde{T} \in C_{w^*}(S, X^*)$ defined by*

$$\tilde{T}(s)x = (Tx)(s), \qquad x \in X,\ s \in S. \tag{3.1.1}$$

Under this mapping, $\mathscr{K}(X, C_0(S))$ *is also isometrically isomorphic to* $C_0(S, X^*)$. *Moreover,* $T \in \mathscr{K}_N(X, C_0(S))$ *iff* $\tilde{T} \in C_0(S, V)$ *for some* N *dimensional subspace* V *of* X.

This result is well known, at least when S is compact (see, e.g., [6, p. 490]).

Let V be a proximinal subspace of the normed linear space Z. A selection for the metric projection P_V is any function $\sigma = \sigma_V : Z \to V$ such that $\sigma(z) \in P_V(z)$ for every $z \in Z$. A *continuous selection* for P_V is a selection which is also continuous.

3.2. Definition. A normed linear space Z is said to have the (CSF) *property* if the metric projection onto each finite dimensional subspace of Z has a continuous selection.

It is easily shown that each strictly convex space has the (CSF) property. Indeed, in this case the metric projections themselves are single-valued and continuous. More generally, any space with the property (P) of Brown [4] has the (CSF) property.

3.3. Theorem. *Let* X *be a normed linear space whose dual space* X^* *has the* (CSF) *property (e.g., if* X^* *is strictly convex). Then* $\mathscr{K}_N(X, C_0(S))$ *is proximinal in* $\mathscr{K}(X, C_0(S))$.

Proof. Let $K \in \mathscr{K}(X, C_0(S))$ and set $\mathscr{K}_N = \mathscr{K}_N(X, C_0(S))$. Using Theorem 3.1, we see that the function $\tau = \tilde{K}$ is norm-continuous and

$$d(K, \mathscr{K}_N) = \inf_{\substack{V \subset X^* \\ \dim V = N}} \inf_{f \in C_0(S, V)} \sup_{s \in S} \|\tau(s) - f(s)\|.$$

Given any N dimensional subspace V of X^*, let σ_V be a continuous selection for the metric projection P_V. Since $\|\sigma_V(\tau(s))\| \leqslant 2\|\tau(s)\|$ and $\tau \in C_0(S, X^*)$, it follows that $\sigma_V \circ \tau \in C_0(S, V)$ and hence

$$\begin{aligned} d(K, \mathscr{K}_N) &= \inf_{\substack{V \subset X^* \\ \dim V = N}} \sup_{s \in S} \|\tau(s) - \sigma_V(\tau(s))\| \\ &= \inf_{\substack{V \subset X^* \\ \dim V = N}} \sup_{s \in S} d(\tau(s), V). \end{aligned}$$

By a result of Garkavi [8], an N dimensional subspace V_0 of X^* exists for which the infimum is attained. Thus

$$d(K, \mathscr{K}_N) = \sup_{s \in S} \|\tau(s) - \sigma_{V_0}(\tau(s))\|.$$

By Theorem 3.1, $\sigma_{V_0} \circ \tau = \tilde{F}_0$ for some $F_0 \in \mathscr{K}_N(X, C_0(S))$ and

$$\|K - F_0\| = \sup_{s \in S} \|\tau(s) - \sigma_{V_0}(\tau(s))\| = d(K, \mathscr{K}_N).$$

That is, F_0 is a best approximation to K. ∎

3.4. *Remarks.* (1) Fakhoury [7] has proved Theorem 3.3 in the particular case when X^* is strictly convex and S compact.

(2) We do not know whether $\mathscr{K}(X, C_0(S))$ can be replaced by $\mathscr{L}(X, C_0(S))$ in Theorem 3.3.

(3) If S has the discrete topology, then *every* function defined on S is continuous. In this case, the same proof as given for Theorem 3.3 (where now c_V can be *any* selection for P_V) establishes the following result.

3.5. THEOREM. *Let S be any set with the discrete topology and X any normed linear space. Then $\mathscr{K}_N(X, c_0(S))$ is proximinal in $\mathscr{K}(X, c_0(S))$. In particular, $\mathscr{K}_N(X, c_0)$ is proximinal in $\mathscr{K}(X, c_0)$.*

We do not know whether $\mathscr{K}_N(X, c_0)$ is proximinal in $\mathscr{L}(X, c_0)$. However, with a certain restriction on X, the answer is affirmative.

3.6. THEOREM. *Let X be a uniformly smooth Banach space (i.e., X^* is uniformly convex). Then $\mathscr{K}_N(X, c_0)$ is proximinal in $\mathscr{L}(X, c_0)$.*

An essential step in the proof of this theorem is the following lemma whose proof can be found in [14].

3.7. LEMMA. *Let X be a uniformly convex Banach space, $r > 0$. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every point $x \in X$ with $\|x\| \leqslant r + \delta$ and every closed subspace V of X a point $y \in V$ with $\|y\| \leqslant \varepsilon$ exists such that*

$$\|x - y\| \leqslant \operatorname{Max}(r, d(x, V)).$$

Proof of Theorem 3.6. By Theorem 3.1, $\mathscr{L}(X, c_0)$ is isometrically isomorphic to the space $c_{w^*,0}(X^*)$ of all X^*-valued sequences $\{x_i\}_{i \in \mathbb{N}}$ which w^*-converges to 0, equipped with the norm of $l_\infty(\mathbb{N}, X^*)$, and $T \in \mathscr{K}_N(X, c_0)$ iff the corresponding sequence is in the set $A = \bigcup C_0(\mathbb{N}, V)$, where the union on the right hand side is taken over all subspaces V of X^* with $\dim V \leqslant N$. We show that even for every $x \in l_\infty(\mathbb{N}, X^*)$ there exists a best approximation in A. Let $x = \{x_k^*\}_{k \in \mathbb{N}} \in l_\infty(\mathbb{N}, X^*)$ be given. Let $R = d(x, A)$, $r_1 = \overline{\lim} \|x_k^*\|$. For every (closed) subspace V of X^* let $r_V = \sup_{k \in \mathbb{N}} \|x_k^* - P_V x_k^*\|$. Clearly

$$d(x, C_0(\mathbb{N}, V)) \geqslant \operatorname{Max}(r_1, r_V).$$

Let $\{\varepsilon_i\}_{i\in\mathbb{N}}$ be a strictly decreasing sequence of positive numbers with $\lim \varepsilon_i = 0$. We show first that there is a strictly increasing sequence $\{k_i\}_{i\in\mathbb{N}}$ of natural numbers such that for every subspace V of X^* x has a best approximation y_V in $C_0(\mathbb{N}, V)$ satisfying

$$\| y_{V,k} \| \leqslant \varepsilon_i$$

for every $i \in \mathbb{N}$ and every $k \geqslant k_i$. To construct such a best approximation, choose for every ε_i, $i \in \mathbb{N}$, a number δ_i for which the conclusion of Lemma 3.7 holds. For every $i \in \mathbb{N}$ there exists a $k_i \in \mathbb{N}$ such that for every $k \geqslant k_i$ the inequality $\| y_k \| < r_i + \delta_i$ holds (the sequence $\{k_i\}$ can obviously be chosen strictly increasing). For $k < k_1$ put $y_{V,k} = P_V x_k^*$. Let $i \in \mathbb{N}$, $k_i \leqslant k < k_{i+1}$. By Lemma 3.7 there exists a $y_k \in V$ such that $\| y_k \| \leqslant \varepsilon_i$ and

$$\| x_k^* - y_k \| \leqslant \text{Max}(r_1, \text{dist}(x_k^*, V)).$$

Put $y_{V,k} = y_k$. It follows immediately from the last inequality that for $y_V = \{ y_{V,k} \}_{k\in\mathbb{N}}$ we have

$$\| x - y_V \| = \sup_{k\in\mathbb{N}} \| x_k^* - y_{V,k} \| \leqslant \text{Max}(r_1, r_V) \leqslant d(x, C_0(\mathbb{N}, V)).$$

Hence y_V is a best approximation of x in $C_0(\mathbb{N}, V)$ with the required property.

Let $\{ V_j \}_{j\in\mathbb{N}}$ be a sequence of subspaces of X^* with $\dim V_j \leqslant N$ and

$$d(x, C_0(\mathbb{N}, V_j)) \leqslant R + 1/j$$

for every $j \in \mathbb{N}$. Let $y_j = y_{V_j}$, $j \in \mathbb{N}$, be the best approximation of x in $C_0(\mathbb{N}, V_j)$ constructed above. By Auerbach's lemma [3], for every $j \in \mathbb{N}$ there is a basis $z_1^j, ..., z_N^j$ of V_j and functionals $f_1^j, ..., f_N^j \in X^*$ such that for every $m = 1, ..., N$ we have $\| z_m^j \| = \| f_m^j \| = 1$ and

$$y_{j,k} = \sum_{m=1}^{N} f_m^j(y_{j,k})\, z_m^j$$

for every $k \in \mathbb{N}$. Without loss of generality assume that each of the sequences $\{z_m^j\}_{j\in\mathbb{N}}$ converges weakly to some $z_m \in X^*$ with $\| z_m \| \leqslant 1$, $m = 1, ..., N$. Let V_0 be the subspace of X^* generated by $z_1, ..., z_N$. Now, we construct an element y of $C_0(\mathbb{N}, V_0)$ for which $\| x - y \| \leqslant R$ holds. Let $k \in \mathbb{N}$. It follows from the above representation that the sequences $\{ f_m^j(y_{j,k}) \}_{j\in\mathbb{N}}$, $m = 1, ..., N$ are bounded. Hence we may without loss of generality assume that $\lim_j f_m^j(y_{j,k}) = f_{m,k}$ for some $f_{m,k} \in \mathbb{R}$, $m = 1, ..., N$. Denote $y_k = \sum_{m=1}^{N} f_{m,k}\, y_m$. Since y_k is the weak limit of the sequence $\{ y_{j,k} \}_{j\in\mathbb{N}}$ and since $y_{j,k}$ satisfies $\| y_{j,k} \| \leqslant \varepsilon_i$ if $k_i \leqslant k < k_{i+1}$ for some $i \in \mathbb{N}$, $y = \{ y_n \}_{n\in\mathbb{N}}$ is in $C_0(\mathbb{N}, V_0)$. Since $\|x_k^* - y_{j,k}\| \leqslant R + 1/j$ for every $j, k \in \mathbb{N}$, we have $\|x_k^* - y_k\| \leqslant R$ for every $k \in \mathbb{N}$. Hence y is a best approximation of x in A. ∎

The proof of the following lemma may be found in [13].

3.8. Lemma. *Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence in l_1 which w^*-converges to 0, $y \in l_1$. Then for every $\varepsilon > 0$ there exists an $i_0 \in \mathbb{N}$ such that for every $i \geqslant i_0$ we have*

$$|\,\|x_i - y\| - \|x_i\| - \|y\|\,| < \varepsilon.$$

3.9. Theorem. *$\mathscr{K}_N(c_0, c_0)$ is proximinal in $\mathscr{L}(c_0, c_0)$.*

Proof. Let $x = \{x_i\}_{i\in\mathbb{N}} \in c_{w^*,0}(l_1)$ (see the proof of Theorem 3.6 for the notation). Denote again by A the set $\bigcup C_0(\mathbb{N}, V)$, where the union is taken over all subspaces V of l_1 with $\dim V \leqslant N$. Let $R = d(x, A)$. Let $\{V_j\}_{j\in\mathbb{N}}$ be a sequence of subspaces of l_1 with $\dim V_j \leqslant N$ and

$$d(x, C_0(\mathbb{N}, V_j)) \leqslant R + 1/j$$

for every $j \in \mathbb{N}$. Using again Auerbach's Lemma [3], every V_j has a basis $y_1^j, ..., y_N^j$, $\|y_m^j\| = 1$, $m = 1, ..., N$ such that every $y \in V_j$ admits the representation

$$y = \sum_{m=1}^{N} f_m^j(y)\, y_m^j,$$

where $f_m^j \in V_j^*$, $\|f_m^j\| = 1$, $m = 1, ..., N$. Without loss of generality assume that y_m^j w^*-converges to some $y_m \in l_1$ with $\|y_m\| \leqslant 1$, $m = 1, ..., N$. Let $V_0 = \operatorname{span}\{y_1, ..., y_N\}$. For every $i, j \in \mathbb{N}$ let z_i^j be an arbitrary best approximation of x_i in V_j. Since the coefficients of $\{z_i^j\}_{j\in\mathbb{N}}$ in the above representation are bounded for every $i \in \mathbb{N}$, we may without loss of generality assume that $w^* - \lim_j z_i^j = z_i$ for some $z_i \in V_0$, $i \in \mathbb{N}$. Since for every i, $j \in \mathbb{N}$

$$\|x_i - z_i^j\| \leqslant d(X, C_0(\mathbb{N}, V_j)) \leqslant R + 1/j$$

holds, we have $\|x_i - z_i\| \leqslant R$. For every $i \in \mathbb{N}$ choose an arbitrary best approximation w_i of x_i in V_0. Obviously $\|x_i - w_i\| \leqslant R$ for every $i \in \mathbb{N}$. We show that $w = \{w_i\}_{i\in\mathbb{N}} \in C_0(\mathbb{N}, V_0)$. Assume the contrary. Then there is a subsequence of $\{w_i\}$, denote it again by $\{w_i\}$, which converges to some $w_0 \neq 0$. Let $\varepsilon = \|w_0\|/4$. Since $\{x_i - (w_i - w_0)\}_{i\in\mathbb{N}}$ w^*-converges to 0 there is, by the previous lemma, an $i_0 \in \mathbb{N}$ such that for every $i \geqslant i_0$ we have

$$|\,\|x_i - (w_i - w_0) - w_0\| - \|x_i - (w_i - w_0)\| - \|w_0\|\,| < \varepsilon$$

and

$$\|w_i - w_0\| < \varepsilon.$$

Hence

$$\begin{aligned}\|x_{i_0} - w_{i_0}\| &= \|x_{i_0} - (w_{i_0} - w_0) - w_0\| > \|x_{i_0} - (w_{i_0} - w_0)\| + \|w_0\| - \varepsilon \\ &\geqslant \|x_{i_0}\| - \|w_{i_0} - w_0\| + \|w_0\| - \varepsilon > \|x_{i_0}\| + \|w_0\| - 2\varepsilon \\ &= \|x_{i_0}\| + 2\varepsilon.\end{aligned}$$

It follows that w_{i_0} cannot be a best approximation of x_{i_0} which is a contradiction. ∎

4. Approximation by Compact Operators

In this section we make a few observations about approximating by compact operators.

If we approximate in $\mathcal{L}(X, c_0)$ by the compact operators $\mathcal{K}(X, c_0)$, rather than $\mathcal{K}_N(X, c_0)$, then there is a substantial amount which can be said (with no restriction on X).

4.1. Theorem. *Let S be any set with the discrete topology and X any normed linear space. For brevity, let $\mathcal{L} = \mathcal{L}(X, c_0(S))$, $\mathcal{K} = \mathcal{K}(X, c_0(S))$, and, for each $T \in \mathcal{L}$, let $d(T) = d(T, \mathcal{K})$ and define σ on $\mathcal{L}$ by*

$$\begin{aligned}[(\sigma T)x](s) &= 0 && \text{if } \|\tilde{T}(s)\| \leqslant d(T) \\ &= \left[1 - \frac{d(T)}{\|\tilde{T}(s)\|}\right](Tx)(s) && \text{otherwise}\end{aligned}$$

for $x \in X$, $s \in S$, where $\tilde{T}$ is defined as in (3.1.1). *Then:*

(1) *$\mathcal{K}(X, c_0(S))$ is proximinal in $\mathcal{L}(X, c_0(S))$.*

(2) $$d(T) = \inf_{F \in J} \sup_{s \in S \setminus F} \|\tilde{T}(s)\|,$$

where J denotes the class of all finite subsets of S.

(3) *For every $T \in \mathcal{L} \setminus \mathcal{K}$, $\mathcal{K}$ is the cone generated by the set $P_{\mathcal{K}}(T) - P_{\mathcal{K}}(T)$. In fact, for each $K \in \mathcal{K}$ with $\|K\| \leqslant \frac{1}{2} d(T)$, $K = T' - T''$ for some T', T'' in $P_{\mathcal{K}}(T)$. In particular, span $P_{\mathcal{K}}(T) = \mathcal{K}$ and $P_{\mathcal{K}}(T)$ is not compact.*

(4) $d_H(P_{\mathcal{K}}(T), P_{\mathcal{K}}(V)) \leqslant 2\|T - V\|$

for each T, V *in* $\mathscr{L}$ *and* 2 *is the smallest constant.* (*Here* d_H *denotes the Hausdorff metric.*) *In particular,* $P_{\mathscr{K}}$ *is Hausdorff continuous.*

(5) $P_{\mathscr{K}}$ *is lower semicontinuous, but* $P_{\mathscr{K}}$ *is not upper semicontinuous at any point of* $\mathscr{L} \setminus \mathscr{K}$.

(6) $P_{\mathscr{K}}^{-1}(0) \equiv \{T \in \mathscr{L} \mid 0 \in P_{\mathscr{K}}(T)\}$ *is nowhere dense.*

(7) $\sigma: \mathscr{L} \to \mathscr{K}$ *is a homogeneous selection for the metric projection* $P_{\mathscr{K}}$ *which is Lipschitz continuous*:

$$\|\sigma T - \sigma V\| \leqslant 2 \|T - V\|,$$

and 2 *is the smallest constant.*

(8) $\|\sigma T\| \leqslant \|T\| - d(T)$ *and* $\|\sigma T\| = \|T\|$ *if and only if* $T \in \mathscr{K}$.

(9) σ *is minimal in norm, i.e.,*

$$\|\sigma T\| = \inf\{\|T'\| \mid T' \in P_{\mathscr{K}}(T)\}, \qquad T \in \mathscr{L}.$$

This even holds pointwise:

$$\|(\sigma T)x\| = \inf\{\|T'x\| \mid T' \in P_{\mathscr{K}}(T)\}, \qquad T \in \mathscr{L},\, x \in X.$$

Proof. Let $C(S, X^*)$ denote the set of all $f: S \to X^*$ with $\|f\| \equiv \sup_{s \in S} \|f(s)\| < \infty$. Then, in the notation defined at the beginning of this section,

$$C_{w^*}(S, X^*) = \{f \in C(S, X^*) \mid \hat{x} \circ f \in c_0(S), x \in X\}$$

and

$$C_0(S, X^*) = \{f \in C(S, X^*) \mid \{s \in S \mid \|f(s)\| \geqslant \varepsilon\} \text{ is finite for every } \varepsilon > 0\}.$$

From [2; Proposition 4.1 and 4.3], it follows that the statements (1)–(9) are valid if $\mathscr{L}$ is replaced by $C(S, X^*)$ and $\mathscr{K}$ by $C_0(S, X^*)$. However, by Theorem 3.1, we may identify $\mathscr{L}(X, c_0(S))$ with $C_{w^*}(S, X^*)$ and $\mathscr{K}(X, c_0(S))$ by $C_0(S, X^*)$. Since $C_{w^*}(S, X^*) \subset C(S, X^*)$, it follows immediately that all of the statements except (6) hold. However, we shall prove in Theorem 5.1 below a much stronger statement than (6). ∎

4.2. *Remark.* From a result of Mach and Ward [15; Theorem 3.1], $\mathscr{K}(X, c_0(S))$ is an M-ideal in $\mathscr{L}(X, c_0(S))$. Thus statements (1) and (4) can also be deduced from the general M-ideal theory (see [1] and [10] resp.). Holmes *et al.* [10] had shown the *existence* of a continuous homogeneous selection for the metric projection onto an M-ideal. Unlike our proof, however, their proof was *non*constructive.

There is a large collection of pairs of normed linear spaces (X, Y) such that $\mathscr{K}(X, Y)$ is an M-ideal in $\mathscr{L}(X, Y)$. For example, (l_p, l_q) for $1 < p \leqslant q < \infty$ [17] and (X, c_0) for any normed space X [15; Theorem 3.1]. From the remark following Proposition 4.1 of [2], we immediately obtain

4.3. COROLLARY. *Let X, Y be normed linear spaces such that $\mathscr{K}(X, Y)$ is a (proper) M-ideal in $\mathscr{L}(X, Y)$. Then the metric projection $P_{\mathscr{K}(X,Y)}$ is Hausdorff continuous and lower semicontinuous, but it is not upper semicontinuous at any point of $\mathscr{L}(X, Y)\backslash\mathscr{K}(X, Y)$.*

5. MISCELLANEOUS FACTS AND SOME OPEN PROBLEMS

If M is a subset of the normed linear space Z, the *kernel* of the metric projection $P_M: Z \to 2^M$ is the set

$$\ker P_M = \{z \in Z \mid 0 \in P_M(z)\} = \{z \in Z \mid \|z\| = d(z, M)\}.$$

If M is a subspace, it is easy to see that $\ker P_M$ is a nonempty closed and proper "cone" in Z, i.e., $\lambda z \in \ker P_M$ whenever $z \in \ker P_M$ and $\lambda \geqslant 0$.

It is usually the case that the kernel of the metric projection onto a proximinal, but not Chebyshev, subspace has an interior. In spite of this, we have

5.1. THEOREM. *If X and Y are any normed linear spaces and M is any subset of $\mathscr{L}(X, Y)$ which contains $\mathscr{K}_1(X, Y)$, then* $\ker P_M$ *is nowhere dense in $\mathscr{L}(X, Y)$. In particular,* $\ker P_{\mathscr{K}(X,Y)}$ *is nowhere dense in $\mathscr{L}(X, Y)$.*

Proof. It suffices to show that $\ker P_M$ contains no ball centered at some nonzero $T \in \ker P_M$. Given any $\varepsilon > 0$, choose $x_0 \in X$, $\|x\| = 1$, such that $\|Tx_0\| > \|T\| - \varepsilon/4$. Choose $x_0^* \in X^*$, $\|x_0^*\| = 1$, such that $x_0^*(x_0) = 1$. Define a mapping $T_\varepsilon: X \to Y$ by

$$T_\varepsilon x = \frac{\varepsilon x_0^*(x)}{2\|Tx_0\|} Tx_0, \qquad x \in X.$$

Then $T_\varepsilon \in M$, $\|T_\varepsilon\| = \varepsilon/2$, and

$$\|T + T_\varepsilon\| \geqslant \|Tx_0 + T_\varepsilon x_0\| > \|T\| = d(T, M) = d(T + T_\varepsilon, M).$$

Thus $T + T_\varepsilon \notin \ker P_M$ and hence the ε-ball centered at T is not contained in $\ker P_M$. ∎

5.2. SOME OPEN PROBLEMS. During the course of our investigation, a number of questions arose naturally. With the intention of bringing these problems to the attention of a wider audience and thus, hopefully, contributing to their eventual solution, we list some of them here.

In Theorem 2.2 we showed that if Y is a *dual* space, then $\mathscr{K}_N(X, Y)$ is proximinal in $\mathscr{L}(X, Y)$.

5.2.1. *Question.* Is $\mathscr{K}_N(X, Y)$ proximinal in $\mathscr{L}(X, Y)$ for *any* pair of normed linear spaces X and Y?

One natural candidate for a counterexample would be when $Y = c_0$. But in this case, by Theorem 3.5, the answer is affirmative whenever X^* is uniformly convex or c_0. This remarks lead to the following specialization of Question 5.2.1.

5.2.2. *Question.* (a) Is $\mathscr{K}_N(X, c_0)$ proximinal in $\mathscr{L}(X, c_0)$ if X is either c or l_∞?

(b) More generally, is it possible to give a useful *characterization* of those normed spaces X such that $\mathscr{K}_N(X, c_0)$ is proximinal in $\mathscr{L}(X, c_0)$?

Such a characterization must, of course, include those spaces X with X^* uniformly convex (by Theorem 3.6).

5.2.3. *Question.* If X^* is uniformly convex, is $\mathscr{K}_N(X, C_0(S))$ proximinal in $\mathscr{L}(X, C_0(S))$?

By a result of Fakhoury [7], the answer is affirmative if $X = l_p$ for $1 < p < \infty$. Also, if S is a countable discrete set, the answer is affirmative by Theorem 3.6.

We have observed (see Remark 2.3(3)) that there are spaces X, Y such that $\mathscr{K}_N(X, Y)$ is proximinal in $\mathscr{L}(X, Y)$ but $\mathscr{K}(X, Y)$ is not proximinal in $\mathscr{L}(X, Y)$. This leads to the converse question.

5.2.4. *Question.* Do there exist spaces X, Y such that $\mathscr{K}(X, Y)$ is proximinal in $\mathscr{L}(X, Y)$, but $\mathscr{K}_N(X, Y)$ is not proximinal in $\mathscr{L}(X, Y)$?

5.2.5. *Question.* Is N "essential" in these theorems? That is, is $\mathscr{K}_N(X, Y)$ proximinal in $\mathscr{L}(X, Y)$ for every N if $\mathscr{K}_N(X, Y)$ is proximinal in $\mathscr{L}(X, Y)$ for some N?

An affirmative answer here would, of course, reduce all such questions of proximinality to the formally simpler question of whether or not $\mathscr{K}_1(X, Y)$ is proximinal in $\mathscr{L}(X, Y)$.

References

1. E. M. Alfsen and E. G. Effros, Structure in real Banach spaces, I, *Ann of Math. II. Ser.* **96** (1972), 98–128.
2. D. Amir and F. Deutsch, Approximation by certain subspaces of the Banach space of continuous vector valued functions, *J. Approx. Theory* **27** (1979), 1–17.

3. S. BANACH, "Théorie des operateurs lineaires," Monogr. Math. Tom. 1, Warzaw 1932, Chelsea, New York, 1955.
4. A. L. BROWN, Best n-dimensional approximation of functions, *Proc. London Math. Soc.* **14** (1964), 577–594.
5. F. DEUTSCH, Existence of best approximations, *J. Approx. Theory* **28** (1980), 132–154.
6. N. DUNFORD AND J. T. SCHWARTZ, "Linear Operators, I," Interscience, New York, 1958.
7. H. FAKHOURY, Approximation des bornés d'un espace de Banach par des compacts et applications à l'approximation des opérateurs bornés. *J. Approx. Theory* **26** (1979), 79–100.
8. A. L. GARKAVI, The best possible net and the best possible cross-section of a set in a normed space, *Amer. Math. Soc. Transl.* **39** (1964), 111–132.
9. R. B. HOLMES AND B. KRIPKE, Best approximation by compact operators, *Indiana Univ. Math. J.* **21**, (1971–1972), 255–263.
10. R. B. HOLMES, B. E. SCRANTON, AND J. D. WARD, Approximation from the space of compact operators and other M-ideals, *Duke Math. J.* **39** (1972), 1–8.
11. K. S. LAU, Approximation by continuous vector-valued functions, *Studio Math.* **68** (1980), 291–298.
12. A. LIMA, M-ideals of compact operators in classical Banach spaces, *Math. Scand.* **44** (1979), 207–217.
13. J. MACH, On the proximinality of compact operators with range in $C(S)$, *Proc. Amer. Math. Soc.* **72** (1978), 99–104.
14. J. MACH, Best simultaneous approximations of bounded functions with values in certain Banach spaces, *Math. Ann.* **240** (1979), 157–164.
15. J. MACH AND J. D. WARD, Approximation by compact operators in certain Banach spaces, *J. Approx. Theory* **23** (1978), 274–286.
16. C. L. OLSEN, Extreme points and finite rank operators in the set of compact approximants, *Indiana Univ. Math. J.* **24** (1979), 409–416.
17. K. SAATKAMP, M-ideals of compact operators, *Math. Z.* **158** (1978), 253–263.
18. H. H. SCHAEFFER, "Banach Lattices and Positive Operators," Springer-Verlag, Berlin/Heidelberg/New York, 1979.