Finitistic Dimensions of Noetherian Rings

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Communicated by J. T. Stafford

Received January 23, 1990

The global dimension gldim(R) of a Noetherian ring R, while a useful invariant when finite, is often infinite. Several homological dimensions have been introduced to handle rings of infinite global dimension. These dimensions include the injective dimension of the ring R as a left R-module, injdim_R(R), and the finitistic global dimensions, lfPD(R) and lFPD(R). The finitistic dimensions are defined as follows: lfPD(R) is the supremum of the projective dimensions pd(M) of left R-modules M of finite projective dimension, and lFPD(R) is the supremum of the projective dimensions pd(M) of left R-modules M which are finitely generated and have finite projective dimension. For a Noetherian ring R, when gldim(R) is finite, all these dimensions are equal (on the right and the left). Clearly, lFPD(R) ≤ lfPD(R), and Bass [B2, Proposition 4.3] has shown that for left Noetherian rings lFPD(R) ≤ injdim_R(R). We will note some further relationships between the dimensions in Proposition 2.1.

More is known about these dimensions when R is a commutative Noetherian ring. Auslander and Buchsbaum [AuBu, Theorem 1.6] showed that fPD(R) = codim(R) ≤ Kdim(R) (where codim(R) is the least upper bound on the lengths of R-sequences). Bass [B2, Proposition 5.4] showed

* Partially supported by the National Science Foundation.

0021-8693/92 $3.00
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that the Krull dimension of $R$ is related to the finitistic dimension by $Kdim(R) \leq FPD(R)$; Raynaud and Gruson [RG, p. 84, Theoreme 3.2.6] showed that $FPD(R) = Kdim(R)$. Even commutative Noetherian rings can have infinite codimension (e.g., [N, Appendix, Example 1]), and hence infinite $fPD$. Commutative Noetherian rings with finite injective dimension form a special class of Cohen–Macaulay rings (see [B2]); there exist commutative Noetherian rings of finite Krull (and hence finitistic) dimension, but infinite injective dimension.

Little is known about the finitistic dimension of a noncommutative Noetherian ring. Our purpose here is to present techniques that allow one to compute (or bound) the finitistic dimension of some Noetherian rings. The computations we have made suggest that $1FPD$ and $rFPD$ are useful invariants of a noncommutative ring.

In the first section of this paper some basic properties of finitistic dimension are given, and the finitistic dimensions of some Noetherian rings are computed. We note that even in nice Noetherian rings $1FPD(R)$ and $1fPD(R)$ can be arbitrarily different, that $1FPD$ behaves nicely with respect to localization while $1fPD$ does not, that $1FPD(R[x])$ and $1fPD(R)$ can be arbitrarily different, that $rFPD(R)$ and $1FPD(R)$ can be arbitrarily different even for nice prime Noetherian rings, and we provide some techniques for computing the FPD in idealizers and pullbacks.

In the second section we relate the injdim$_R(R)$ and the FPD($R$), and use the injective dimension to show that the FPD of the universal enveloping algebra of a finite dimensional Lie superalgebra is the dimension of the Lie algebra of even terms (Proposition 2.3).

In the third section we relate the Krull dimension of a Noetherian semiprime PI ring to its finitistic dimension. We will follow the approach of [ReSS] in which it is shown that for a semiprime Noetherian PI ring $R$, $Kdim(R) \leq gldim(R)$ (this result has been extended to all Noetherian PI rings by results in [BrW, GS]). Here (Proposition 3.3) we show that $Kdim(R) \leq \min\{1FPD(R), rFPD(R)\} + 1$ for any semiprime Noetherian PI ring. We note that K. Brown [Br] has proved that for many FBN rings, $Kdim(R) \leq injdim_R(R)$.

1. FINITISTIC DIMENSION

In this section we will examine some basic properties of the finitistic dimensions: the effects of localization, the relationship between $1fPD$ and $1FPD$, the relationship between the finitistic dimension of a ring and the finitistic dimensions of certain subrings and factor rings, and the lack of left–right symmetry in finitistic dimension; we will also compute the finitistic dimensions of some idealizer rings and “pullbacks.”
The following well-known change of rings theorem will be used throughout (see, e.g., [Rot, p. 245, Theorem 9.32]).

**Lemma 1.1.** Let \( \varphi : R \to S \) be a ring homomorphism, and let \( M \) be a left \( S \)-module. Then \( \text{pd}_R(M) \leq \text{pd}_S(M) + \text{pd}_R(S) \).

Jensen [J, p. 164, Proposition 6] proved that if \( R \) is any ring in which \( \text{lfpd}(R) \) is finite, then any module with finite flat (weak) dimension \( \text{fd}(M) \) has finite projective dimension \( \text{pd}(M) \). It follows from this result that \( \text{lfpd} \) shares with global dimension the property of behaving nicely with localization.

**Theorem 1.2.** Let \( Q \) be the localization of any ring \( R \) with respect to a left denominator set \( \mathcal{M} \). If \( \text{fd}_R(Q) < \infty \), then \( \text{lfpd}(Q) \leq \text{lfpd}(R) \).

**Proof.** Suppose that \( \text{lfpd}(R) < \infty \), and let \( M \) be a left \( Q \)-module with \( n = \text{pd}_Q(M) < \infty \). Let \( 0 \to P_n \to P_{n-1} \to \cdots P_0 \to M \to 0 \) be a projective resolution of \( M \) as a \( Q \)-module; since \( \text{fd}_R(Q) < \infty \), it follows from [J] that \( \text{pd}_R(P_i) < \infty \), so \( \text{pd}_R(M) < \infty \). Applying \( Q \otimes_R - \) to a projective resolution of \( RM \) gives a projective resolution of \( Q \otimes_R M \) as a \( Q \)-module, so \( \text{pd}_Q(M) \leq \text{pd}_R(M) \leq \text{lfpd}(R) \).

We note that the hypotheses of Theorem 1.2 are satisfied if \( \mathcal{M} \) is a right and left denominator set.

It is perhaps surprising that \( \text{fPD} \) does not behave nicely with respect to localization, as the following example shows. This example also shows that \( \text{fPD}(R) \) and \( \text{FPD}(R) \) can be arbitrarily different, even in a commutative local Noetherian ring; this is in contrast to the global dimension of a Noetherian ring, which can be computed by considering only cyclic modules.

**Example 1.3.** Let \( k \) be any field, let \( T \) be the ring of power series \( T = k[[x_1, \ldots, x_{n+1}]] \), and let \( S = k[[y]]/(y^2) \). Let \( R \) be the subring of \( S \oplus T \), \( R = \{ (f(y), g(x_1, \ldots, x_{n+1})) : f(0) = g(0, \ldots, 0) \} \); note that \( R \) is a commutative local Noetherian ring. By [B1, Corollary 5.6] \( \text{fPD}(R) = 0 \), since \( y, 0 \in \text{ann}(\text{rad}(R)) \). Let \( T_{x_1} \) denote the localization of \( T \) at the multiplicatively closed set \( \{ x_1^m : m > 0 \} \). Choosing \( \mathcal{M} = \{(0, x_1)^m : m \in \mathbb{N} \} \) gives \( Q \approx T_{x_1} \), a ring of global dimension \( n \) (and hence \( \text{fPD}(Q) = n \)). Note further that by [GR], \( \text{FPD}(R) = \text{Kdim} R = n + 1 \), so that even in a commutative local Noetherian ring \( R \) the two finitistic dimensions \( \text{fPD}(R) \) and \( \text{FPD}(R) \) can differ by an arbitrary amount, and \( \text{fPD}(R) \) can be smaller than the Krull dimension of \( R \). It can also be shown using [AuBu, Theorem 2.11] that \( \text{fPD}(R[x]) - \text{fPD}(R) \geq n + 1 \).
We note that an example of Auslander and Buchsbaum [AuBu, p. 202-3] also shows that $fPD(R)$ and $FPD(R)$ can be arbitrarily different in a commutative Noetherian local ring. Their example also shows that $fPD(R[x]) - fPD(R)$ can be arbitrarily large [AuBu, p. 204-5]; we note that $lFPD(R[x]) = lFPD(R) + 1$ (see [Rot, pp. 248-249; Co, Theorem 2.4]).

Example 1.3 was not a prime ring; the following example shows the same sort of pathology can occur in a Noetherian local prime PI ring which is module-finite over its center.

**Example 1.4.** Let $D = k[t, x_1, \ldots, x_n]$ and let $R = \left(\begin{smallmatrix} P^* & 0 \\ 0 & \delta \end{smallmatrix}\right)$, where *-entries have the same constant term. The ring $R$ is a Noetherian local prime PI ring which is module-finite over its center. Consider the ring $S = R/\langle T \rangle$, where $T = (t_0, \ldots)$; let $\nu: R \to S$ be the natural homomorphism. By [B1, Corollary 5.6.], since $\nu(\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right))(\text{rad}(S)) = 0$, $lFPD(S) = 0$, and so by [GrKK, Lemma 2.7] $lFPD(R) = 1$. Let $R_T$ (resp. $D_T$) be the localization of $R$ (resp. $D$) at the central multiplicatively closed set $\{T^m : m > 0\}$ (resp. $\{t^m : m > 0\}$); then $R_T \approx M_2(D_T)$, a ring of global dimension $\dim$, so that $\dim = lFPD(R_T) = 1$. Let $R_\alpha$ (resp. $D_\alpha$) be the localization of $R$ (resp. $D$) at the central multiplicatively closed set $\{T^m : m > 0\}$ (resp. $\{t^m : m > 0\}$); then $R_\alpha \approx M_2(D_\alpha)$, a ring of global dimension $\dim + 1$. Note that $Gdim(R) = \dim + 1$.

The following proposition [KK2, Proposition 3] was used to find upper bounds on the global dimension of a ring which is a "pullback" (or "fibre product," see [Mi]); using a particular example we will illustrate how the proposition can be used to find upper bounds on the finitistic dimension of a ring. As one example, the proposition can be used to compute $lFPD(R)$ when the ring $R$ shares an ideal $A$ with an overring $S$, and the syzygies of projective resolutions of $A$ and $S$ as right $R$-modules eventually have summands which repeat periodically: in this context the proposition becomes a change of rings theorem since it uses homological properties of the factor ring $R/A$ and the overring $S$ to bound $lFPD(R)$.

**Proposition 1.5 [KK2, Proposition 3].** Let $R$ be a pullback of $R_1$ and $R_2$ over $R'$. Let $M$ be a left $R$-module such that $\text{Tor}_{i+m}^R(R_i, M) = 0$ for $m > 0$, $i = 1, 2$. Then $\text{pd}_R(M) + m \leq \max_i \{n_i + \text{pd}_{R_i}(R_i \otimes_{R_j} \text{Im}(f_{n_j}))\}$, where $\to P_0 \to \cdots \to P_i \to M \to 0$ is a projective resolution of $M$.

We modify Example 1.4 above to produce an example to illustrate the technique described above.

**Example 1.6.** Let $D = k[[t]]$ and let $R = \left(\begin{smallmatrix} P^* & 0 \\ 0 & \delta \end{smallmatrix}\right)$, where *-entries have the same constant term. Let $R_1 = M_2(k[[t]])$, $A = M_2(tk[[t]])$, $R_2 = R/A \approx k$, and $R' = R_1/A$; since $A$ is an ideal in both $R$ and $R_1$, as noted in [KK2], $R$ is a pullback of $R_1$ and $R_2$ over $R'$. Note further that $R_1 \approx A \approx \epsilon_{11}R_1 \oplus$
Consider the following surjection \( \varphi: R \oplus R \rightarrow e_{11} R_1 \) given by
\[
\varphi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} + \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \right) = (a + b_{11} - d_{21}, -c + b_{12} - d_{22}),
\]
where \( a, c \in k \) and \( b_{ij}, d_{ij} \in tk[[t]] \). Note that kernel \( \varphi \cong R_1 \oplus e_{11} R_1 \), so that the exact sequence \( 0 \rightarrow R_1 \oplus e_{11} R_1 \rightarrow R \oplus R \rightarrow e_{11} R_1 \rightarrow 0 \) can be extended to an exact sequence of arbitrary length
\[
0 \rightarrow \Omega_i \rightarrow F_{i-1} \rightarrow \cdots \rightarrow F_0 \rightarrow e_{11} R_1 \rightarrow 0
\]
with \( F_j \) a free right \( R \)-module and \( \Omega_i = F'_i \oplus e_{11} R_1 \), for \( F'_i \) a free right \( R_1 \)-module. Then \( \text{Tor}_i^R(e_{11} R_1, M) \cong \text{Tor}_i^R(\Omega_i, M) \) for \( 1 \leq i \leq m - 1 \). If \( \text{pd}(M) < \infty \), then by choosing \( m > \text{fd}_R(M) \), we have \( \text{Tor}_i^R(\Omega_{m-1}, M) = 0 \) and hence \( \text{Tor}_i^R(e_{11} R_1, M) = 0 \) (recall that \( e_{11} R_1 \) is a summand of \( \Omega_{m-1} \)).

We next note two change of rings theorems relating the finitistic dimensions of a ring and a factor ring of the ring. The first result was proved for global dimension by Fields.

**Theorem 1.7.** Let \( I \) be an idempotent ideal of \( R \).

(i) \([F, \text{Theorem 1}]\) If \( I \) is projective as a left \( R \)-module or flat as a right \( R \)-module, then \( \text{pd}_{R/I}(M) \leq \text{pd}_R(M) \) for a left \( R/I \)-module \( M \).

(ii) If \( I \) is projective as a left \( R \)-module, then \( \text{lFPD}(R/I) \leq \text{lFPD}(R) \).

**Proof:** To see (ii), if \( \text{pd}_{R/I}(M) < \infty \), then by Lemma 1.1, \( \text{pd}_R(M) \leq \text{pd}_{R/I}(M) + \text{pd}_R(R/I) \leq \text{pd}_{R/I}(M) + 1 \), so \( \text{pd}_R(M) < \infty \), and the result follows from (i).

The proof of Small's change of rings theorem \([S, \text{Theorem 1}]\) for global dimension gives parts (i) and (ii) of the following theorem; the theorem below is useful when forming a factor ring by an ideal of finite homological dimension (on the opposite side) or, more generally, by an ideal whose projective resolution eventually has syzygies whose summands repeat periodically (recall the computation of \( \text{lFPD}(R) \) in Example 1.6).

**Theorem 1.8.** Let \( R \) be a ring and let \( I \) be an ideal of \( R \).

(i) If \( I \) is nilpotent, \( \text{lFPD}(R) \leq \text{lFPD}(R/I) + \text{fd}(R/I)_R \).
(ii) If $R$ is left Noetherian and $I \subseteq \text{rad}(R)$ then $\text{lfPD}(R) \leq \text{lfPD}(R/I) + \text{fd}(R/I)_R$.

(iii) Suppose in either (i) or (ii) $R$ is right Noetherian and $\text{fd}(R/I)_R$ is infinite, but that $I$ has a projective resolution as a right $R$-module having the following properties. Let the resolution be

$$
\cdots \rightarrow P_{n-1} \xrightarrow{f_n} P_n \rightarrow I \rightarrow 0
$$

with $\Omega_i(I) = \ker f_{i-1}$. Suppose that there exists an integer $c$, such that given any positive integer $N$, every indecomposable summand $S$ of $\Omega_{c+k}(I)$ for $k \geq 0$ is a summand of $\Omega_{n(S)}(I)$ for some $n(S) \geq N$, then $\text{lfPD}(R) \leq \text{lfPD}(R/I) + c$ in (i) (and $\text{lfPD}(R) \leq \text{lfPD}(R/I) + c$ in (ii)).

**Proof.** For (iii) note that if $\text{pd}_R(M) < \infty$ then by the reasoning used in Example 1.6 above $\text{Tor}_p^{R}(R/I,M) = 0$ for all $p > 0$, and hence $\text{Tor}_p^{R}(R/I,\Omega_c(M)) = 0$, where $\Omega_c(M)$ is the $c$th syzygy in a projective resolution of $M$. Hence by [S, Lemma 1] $\text{pd}_R(R/I,\Omega_c(M)/I\Omega_c(M)) = \text{pd}_R(\Omega_c(M))$, which is finite. Thus $\text{pd}(M) = \text{pd}_R(R/I,\Omega_c(M)/I\Omega_c(M)) + c$, giving the result.

For a Noetherian ring $R$, $\text{rgldim } R = \text{lgldim } R$; it is well known that the finitistic dimensions need not be left–right symmetric for finite dimensional algebras. In [GrKK, Example 2.3] a finite dimensional algebra $A$ is constructed with $\text{rFDP}(A) = \text{rfPD}(A) = m$ and $\text{lFDP}(A) = \text{lfPD}(A) = s$ for arbitrary finite numbers $m$ and $s$. The lemma below shows that this algebra $A$ can be used to produce a Noetherian prime affine PI ring which is module-finite over its center and has left and right finitistic dimensions which are arbitrarily different.

**Lemma 1.9.** Let $A$ be a finite dimensional $k$-algebra, which, without loss of generality, we assume is a subring of $M_n(k)$. Let $D = k[x]$ and let $R$ be a subring of $M_n(D)$ such that $R/M_n((x)) \cong A$. Let $\Gamma = \begin{bmatrix} D & (x) \cdots (x) \\ D & \\ \vdots & \\ D & R \end{bmatrix} \subseteq M_{n+1}(D)$.

Then $\text{rFDP}(A) \leq \text{rFDP}(\Gamma) \leq \text{rFDP}(A) + 2$ (and similarly on the left).

**Proof.** The lower bound holds by Theorem 1.7 since $I = \begin{bmatrix} D & (x) \cdots (x) \\ \vdots \\ D & M_n((x)) \end{bmatrix}$.
is an idempotent ideal which is a right (and left) projective \( I \)-module with \( I/I \cong A \).

The upper bound holds by the following argument, similar to that working for global dimension \([\text{KK3, Proposition 2.4}]\). Let \( M \) be a right \( I \)-module with \( \text{pd}(M) < \infty \). Start a projective resolution of \( M \)

\[
0 \to K \to F \to M \to 0
\]

with \( F \) a free \( I \)-module. Then \( \text{pd}(K) < \infty \).

Consider the short exact sequence \( 0 \to KeI \to K \to K/KeI \to 0 \), where \( e \) is the matrix unit \( e_{11} \). Since \( I = IeI \), we have \( KI = KKeI = KeI \), and therefore \( K/KeI \) is a right \( I/I \)-module and \( \text{pd}(K/KeI) \leq \text{pd}(K/KeI)_{I/I} + \text{pd}(I/I)_{I} \leq \text{pd}(K/KeI)_{I/I} + 1 \), since \( I \) is projective. The result will follow if we show that \( K/KeI \) is \( I \)-projective, for then \( \text{pd}(K/KeI)_{I/I} < \infty \), and so Field's Theorem (Theorem 1.7(i)) gives \( \text{pd}(K/KeI)_{I/I} \leq \text{pd}(K/KeI)_{I/I} + 1 \), so that \( \text{pd}(K) \leq \text{rFPD}(I/I) + 1 \), and hence \( \text{pd}(M) \leq \text{rFPD}(I/I) + 2 \).

To show that \( KeI \) is \( I \)-projective, let \( \{b_{\alpha}\} \) be a basis for \( Ke \) over \( D \) as a right \( D \)-module; such a basis exists since \( Ke \) is a finitely generated torsion free module over the principal ideal domain \( D \). We first show that \( b_{\alpha}eI \simeq eI \) as right \( I \)-modules for each \( b_{\alpha} \). Consider the map \( \varphi : eI \to b_{\alpha}eI \) given by \( \varphi(e_{\gamma}) = b_{\alpha}e_{\gamma} \). If \( e_{\gamma} \in \ker(\varphi) \) then \( b_{\alpha}e_{\gamma} = 0 \) implies that \( b_{\alpha}e_{\gamma}e = 0 \) and hence \( e_{\gamma}e = 0 \) since \( \{b_{\alpha}\} \) is a basis over \( D \). Thus \( \ker(\varphi) \) is a right ideal of \( I \) with the property that \((\ker(\varphi))e = 0\), so \( e_{\gamma} \in \ker(\varphi) \) implies \( e_{\gamma}eI = 0 \). This implies that \( e_{\gamma} = 0 \) since \( I \) is prime, and hence \( \ker(\varphi) = 0 \). Next we show that \( KeI \simeq \bigoplus b_{\alpha}eI \). It is clear that \( \{b_{\alpha}\} \) is a generating set for \( KeI \) as a right \( I \)-module. Consider the map \( \Psi : \bigoplus b_{\alpha}eI \to KeI \) defined on each summand by \( \Psi(b_{\alpha}e_{\gamma}) = b_{\alpha}e_{\gamma} \). Suppose \( \Psi(\Sigma b_{\alpha}e_{\gamma}) = 0 \); then \( \Sigma b_{\alpha}e_{\gamma}e = 0 \), so \( e_{\gamma}e = 0 \) since \( \{b_{\alpha}\} \) is a basis for \( Ke \) over \( eIe \). Thus \((\ker(\Psi))e = 0 \). Since \( \bigoplus b_{\alpha}eI \) is \( I \)-projective, \( \ker(\Psi) \subseteq I^{(v)} \) for some \( v \). Let \( \pi_{\beta} \) be the projection of \( I^{(v)} \) onto the \( \beta \)th component. Since \( \pi_{\beta}(\ker(\Psi)) \) is a right ideal of \( I \) with \( \pi_{\beta}(\ker(\Psi))e = 0 \), again, since \( I \) is prime, we have \( \pi_{\beta}(\ker(\Psi)) = 0 \). Hence \( \ker(\Psi) = 0 \), and \( KeI \) is \( I \)-projective.

We call a subring \( R \) of a ring \( S \) a subidealizer in \( S \) of a right ideal \( A \) of \( S \) if \( A \) is a (two-sided) ideal of \( R \); we call \( A \) generative if \( SA = S \). When \( R \) is a subidealizer in \( S \), often \( R \) and \( S \) have related homological properties (e.g., see [Ro1, Ro2, RoS, G, KK1, St]). We first give a generalization of a result on global dimension [RoS, Theorem 2(ii) and Corollary 3] (see also [H, Kor. 3.8]).

**Proposition 1.10.** Suppose that \( R \) is a subidealizer in \( S \) of a generative right ideal \( A \) of \( S \) with \( \text{pd}(R/A) < \infty \). Then \( \max\{\text{IFPD}(R/A), \text{IFPS}(S)\} \leq \text{IFPD}(R) \leq \max\{\text{IFPD}(R/A) + 1, \text{IFPD}(S) + \text{pd}(R/A)\} \), so that \( \text{IFPD}(R) \leq \text{IFPD}(R/A) + \text{IFPD}(S) + 1 \).
Proof. As in [McR, p. 153, Lemma 5.5.7], the existence of the generative right ideal $A$ implies that $S_R$ is projective, that $R_A$ is projective, that $S \otimes_R R \simeq S$ under multiplication, that [McR, p. 251, Proposition 7.5.11] for a right $S$-module $M$, pd$(M_R) = \text{pd}(M_S)$, and that for a left $S$-module $N$, $\text{pd}_S(N) \leq \text{pd}_R(N)$. As $A$ is an idempotent ideal, by Field's result (Theorem 1.7(i)) we obtain $\text{lFPD}(R/A) \leq \text{lFPD}(R)$. Since $\text{pd}_R(S) < \infty$, if $N$ is a left $S$-module with $\text{pd}_S(N) < \infty$, then $\text{pd}_R(N) < \infty$ and $\text{pd}_S(N) \leq \text{pd}_R(N)$, so that $\text{lFPD}(S) \leq \text{lFPD}(R)$.

Let $K$ be a left $R$-submodule of a free $R$-module (and hence $K$ is an $R$-submodule of a free $S$-module $F = \bigoplus S$), and assume that $\text{pd}_R(K) < \infty$. Since $S$ is a projective right $R$-module, $S \otimes_R K$ embeds in $S \otimes_R F \simeq \bigoplus S \otimes_R S \simeq \bigoplus S$; note that the image of $S \otimes_R K$ under this $S$-isomorphism is $SK$. Since $S_R$ is projective $\text{pd}_S(SK) = \text{pd}_S(S \otimes_R K) \leq \text{pd}_R(K)$. Further note that $\text{pd}_R(SK) \leq \text{pd}_S(S \otimes_R K) + \text{pd}_R(S) < \infty$, and, in fact, $\text{pd}_R(SK) \leq \text{lFPD}(S) - 1 + \text{pd}_R(S)$ since $SK$ is a submodule of a free $S$-module. The exact sequence $0 \to K \to SK \to SK/K \to 0$ shows that $\text{pd}_R(SK/K) < \infty$. Next note that $SK/K$ is a left $R/A$-module; by Field's Theorem (Theorem 1.7(i)) $\text{pd}_{R/A}(SK/K) \leq \text{pd}_R(SK/K) < \infty$. Since $R_A$ is a projective we have $\text{pd}_R(SK/K) \leq \text{pd}_{R/A}(SK/K) + 1 \leq \text{IFPD}(R/A) + 1$. When $\text{pd}_R(SK) \leq \text{pd}_R(K)$, then $\text{pd}_R(K) \leq \text{pd}_R(SK) \leq \text{IFPD}(S) - 1 + \text{pd}_R(S)$. In either case, $\text{pd}_R(K) \leq \max\{\text{IFPD}(R/A), \text{IFPD}(S) - 1 + \text{pd}_R(S)\}$ and hence $\text{IFPD}(R) \leq \max\{\text{IFPD}(R/A) + 1, \text{IFPD}(S) + \text{pd}_R(S)\}$. As in [RoS, Corollary 3] the exact sequence $0 \to R \to S \to S/R \to 0$ of $R$-modules gives $\text{pd}_R(S) \leq \text{pd}_{R/A}(S/R) \leq \text{pd}_{R/A}(S/R) + \text{pd}_R(R/A) \leq \text{pd}_{R/A}(S/R) + 1$. Note that $S/R$ is a left $R/A$-module; again $\text{pd}_R(S/R) < \infty$ and Field's Theorem (Theorem 1.7(i)) gives $\text{pd}_{R/A}(S/R) < \infty$, so that $\text{pd}_R(S) \leq \text{IFPD}(R/A) + 1$, and $\text{IFPD}(R) \leq \text{IFPD}(S) + \text{IFPD}(R/A) + 1$.

When $R$ is a right subidealizer at a generative right ideal $A$ of $S$ and at a generative left ideal $B$ of $S$ we can say more.

**Corollary 1.11.** If $R$ is a subidealizer at a generative right ideal $A$ of $S$ and at a generative left ideal $B$ of $S$ then $\max\{\text{IFPD}(S), \text{IFPD}(R/A)\} \leq \text{IFPD}(R) \leq \max\{\text{IFPD}(S), \text{IFPD}(R/A) + 1\}$ and $\max\{\text{IFPD}(R), \text{rFPD}(R/B)\} \leq \text{rFPD}(R) \leq \max\{\text{IFPD}(S), \text{rFPD}(R/B) + 1\}$.

**Proof.** The result follows from Proposition 1.10 since the existence of $B$ gives $\text{pd}_R(S) = 0$.

As a corollary we obtain a generalization of a result known to hold for global dimension [KK1, Corollary 1.7].

**Corollary 1.12.** Let $R$ be any ring, and let $I$ be an ideal of $R$. Then $\text{lFPD}[R, I] = \max\{\text{IFPD}(R), \text{IFPD}(R/I) + 1\}$ (and similarly on the right).
Proof. The ring $A = R/I$ is a subidealizer of the generative right ideal $A = M_2(R)$, and of the generative left ideal $B = R/I$ of $M_2(R)$. Thus by Corollary 1.11, $l\text{FPD}(R) \leq l\text{FPD}(A)$ and $l\text{FPD}(A) \leq \max\{l\text{FPD}(R), l\text{FPD}(R/I) + 1\}$.

To show $l\text{FPD}(R/I) + 1 \leq l\text{FPD}(A)$, let $M$ be a left $R/I$ module with $pd_{R/I}(M) = l\text{FPD}(R/I)$. Begin a projective resolution of $M$ over $R$, $0 \to K 	o P 	o M 	o 0$ with $P$ a projective $R$-module. Note that $IP \subseteq K$, and hence $N = [P, K]$ is a left $A$-module. By [KK1, Lemma 1.21], $L = [P, IP]'$ is a projective left $A$-module. Note that $0 \to K/IP 	o P/IP 	o M \to 0$ is an exact sequence of $R/I$-modules with $P/IP$ a projective $R/I$-module, and hence $pd_{R/I}(K/IP) < \infty$. Next consider the exact sequence of $A$-modules $0 \to L \to N \to \text{coker} \to 0$. One can check that $\text{coker} \cong K/IP$ as $R/I$-modules, and hence $pd_{R/I}(\text{coker}) < \infty$. The action of $R/I$ on $\text{coker}$ is the same as the action of $A/A \cong R/I$ on $\text{coker}$, so $pd_A(\text{coker}) \leq pd_{R/I}(\text{coker}) + \text{pd}_A(A/A) \leq pd_{R/I}(\text{coker}) + 1 < \infty$, and hence $pd_A(N) < \infty$. Then [KK1, Lemma 1.3] shows that $pd_{R/I}(M) \leq pd_A(N)$. The left $A$-module $Q = [P, P]'$ is projective [KK1, Lemma 1.1], and the sequence $0 \to N \to Q \to Q/N \to 0$ gives an $A$-module $Q/N$ with $l\text{FPD}(R/I) + 1 \leq \text{pd}(Q/N) < \infty$.

We next illustrate how this result can be used to produce examples of Noetherian prime $PI$ rings with homological properties similar to those of nonprime rings. In [GrKK, Example 1.11] a finite dimensional $k$-algebra $A'$ is given with $l\text{FPD}(A') = n - 1$ and $r\text{FPD}(A') = 0$. We will use this example to produce a Noetherian affine prime $PI$ ring $A$ which is module-finite over its center, with $l\text{FPD}(A) = n - 1$ and $r\text{FPD}(A) \leq 3$ for $n \geq 2$.

**Example 1.13.** Let $C$ be the ring of $n$ by $n$ triangular matrices of the form

$$C = \begin{bmatrix} k & k & \cdots & k & k \\ 0 & k & \cdots & k & k \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k & 0 & \cdots & 0 & S \\ 0 & 0 & \cdots & 0 & S \end{bmatrix},$$

where $S = k[x]/(x^2)$

and $x \in S$ acts trivially on $k$; then the f.d. $k$-algebra $A'$ described above is isomorphic to $C/\text{rad} C)^2$. Now let

$$R = \begin{bmatrix} D & D & \cdots & D & D \\ (x) & D & \cdots & D & D \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (x) & (x) & \cdots & (x) & D & D \\ (x^2) & (x^2) & \cdots & (x^2) & D \end{bmatrix}$$
where \( D = k[x] \), and let

\[
I = \begin{bmatrix}
(x) & (x) & D & \cdots & D & D \\
(x) & (x) & (x) & D & D \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(x) & (x) & (x) & \cdots & (x) & (x) \\
(x^2) & (x^2) & (x^2) & \cdots & (x^2) & (x^2)
\end{bmatrix}
\]

Then \( R/I \cong C/(\text{rad } C)^2 = A' \). The ring \( A = R_\mathcal{R} \) is an affine Noetherian prime PI ring, module-finite over its center. To compute the finitistic dimensions of \( A \) we will first compute \( \text{rFPD}(R) \) and \( \text{lFPD}(R) \). Now \( R \) is itself a subidealizer of the generative right ideal

\[
A = \begin{bmatrix}
D & \cdots & D \\
(x) & \cdots & (x) \\
\vdots & \vdots & \vdots \\
(x) & \cdots & (x) \\
(x^2) & \cdots & (x^2)
\end{bmatrix}
\]

and the generative left ideal

\[
B = \begin{bmatrix}
(x^2) & \cdots & (x^2) & D \\
\vdots & \vdots & \vdots & \vdots \\
(x^2) & \cdots & (x^2) & D
\end{bmatrix}
\]

of \( M_n(D) \).

Since \( R/A \) is a monomial algebra, using methods of [GrKK] one can compute that \( \text{lFPD}(R/A) = 1 \), so that \( \text{lFPD}(R) \leq \max\{1, 1 + 1\} = 2 \). To compute \( \text{rFPD}(R) \), note that \( R/B \) is a subidealizer in \( M_{n-1}(S) \) of the generative right ideal

\[
A'' = \begin{bmatrix}
S & \cdots & S \\
x/x^2 & \cdots & x/x^2 \\
\vdots & \vdots & \vdots \\
x/x^2 & \cdots & x/x^2
\end{bmatrix}
\]

and the generative left ideal

\[
B'' = \begin{bmatrix}
x/x^2 & \cdots & x/x^2 & S \\
\vdots & \vdots & \vdots & \vdots \\
x/x^2 & \cdots & x/x^2 & S
\end{bmatrix}
\]
rFPD\left( R/B^{n} \right) \leq \max\{ rFPD(M_{n-1}(S)), rFPD(T_{n-2}(k)) + 1 \} = \max\{ 0, 1 + 1 \} = 2. \quad \text{Thus} \quad rFPD(R) \leq \max\{ rFPD(D), rFPD(R/B) + 1 \} = \max\{ 1, 3 \} = 3, \quad \text{and hence} \quad A \text{ has } lFPD(A) = \max\{ lFPD(R), lFPD(R/D) + 1 \} = \max\{ lFPD(R), lFPD(A') + 1 \} = \max\{ 2, n \} = n \text{ for } n \geq 2, \quad \text{and } rFPD(A) = \max\{ rFPD(R), rFPD(A') + 1 \} \leq \max\{ 3, 1 \} \leq 3.

2. Injective Dimension and FPD

In this section we will note the relationship between injdim\(_{R}(R)\) and lFPD\(_{R}(R)\), and then use results of Bjork on injective dimension to compute the lFPD of the universal enveloping algebra of a finite dimensional Lie superalgebra over a field of characteristic zero.

Bass [B2, Proposition 4.3] proved that lFPD\(_{R}(R) \leq injdim\(_{R}(R)\) holds for left Noetherian rings; we have noted that examples exist in which injdim\(_{R}(R)\) is infinite, but lFPD\(_{R}(R)\) is finite. A. Zaks [Z, p. 84, Lemma A] has noted that injective dimension, unlike finitistic dimension, is symmetric for Noetherian rings when it is finite on both sides. This fact, combined with results of Bass, shows that for Noetherian rings, when the injective dimension is finite on both sides it is the same as the finitistic dimension (on both sides). We know of no example of a Noetherian ring with finite injective dimension on one side and infinite injective dimension on the other side; Auslander has noted that if a finite dimensional algebra with these properties exists, it would be a finite dimensional algebra \( A \) with fPD\(_{A}(A)\) infinite. Any Noetherian ring with differing rFPD and lFPD must then have infinite injective dimension on at least one side.

**PROPOSITION 2.1.** Suppose that \( R \) is a Noetherian ring with injdim\(_{R}(R) \leq \infty \) and injdim\(_{R}(R) \leq \infty \). Then:

(a) \[ Z \text{ injdim}(R)_{R} = injdim_{R}(R). \]

(b) \[ injdim_{R}(R) = lFPD_{R}(R) = rFPD_{R}(R). \]

**Proof.** By [B2, Proposition 4.3] lFPD\(_{R}(R) \leq injdim\(_{R}(R)\), and by [M, Theorem 1] lFID\(_{R}(R) = rFWD\(_{R}(R)\), where lFID\(_{R}(R)\) is the supremum of the injective dimension of left modules of finite injective dimension, and rFWD\(_{R}(R)\) is the supremum of the flat (weak) dimension of right modules of finite flat dimension. By Jensen’s result [J, Proposition 6] rFWD\(_{R}(R) \leq rFPD\(_{R}(R)\) because when rFPD\(_{R}(R)\) is finite, any module \( M \) of finite flat dimension will have finite projective dimension, and fd\(_{R}(M) \leq pd\(_{R}(M)\). Thus, if injdim\(_{R}(R)\) is finite then lFID\(_{R}(R)\) is finite, and we have lFID\(_{R}(R) = rFWD\(_{R}(R) \leq rFPD\(_{R}(R) \leq injdim\(_{R}(R)\) < \infty \). Hence by [B2, Proposition 4.2] these dimensions are all equal. \[ \square \]
It is known that if $R = k[G]$ is the group ring of a polycyclic-by-finite group $G$, then $\text{injdim}_R(R) = \text{the Hirsch number of } G$; it is also known that if $R$ is the trace ring of $m \times n$ generic matrices, then $\text{injdim}_R(R) = \text{Kdim}(R)$. Hence Proposition 2.1 computes the finitistic dimensions of these rings.

We note the following special case of a result of Bjork which we will use to compute the injective dimension (and hence FPD) of the universal enveloping algebra of a finite dimensional Lie superalgebra (see [Be] for definitions). As noted in [Be] the global dimension of the universal enveloping algebra of a Lie superalgebra can be infinite (see also [AL]). The global dimension of the universal enveloping algebra of a finite dimensional Lie algebra $L_0$ is equal to the vector space dimension of $L_0$ [CE, p. 283, Theorem 8.2].

**Theorem 2.2 [Bj, Corollary 3.12].** Let $R$ be a filtered ring with discrete filtration $\Sigma_0 \subseteq \Sigma_1 \subseteq \ldots$, and suppose that the associated graded ring $\text{gr}(R)$ is Noetherian. Then $\text{injdim}_R(R) \leq \text{injdim}_{\text{gr}(R)}(\text{gr}(R))$.

It is well known that $\text{gldim}(R) \leq \text{gldim}(\text{gr}(R))$ [McR, 7.6.18]; it is not difficult to see that such a result does not hold for $\text{FPD}(R)$. For example, let $R$ be a finite dimensional $k$-algebra of finite global dimension $n = \text{gldim}(R) > 0$. Choose the filtration $\Sigma_0 = k$, $\Sigma_i = R$ for $i \geq 1$. Then $\text{gldim}(\text{gr}(R)) = \infty$; but, $\text{FPD}(\text{gr}(R)) = 0$ because $\text{gr}(R)$ is local. Hence, $\text{FPD}(\text{gr}(R)) = 0 < \text{FPD}(R) = n$. Note that $\text{injdim}_{\text{gr}(R)}(\text{gr}(R))$ is infinite.

**Proposition 2.3.** Let $L = L_0 \oplus L_1$ be a finite dimensional Lie superalgebra over a field of characteristic zero, with $L_0$ the Lie algebra of even elements. Let $U = U(L)$ be the universal enveloping algebra of $L$ and $U_0 = U(L_0)$ the universal enveloping algebra of $L_0$. Then $\text{gldim}(U_0) = \text{FPD}(U) = \text{injdim}_U(U) = \dim(L_0)$ (and similarly on the right).

**Proof.** As shown in [Be, p. 12], $\text{Gr } U = A(Y_1, \ldots, Y_n)[X_1, \ldots, X_m]$, the polynomial ring in $m = \dim(L_0)$ (commuting) indeterminates over the exterior algebra $A(Y_1, \ldots, Y_n)$. Since $A(Y_1, \ldots, Y_n)$ is a local finite dimensional algebra with simple socle it follows from [Di, (3.1)] (or see [B3, (2.8)]) that the exterior algebra is self-injective, and from [B2, Theorem 2.2] that $A(Y_1, \ldots, Y_n)[X_1, \ldots, X_m]$ has injective dimension $m = \dim(L_0)$; hence by Theorem 2.2 $\text{FPD}(U) \leq \text{injdim}_U(U) = \dim(L_0)$. It follows from [Co, Theorem 1.4] that $m = \text{gldim}(U_0) \leq \text{FPD}(U)$ because $U$ is free over $U_0$ (on both sides).
3. KRULL DIMENSION AND FPD

For a commutative Noetherian ring $R$ the Krull dimension of $R$ and the finitistic dimension $\text{FPD}(R)$ are equal [RG]. In a series of papers by Resco, Small, and Stafford [ReSS], K. Brown and Warfield [BrW], and Goodearl and Small [GS], it is shown that for a Noetherian PI ring $R$, the inequality $\text{Kdim}(R) \leq \text{gldim}(R)$ holds; it remains an open question whether this inequality holds in all (right and left) Noetherian rings. K. Brown and C. Hajarnavis [BrH] have related $\text{Kdim}(R)$ and $\text{injdim}_R(R)$ for Noetherian rings integral over their centers, and Brown [Br] has proved that for a certain class of FBN rings (a class including many Noetherian PI rings; see [Br, Theorem D]) the inequality $\text{Kdim}(R) \leq \text{injdim}_R(R)$ holds. Here we will prove a relationship between the Krull dimension and the finitistic dimension; we prove $\text{Kdim}(R) \leq \min\{\text{rFPD}(R), \text{lFPD}(R)\} + 1$ for semiprime Noetherian PI rings. As examples in the Section 1 indicate, the right and left finitistic dimensions of a ring can be quite different.

We will follow the approach of [ReSS]: we consider the case first of an Azumaya algebra, then of a Noetherian prime PI Jacobson ring, then of the “monic localization” of a Noetherian prime PI ring, and finally of a Noetherian semiprime PI ring. The possibly unnecessary addition of one in our upper bound is due to our inability to obtain results for the FPD of a “monic localization” which are as precise as the results known to hold for global dimension.

**Lemma 3.1.** If $R$ is a Noetherian Azumaya algebra then $\text{rFPD}(R) = \text{lFPD}(R) = \text{Kdim}(R)$.

**Proof.** If $C$ is the center of $R$, by standard properties of Azumaya algebras $\text{Kdim}(R) = \text{Kdim}(C)$, and by [RG] $\text{Kdim}(C) = \text{FPD}(C)$. Standard properties of Azumaya algebras and [Co, Theorem 1.4] give $\text{FPD}(C) \leq \text{lFPD}(R)$.

The reverse inequality also follows in the usual manner. If $M$ is an $R$-module with $\text{pd}_R(M) < \infty$, then $\text{pd}_C(M) < \infty$, and $\text{pd}_C(M) = \text{pd}_R(M)$ follows from the “projective lifting” property of Azumaya algebras (see e.g., [DI, p. 48]).

**Corollary 3.2.** In a prime Noetherian Jacobson PI ring, for every $n \leq \text{Kdim}(R)$ there exists a regular prime with height $(P)=n$. Therefore $\text{Kdim}(R) \leq \text{lFPD}(R)$.

**Proof.** The first statement is Corollary 1.3 of [ReSS]. Choosing $p$ to be a regular prime with height $(p)=\text{Kdim}(R)$, then the prime $p$ is localizable, $R_p$ has unique maximal ideal $pR_p$, and $R_p$ is an Azumaya algebra. Thus
\( \text{ht}(p) = \text{ht}(pR_p) = \text{Kdim}(R_p) = \text{IFPD}(R_p) \leq \text{IFPD}(R) \), the last inequality by Theorem 1.2.

The following proposition follows as in [ReSS, Theorem 2.5].

**Proposition 3.3.** If \( R \) is a Noetherian semiprime PI ring, then \( \text{Kdim } R \leq \min\{\text{IFPD}(R), \text{rIFPD}(R)\} + 1 \).

**Proof:** When \( R \) is a Noetherian prime PI ring then \( \mathcal{M} = \{f(x) : f(x) \in R[x] \text{ is a monic polynomial}\} \) is a right and left denominator set in \( R[x] \), \( R<x> = R[x]_\mathcal{M} \) is a prime Noetherian Jacobson PI ring [ReSS, Theorem 2.8] with \( \text{Kdim } R<x> = \text{Kdim } R \) [ReSS, Theorem 2.4], \( \text{gldim } R<x> = \text{gldim } R \) [ReSS, Theorem 2.5], and \( R<x> \) is a faithfully flat left \( R \)-module [ReSS, Lemma 2.5]. Since FPD does not increase with this localization (Theorem 1.2), \( \text{IFPD } R<x> \leq \text{IFPD}(R[x]) = \text{IFPD}(R) + 1 \). By Corollary 3.2, \( \text{Kdim } R = \text{Kdim } R<x> \leq \text{IFPD } R<x> \leq \text{IFPD}(R) + 1 \) (similarly on the right).

The semiprime case follows from the prime case as in [ReSS, Theorem 3.2].

Clearly the bound in the previous theorem could be improved if one could prove that \( \text{IFPD}(R<x>) = \text{IFPD}(R) \), which is true when \( R \) is commutative, or when \( R \) has finite global dimension [ReSS, Theorem 2.5]. When \( R \) has finite finitistic dimension we have been able to obtain only the bounds: \( \text{IFPD}(R) - \text{pd}_R(R<x>) \leq \text{IFPD}(R(x)) \leq \text{IFPD}(R) + 1 \) (the lower bound from [Co, Proposition 1.5]). It remains an open question whether in a Noetherian ring \( R \) the finitistic dimension \( \text{IFPD}(R) \) can be less than the Krull dimension of \( R \).

**References**


Kirkman, Kuzmanovich, and Small


