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Bilinear Forms on G-Algebras

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TO J. A. GREEN, ON HIS 60TH BIRTHDAY

INTRODUCTION

It is a pleasure to acknowledge the influence of J. A. Green on this paper. Our original aim was to produce a proof of the main results of [Ro] using an endomorphism ring approach suggested by Green. It soon became apparent that (as we and Green had suspected) there was a common theme linking parts of Green's paper on Scott modules [Gr2] with the papers [Br1, Ro] (although the results of these papers are independent).

It turns out that the most convenient setting to work with is that of G-algebras (see [Gr1]) endowed with G-stable bilinear forms somewhat analogous to that introduced by Green in [Gr2].

The paper is divided into two more or less independent parts, the only overlap being when we combine the results of both sections to prove the main result of [Ro].

In the first section, we develop general properties of bilinear forms on G-algebras, include some applications which illustrate how our techniques unify some apparently diverse results (such as a recent result of Benson and Carlson on the multiplicity of the Scott module in $M \otimes M^*$ (see [Be-Ca]), results on the number of blocks of KG with given defect groups, results about height 0 characters), and give explicit formulae for the ranks of the forms we define.

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In the second section, we examine links between group algebras and Hecke algebras to produce (via the results of the first section) a proof of the main result of [Ro]. We also obtain some general results linking the structure of Hecke algebras with group-theoretic properties (such as control of fusion) and mention some consequences for the structure of permutation modules and their summands.

Notation

For an algebra A, we let $J(A)$ denote the Jacobson radical of A, and $A^{\times}$ denote the group of multiplicative units of A. When G is a finite group, R is a ring, and X is a subset of G, we let $\mathscr{S}X$ denote $\sum_{g \in X} g$ in RG.

When A, B are subgroups of G, we let $[A\backslash G/B]$ denote a full set of representatives for the (A, B) double coset representatives in G, and we abbreviate $[1\backslash G/B]$ to $[G/B]$, $[A\backslash G/1]$ to $[A\backslash G]$.

When M is an RG-module, and H is a subgroup of G, we let M^H denote the set of H-fixed points of M, and we define $\mathrm{Tr}_H^G\colon M^H \to M^G$ by $\mathrm{Tr}_H^G(m) = \sum_{g \in [G/H]} gm$.

1. Bilinear Forms on G-Algebras

A. *Generalities*

Let G be a finite group, p be a prime, k an algebraically closed field of characteristic p. Let A denote a finite dimensional G-algebra over k (see [Gr1]), and suppose further that A is endowed with a linear form $t\colon A \to k$ such that for all $x, y \in A$, all $g \in G$, we have

$$t(xy) = t(yx), \qquad t(g \cdot x) = t(x).$$

For any p subgroup, P, of G, let A_P^G denote $\mathrm{Tr}_P^G(A^P)$, a two-sided ideal of A^G (see [Gr1]). We set

$$A(P) = A^P \Big/ \sum_{Q < P} A_Q^P,$$

and we denote by $\mathrm{Br}_P\colon A^P \to A(P)$ the canonical epimorphism, which we call the Brauer morphism associated with P (see [Br2]). The algebra $A(P)$ is clearly an $\bar{N}(P)$ algebra (where $\bar{N}(P) = N_G(P)/P$), and the algebra morphism Br_P is then an $\bar{N}(P)$-morphism.

Since

$$\mathrm{Tr}_P^G(x) = \sum_{[P\backslash G/P]} \mathrm{Tr}_{P \cap {}^gP}^P (g \cdot x) \qquad \text{for} \quad x \in A^P$$

it is clear that (see [Br2])

$$\mathrm{Br}_P \circ \mathrm{Tr}_P^G = \mathrm{Tr}_1^{\bar{N}(P)} \circ \mathrm{Br}_P. \tag{1.1}$$

Moreover, we see that $\ker \mathrm{Br}_P \subset \ker t$, and thus t defines a linear form $t_P: A(P) \to k$ which is still "symmetric," and $\bar{N}(P)$ stable.

DEFINITION. We define the bilinear form

$$\rho_{P,G}^{A,t}: \quad A_P^G \times A_P^G \to k$$

as follows: if $x = \mathrm{Tr}_P^G(x')$ and $y = \mathrm{Tr}_P^G(y')$ $(x', y' \in A^P)$, then

$$\rho_{P,G}^{A,t}(x, y) = t(xy').$$

(1.2) PROPOSITION. (1) *The form $\rho_{P,G}^{A,t}$ is well defined, symmetric, and associative.*

(2) *Whenever $x, y \in A_P^G$, we have*

$$\rho_{P,G}^{A,t}(x, y) = \rho_{1,\bar{N}(P)}^{A(P),t_P}(\mathrm{Br}_P(x), \mathrm{Br}_P(y)).$$

Proof. Let $x = \mathrm{Tr}_P^G(x')$ and $y = \mathrm{Tr}_P^G(y')$. By definition of t_P, we have $t(xy') = t_P(\mathrm{Br}_P(xy')) = t_P(\mathrm{Br}_P(x)\,\mathrm{Br}_P(y'))$.

By (1.1), we also know that

$$\mathrm{Br}_P(x) = \mathrm{Tr}_1^{\bar{N}(P)}(\mathrm{Br}_P(x')) \qquad \text{and} \qquad \mathrm{Br}_P(y) = \mathrm{Tr}_1^{\bar{N}(P)}(\mathrm{Br}_P(y')).$$

We then see that in order to prove (1), we may as well assume that $P = 1$ (replacing G, P, A by $\bar{N}(P)$, 1, $A(P)$), and also that (2) is clear.

Now we assume $P = 1$. Since t is G-stable, we have

$$t(\mathrm{Tr}_1^G(x')\, y') = t(x'\, \mathrm{Tr}_1^G(y'));$$

this shows that $t(xy') = t(xy'')$ as soon as $\mathrm{Tr}_1^G(y') = \mathrm{Tr}_1^G(y'')$, and so that the form is well defined. It is symmetric since $t(xy') = t(y'x)$, and associative since $zy = \mathrm{Tr}_1^G(zy')$ for all $z \in A^G$, and so

$$\rho_{1,G}^{A,t}(xz, y) = t(xzy') = \rho_{1,G}^{A,t}(x, zy).$$

The next result is really a special case of Green's main result in [Gr2].

(1.3) PROPOSITION. *Suppose that (A, t) is a symmetric algebra. Then for any p-subgroup P, of G, the rank of the form $\rho_{P,G}^{A,t}$ is the multiplicity of the Scott module with vertex P as a summand of A (where A is considered just as kG-module).*

Proof. In [Gr2], J. A. Green proved that, given a kG-module M, the multiplicity of the Scott module with vertex P as a summand of M is given by the rank of the bilinear form $\langle\ ,\ \rangle_P$: $M_P^G \times (M^*)_P^G \to k$ defined by

$$\langle \mathrm{Tr}_P^G(m), \mathrm{Tr}_P^G(\phi)\rangle_P = \phi(\mathrm{Tr}_P^G(m)) \qquad \text{for } m \in M \text{ and } \phi \in (M^*)^P.$$

Let A^* denote the (k-vector space) dual of A. Since (A, t) is a symmetric algebra, there is a k-vector space isomorphism β: $A \to A^*$ given by $\beta(y)(x) = t(xy)$ for all $x, y \in A$. Since t is G-stable, β is easily seen to be an isomorphism of kG-modules. In particular, $\beta(A^P) = (A^*)^P$.

Now $\rho_{P,G}^{A,t}(\mathrm{Tr}_P^G(x'), \mathrm{Tr}_P^G(y')) = t(\mathrm{Tr}_P^G(x')\, y') = \beta(y')(\mathrm{Tr}_P^G(x') = \langle \mathrm{Tr}_P^G(x'), \mathrm{Tr}_P^G(\beta(y'))\rangle_P$, where $\langle\ ,\ \rangle_P$ denotes Green's form from $A_P^G \times (A^*)_P^G$ to k. Thus the forms $\rho_{G,G}^{A,t}$ and $\langle\ ,\ \rangle_P$ have the same rank, which suffices to complete the proof of (1.3).

As an example, let us consider the case where $A = kG$, the group algebra of G over k, endowed with the usual form

$$t\colon \sum_{g \in G} x(g)\, g \to x(1).$$

Then the form $\rho_{P,G}^{kG,t}$ is just the form denoted by $\bar{\beta}_{P,G}$ in [Br2]. Let us recall (see [Gr1]) that $(kG)_P^G$ has for basis the family $(\mathscr{S}C)$ where C runs over the set of conjugacy classes of G with a defect group contained in P. As done in [Br2], it is then immediate to check that

$$\rho_{P,G}^{kG,t}(SC, SC')$$
$$= \begin{cases} 0 & \text{if } C \text{ or } C' \text{ has defect smaller than } P \\ \delta_{C',C^{-1}} \cdot |C|_{p'}/|G|_{p'} & \text{if } C \text{ and } C' \text{ have defect } P, \end{cases}$$

from which it follows that rad $\rho_{P,G}^{kG,t}$ equals $\ker \mathrm{Br}_P \cap (kG)_P^G$.

Now if e is any idempotent of ZkG, it follows that

$$\mathrm{rk}\ \rho_{P,G}^{kG,t} = \dim \mathrm{Br}_P((kG)_P^G e) = \dim(kC_G(P)_1^{\bar{N}(P)}\, \mathrm{Br}_P(e)),$$

the "multiplicity of P associated with e." Either as in Green [Gr2] by applying (1.3), or as in [Br2], we then deduce immediately the following result of Burry:

(1.4) COROLLARY. *Let b be a primitive idempotent of ZkG, and let us consider the block $kG \cdot b$ as a kG-module by the conjugation action. Then for any p-subgroup, P, of G, the multiplicity of P as a lower defect group for b is the same as the multiplicity of the Scott module with vertex P as a summand of $kG \cdot b$.*

B. *When* $J(A) \subset \ker t$

We now return to the situation where A need not be a symmetric algebra. In fact, we assume from now on that $J(A) \subset \ker t$.

We need to recall a definition.

DEFINITION (Puig [Pu]). A point of A is an $A^{\times}$-conjugacy class of primitive idempotents of A.

Of course, the points of A are in natural bijection with the maximal two-sided ideals of A, and with the isomorphism classes of simple A-modules. For a point, s, of A, we let χ_s denote the character of A afforded by the corresponding simple A-module $Ai/J(A)i$ where i is an idempotent in s. We let $t(s)$ denote $t(i)$ (which depends only on s, not on the particular i chosen).

(1.5) LEMMA. *We have* $t = \sum_s t(s)\, \chi_s$, *where* s *runs over the points of* A.

Proof. Since $J(A) \subset \ker t$, and since there is (up to scalar multiples) only one symmetric linear form on a matrix algebra, we may write $t = \sum_s \alpha_s \chi_s$, where each $\alpha_s \in k$. It is easy to see that $\chi_s(s') = \delta_{s,s'}$ whenever s, s' are points of A, so that $\alpha_s = t(s)$ for each point s.

It is clear from the above statement that $t(x) = 0$ whenever x is a nilpotent element of A. In particular, whenever A' is a subalgebra of A, we have $J(A') \subset \ker(t|_{A'})$, so the arguments of (1.5) may be applied with A' in place of A and $t|_{A'}$ in place of t. This remark applies in particular to the algebra A^P where P is a p-subgroup of G.

The Brauer morphism $\mathrm{Br}_P\colon A^P \to A(P)$ identifies points of $A(P)$ with certain points of A^P, namely those s such that $\mathrm{Br}_P(s) \neq \{0\}$. We set $s_P = \mathrm{Br}_P(s)$. Then we have

$$t_P = \sum_s t_P(s_P)\, \chi_{s_P}, \tag{1.6}$$

where s runs the points of A^P such that $s_P \neq \{0\}$.

Considering A just as a P-algebra, we may define (following Green, see [Gr1]) the defect groups in P of a point of A^P: if i is a primitive idempotent of A^P, its defect groups are the subgroups Q of P, minimal subject to $i \in A_Q^P$; the defect groups of i are unique up P-conjugacy (see [Gr1]), and they obviously depend only on the $(A^P)^{\times}$-conjugacy class of i, i.e., on the point of i. By Rosenberg's lemma, it is then clear that (see [Pu])

(1.7) whenever s is a point of A^P, we have $s_P \neq 0$ if and only if s has defect group P in P.

The next result gives the "local calculation" of $\rho^{A,t}_{P,G}$.

For s a point of A^P, we denote by V_s the associated simple A^P-module, by $\sigma_s: A^P \to \mathrm{End}_k V_s$ the associated morphism, by $\mathrm{tr}_s: \mathrm{End}_k V_s \to k$ the ordinary trace form, and by $\bar{N}(P, s)$ the stabilizor of s in $\bar{N}(P)$.

(1.8) PROPOSITION. (1) *For $x, y \in A^G_P$, we have*

$$\rho^{A,t}_{P,G}(x, y) = \sum_{(s)} t(s)\, \rho^{\mathrm{End}_k V_s, \mathrm{tr}_s}_{1, \bar{N}(P,s)}(\sigma_s(x), \sigma_s(y)),$$

where s runs over a set of representatives for the $\bar{N}(P)$-conjugacy classes of points of A^P with defect group P in P, and such that $t(s)$ (i.e., $t_P(s_P)) \neq 0$.

(2) *We have*

$$\mathrm{rk}\, \rho^{A,t}_{P,G} = \sum_{(s)} \dim_k (\mathrm{End}_k V_s)^{\bar{N}(P,s)}_1,$$

where the sum is taken over the same set as in (1).

Proof. (1) By (1.2)(2), it is immediate that we are reduced to the case where $P = 1$ (replacing, as now usual, A, G, P by $A(P)$, $\bar{N}(P)$, 1). In that case, we denote $\bar{N}(P, s)$ by G_s.

Let $x = \mathrm{Tr}^G_1(x')$ be an element of A^G_1. Then, by the Mackey-type formula, $x = \mathrm{Tr}^{G_s}_1(\sum_{[G_s \backslash G]} \sigma_s(g \cdot x'))$, and since σ_s is a G_s-morphism we get

$$\sigma_s(x) = \mathrm{Tr}^{G_s}_1\Bigl(\sum_{[G_s \backslash G]} \sigma_s(g \cdot x')\Bigr).$$

By definition of the form ρ, we then have to prove

$$t(x'y) = \sum_{(s)} t(s)\, \mathrm{tr}_s\Bigl(\sum_{[G_s \backslash G]} \sigma_s(g \cdot x')\, \sigma_s(y)\Bigr),$$

i.e.

$$t(x'y) = \sum_{(s)} t(s)\, \chi_s\Bigl(\sum_{[G_s \backslash G]} (g \cdot x')\, y\Bigr).$$

By Lemma (1.5), we know that

$$t = \sum_s t(s)\, \chi_s = \sum_{(s)} t(s) \sum_{[G_s \backslash G]} \chi_{g^{-1}(s)}.$$

Thus it suffices to prove that whenever s is a point of A, we have

$$\sum_{[G_s \backslash G]} \chi_{g^{-1}(s)} = \sum_{[G_s \backslash G]} \chi_s \circ g, \qquad \text{which is trivial.}$$

(2) Again we may reduce to the case $P=1$, and we do so. The rank of the form $\rho_{1,G_s}^{\mathrm{End}_k V_s, \mathrm{tr}_s}$ is clearly equal to $\dim_k(\mathrm{End}_k V_s)_1^{G_s}$. The assertion follows from the fact that the map

$$\prod_{(s)} \sigma_s\colon \quad A_1^G \to \prod_{(s)} (\mathrm{End}_k V_s)_1^{G_s} \qquad \text{is onto.}$$

C. *Applications*

We continue to assume that $J(A) \subset \ker t$.

First, we need to recall some ideas of Puig [Pu] extending to G-algebras the notion of source (which was previously defined by Green for modules).

The next statement is one of Puig's basic results ([Pu], Theorem 1.2).

When s is a point of A^P, we denote by $A^P \cdot s \cdot A^P$ the two-sided ideal generated by elements of s (recall that s may be viewed as a set of idempotents of A^P).

(1.9) *Suppose that A^G is local, and let P be a defect group of* 1 *in G.*

(1) *There exists a point s of A^P such that $1 \in \mathrm{Tr}_p^G(A^P \cdot s \cdot A^P)$.*

(2) *Such a point has defect P in P, and any other point of A^P with defect P in P is $\bar{N}(P)$-conjugate to s.*

We sketch a proof here for the convenience of the reader. See [Pu] to view this result in the general context of "pointed groups."

(1) is an easy consequence of Rosenberg's lemma, and by transitivity of the relative trace, we see that s must have defect group P in P. Let s and s' two points of A^P such that $A^G = \mathrm{Tr}_P^G(A^P \cdot s \cdot A^P)$ and s' has defect P in P, and let i' be an element of s'. Then

$$i' \in \mathrm{Tr}_p^G(A^P \cdot s \cdot A^P) \subset \sum_{[P \backslash G / P]} \mathrm{Tr}_{P \cap {}^g P}^P(g \cdot (A^P \cdot s \cdot A^P)).$$

By Rosenberg's lemma, we see that there exists $g \in G$ such that $i' \in \mathrm{Tr}_{P \cap {}^g P}^P\,(g \cdot (A^P \cdot s \cdot A^P))$, and since i' has defect P, we see that $g \in N(P)$. Thus $i' \in A^P \cdot g(s) \cdot A^P$, from which it follows that $s' = g(s)$, as required.

If A^G is local and if s is a point of A^P such that (1.9)(1) holds, s is called a *source* of the G-algebra A.

The following lemma will be crucial in the applications.

(1.10) LEMMA. *Suppose that A^G is local.*

(1) *If $\rho_{P,G}^{A,t} \neq 0$, then P is contained in a defect group of 1_A in G.*

(2) *If P is a defect group of 1_A in G, then $\rho_{P,G}^{A,t} \neq 0$ if and only if $t(s) \neq 0$ for s a source of the G-algebra A. In that case, we have*

$$\mathrm{rk}\, \rho_{P,G}^{A,t} = \dim_k(\mathrm{End}_K V_s)_1^{\bar{N}(P,s)}.$$

(3) *If* $A^G \subset Z(A)$, *then* $\rho_{P,G}^{A,t} \neq 0$ *if and only if P is a defect group of* 1_A *in G and* $t(s) \neq 0$ *for s a source of A. In that case,* $\operatorname{rk} \rho_{P,G}^{A,t} = 1$.

Proof. (1) If P is not contained in a defect group of 1_A in G, then $A(P) = 0$ and it follows from (1.2)(2) that $\rho_{P,G}^{A,t} = 0$.

(2) By (1.9), there is a single $\bar{N}(P)$-conjugacy class of points of A^P with defect group P in P (namely, the class of a source). Then by (1.8)(2), we see that

$$\operatorname{rk} \rho_{P,G}^{A,t} = \begin{cases} \dim_k(\operatorname{End}_k V_s)_1^{\bar{N}(P,s)} & \text{if} \quad t(s) \neq 0 \\ 0 & \text{if} \quad t(s) = 0. \end{cases}$$

But with (1.1) it is easy to see [Pu, Proposition 1.3] that

$$\sigma_s(\operatorname{Tr}_P^G(x)) = \operatorname{Tr}_1^{\bar{N}(P,s)}(\sigma_s(x)) \qquad \text{for} \quad x \in A^P \cdot s \cdot A^P.$$

In this case, we may take x such that $\operatorname{Tr}_P^G(x) = 1$, from which it follows that $1 = \operatorname{Tr}_1^{\bar{N}(P,s)}(\sigma_s(x))$, whence $(\operatorname{End}_k V_s)_1^{\bar{N}(P,s)} \neq 0$.

(3) By what has been just proved, it suffices to prove that if $A^G \subset Z(A)$ and if P is strictly contained in a defect group of 1_A, then $\rho_{P,G}^{A,t} = 0$. Indeed, we then have $A_P^G \subset J(A^G)$, and thus each element of A_P^G is central and nilpotent. We use the fact that t vanishes on each nilpotent element and the definition of $\rho_{P,G}^{A,t}$ to get the desired result.

Moreover, since A^G is central in A, it is mapped into the center of $\operatorname{End}_k V_s$ by the morphism σ_s; thus we have $\sigma_s(A_P^G) = 0$ or k. Since it is not zero by assertion (2), it is k, and so $\dim_k(\operatorname{End}_k V_s)_1^{\bar{N}(P,s)} = 1$.

Remarks. (1) The assumption of the third assertion of (1.10) can be weakened: by (1.2)(2), it is enough to assume, for example, that $A(P)^{\bar{N}(P)} \subset Z(A(P))$.

(2) In [Pi–Pu], Picaronny and Puig study G-algebras A such that A^G is local and $(\operatorname{End}_k V_s)_1^{\bar{N}(P,s)} = k$ for a source s of A. For such an algebra, we see that $\operatorname{rk} \rho_{P,G}^{A,t} = 1$ or 0 according to the fact that $t(s) \neq 0$ or $t(s) = 0$. If the algebra satisfies the condition (K) of [Pi–Pu], it is then clear that we are in the first case.

C1. *Applications to the group algebra.* For the next results, let $\mathcal{O}$ be a complete discrete valuation ring of characteristic 0, with maximal ideal $\mathcal{P}$ and such that $\mathcal{O}/\mathcal{P} = k$.

Let t be a linear form on kG which is a class function on G and whose kernel contains $J(kG)$. Then (see (1.5)) t is a linear combination of the characters of the irreducible kG-modules. Thus there exists a linear combination with coefficients in $\mathcal{O}$ of characters of ($\mathcal{O}$-free) $\mathcal{O}G$-modules, say χ, such that $t = \bar{\chi}$, the reduction of $\chi \bmod \mathcal{P}$.

(1.11) PROPOSITION. *Let χ be an $\mathcal{O}$-linear combination of characters of $\mathcal{O}G$-modules, let b be a primitive idempotent (block) of $Z\mathcal{O}G$ with defect P in G, and with source s. Then the following assertions are equivalent*:

(i) $\chi(b)/|G:P|$ *is invertible in* $\mathcal{O}$.

(ii) $\bar{\chi}(s) \neq 0$.

(iii) *The form* $\rho_{P,G}^{kG\bar{b},\bar{\chi}}$ *is nonzero.*

Proof. First, we remark that if $b = \mathrm{Tr}_P^G(b')$ for $b' \in (\mathcal{O}G)^P \cdot b$, we then have $\chi(b) = |G:P|\ \chi(b')$ and $\rho_{P,G}^{kG\bar{b},\bar{\chi}}(\bar{b}, \bar{b}) = \bar{\chi}(b')$.

We prove that (i)$\Leftrightarrow$(iii), for we know by Lemma (1.10)(2) that (ii)$\Leftrightarrow$(iii). For that, it suffices to check that $\rho_{P,G}^{kG\cdot\bar{b},\bar{\chi}}$ is nonzero if and only if it is nonzero on $(\bar{b}, \bar{b})$. But since $ZkG \cdot \bar{b} = (kG \cdot \bar{b})_P^G = k \cdot \bar{b} \oplus J(ZkG \cdot \bar{b})$ and since $J(ZkG \cdot \bar{b}) \subset \mathrm{rad}\ \rho_{P,G}^{kG\cdot\bar{b},\bar{\chi}}$, we are done.

Remark. We let the reader generalize (1.11) to more general G-algebras, and look for the connections with [Pi–Pu].

We may also note that s corresponds to a point of $A(P) = kC_G(P)$ when $A = kG$, hence to an isomorphism class of projective indecomposable $kC_G(P)$-modules (or, equivalently, to an isomorphism class of simple $kC_G(P)$-modules). Let Φ be the Brauer character of such a projective indecomposable $kC_G(P)$-module (Φ is the projective character associated with a "root" of b). Then we have

$$\bar{\chi}(s) = \langle \mathrm{Res}_{C_G(P)}^G \chi, \Phi \rangle_{C_G(P)} \pmod{\mathscr{P}}. \tag{1.12}$$

Indeed, we have (see Subsection A) $\bar{\chi}(s) = \bar{\chi}_P(s_P) = \bar{\chi}_P(i_P)$ where i_P is an idempotent element of s_P. Since Φ is just the character of the projective $kC_G(P)$-module $kC_G(P)\, i_P$, the assertion (1.12) is now clear.

We now arrive at one of the main applications of our forms $\rho_{P,G}^{A,t}$.

(1.13) PROPOSITION. *Let χ be an $\mathcal{O}$-linear combination of characters of G. Whenever P is a p-subgroup of G, the rank of $\rho_{P,G}^{kG,\bar{\chi}}$ is the number of blocks with defect group P of G and such that $\chi(b)/|G:P|$ is invertible in $\mathcal{O}$.*

Proof. Since kG is the direct product of its block algebras, it is clear by Lemma (1.10)(3) and Proposition (1.12).

As an example, we may take for χ the character of the permutation module of G modulo a Sylow p-subgroup. Let χ_p denote this character. Then it is easy to see that $(1/|G|_{p'})\ \bar{\chi}_p$ is just the characteristic function of the set G_p of p-elements of G, and so the form $\rho_{P,G}^{kG,\bar{\chi}_p}$ is the form denoted by $\rho_{P,G}$ in [Br1].

The main advantage of this character χ_p is the following property:

(1.1) *Whenever b is a primitive idempotent of G with defect P in G, then $\chi_p(b)/|G:P|$ is invertible in $\mathcal{O}$.*

This property is known, but for the convenience of the reader, we sketch a proof. By (1.11), it suffices to prove that $\rho_{P,G}^{kG\bar{b},\bar{\chi}_p}$ is nonzero. Since $\mathrm{Br}_P(\bar{b}) \neq 0$, there exists a p-regular element g_0 of $C_G(P)$ such that, if $\bar{b} = \sum_{g \in G} \bar{b}(g)\, g$, then $\bar{b}(g_0^{-1}) \neq 0$. We have $\rho_{P,G}^{kG\bar{b},\bar{\chi}_p}(\bar{b}, \mathrm{Tr}_P^G(g_0)) = \chi_p(\bar{b}g_0)$. Thus it suffices to prove that $\bar{\chi}_p(\bar{b}g_0) = |G|_{p'} \cdot \bar{b}(g_0^{-1})$. This results from the fact that the multiplication by b preserves $(\mathcal{O}G_{p'})^G$, and so that

$$\chi_p(bg_0) = \frac{1}{|G:C_G(g_0)|}\, \chi_p(b \cdot \mathrm{Tr}_{C_G(g_0)}^G(g_0)) = |G|_{p'} \cdot b(g_0^{-1}).$$

As an immediate application of (1.13) and (1.14), we get (see [Br1, Theorem (1.7)]).

(1.15) COROLLARY. *The rank of the form $\rho_{P,G}^{kG,\bar{\chi}_p}$ is equal to the number of blocks of G with defect P.*

In Section 2 (see (2.11)), we shall use some information about Hecke algebras to get the explicit information about rk $\rho_{P,G}^{kG,\bar{\chi}_p}$ which was the substance of [Ro].

C2. *Applications to module theory.* Let M be any kG-module, and let A denote the G-algebra $\mathrm{End}_k M$. In this case, the language of G-algebras reduces to the following (see [Gr1, Pu]): the module M is indecomposable if and only if A^G is local; it has vertex P if and only if 1_A has defect P in G; the summand $i \cdot M$ of $\mathrm{Res}_P^G M$ (i an idempotent of $A^P = \mathrm{End}_{kP} M$) is a source of M if and only if the $(A^P)^\times$-conjugacy class of i is a source of A.

Moreover, if we take for t the usual trace form on $\mathrm{End}_k M$, then $t(s)$ is just the value mod p of the dimension of the module $i \cdot M$ for $i \in S$. In particular, if s is a source of A, then $t(s)$ is the dimension (mod p) of a source module of M.

By Proposition (1.3), we then see that the following result, due to Benson and Carlson [Be–Ca] is just a particular case of (1.10).

(1.16) PROPOSITION. *Let M be an indecomposable kG-module with vertex P. Then the Scott module with vertex P is a summand of $\mathrm{End}_k M = M \otimes_k M^*$ if and only if the source of M has dimension prime to p.*

Moreover, we may note that (1.10) gives the multiplicity of this Scott module. Indeed, let $\mathrm{End}_k V_s$ be the simple quotient of the algebra $\mathrm{End}_{kP} M$ corresponding to a source s of M. The group denoted by $N(P, s)$ is just the inertial group of that source in $N_G(P)$, and $\bar{N}(P, s)$ acts on the simple quotient $\mathrm{End}_k V_s$. The dimension of $(\mathrm{End}_k V_s)_1^{\bar{N}(P,s)}$ is the multiplicity of the projective cover of the trivial module for $\bar{N}(P, s)$ as a summand of $\mathrm{End}_k V_s$, and thus (1.16) may be completed by

(1.17) *If the Scott module with vertex P is a summand of* $\mathrm{End}_k M$, *then its multiplicity is equal to the multiplicity of the Scott module with vertex* 1 *as a summand of the $k\bar{N}(P, s)$-module* $\mathrm{End}_k V_s$.

Finally, we turn our attention to the case where P is a Sylow p-subgroup of G. We state first an easy consequence of (1.3) and (1.10)

(1.18) PROPOSITION. *Let A be a G-algebra such that A^G is local, and assume that (A, t) is symmetric (and t G-stable). Then the trivial kG-module is a summand of A if and only if $t(1_A) \neq 0$.*

Proof. If $t(1_A) \neq 0$, it is clear that $A = k1_A \oplus \ker t$, and so the trivial kG-module is a summand of the kG-module A. Reciprocally, let us assume that the trivial kG-module (i.e., the Scott module with vertex P) is a summand of A. By (1.10)(1), we see that 1_A has defect P. Let s be a source. By (1.3) and (1.10) it suffices to prove that if $t(s) \neq 0$, then $t(1_A) \neq 0$. Indeed, let $i \in s$; we have $\mathrm{Tr}_P^G(i) = \alpha 1_A + j$ where $\alpha \in k$, $j \in J(A^G)$. Thus $|G:P|\ t(i) = \alpha t(1_A)$, from which the desired result is clear.

The following result of Benson [Be–Ca], inspired by a remark of Landrock [La] is now a particular case of (1.18) (taking $A = \mathrm{End}_k(M)$ and $t = \mathrm{tr}$).

(1.19) COROLLARY. *Let M be an indecomposable kG-module. Then the trivial module is a summand of $M \otimes M^*$ if and only if $p \nmid \dim_k M$.*

In [Be–Ca], Benson and Carlson prove a deeper result, namely that whenever M and N are indecomposable kG-modules, the trivial kG-module is a summand of $M \otimes N^*$ if and only if $M \simeq N$ and $p \nmid \dim_k M$. It is easy to check that this is equivalent to the following statement:

(1.20) *Let M be any kG-module, and let $(m_i)_{i \in I}$ be the multiplicities of the isomorphism classes of indecomposable kG-modules $(M_i)_{i \in I}$ as components of M. Then the multiplicity of the trivial kG-module as a summand of $M \otimes M^*$ equals $\sum m_i^2$, where the sum is taken over those i for which $p \nmid \dim_k M_i$.*

The preceding result admits the following generalization, due to Puig (private communication).

Let A be any G-algebra. If b is a point of A^G, corresponding to the irreducible representation $\sigma_b\colon A^G \to \operatorname{End}_k V_b$, we call multiplicity of b and denote by $m(b)$ the dimension of V_b over k (see [Pu]).

(1.21) PROPOSITION (Puig). *Let P be a Sylow p-subgroup of G. Then* $\operatorname{rk} \rho_{P,G}^{A,t} = \sum m(b)^2$, *where the sum is taken over all points b of A^G such that* $t(b) \neq 0$.

Proof. First, we need to quote an elementary fact from general pointed group theory. If P is any p-subgroup of G and s a point of A^P with defect P in P, we denote by $\mathscr{P}(A^G; (P, s))$ the set of all points of A^G with defect group P and source s (i.e., the set of points b of A^G such that P_s is a local pointed group which is maximal in G_b, see [Pu]).

(1.22) LEMMA (see [Pu, Proposition 1.3]). (1) *With the preceding notation, we have*

$$(\operatorname{End}_k V_s)_1^{\bar{N}(P,s)}/J((\operatorname{End}_k V_s)_1^{\bar{N}(P,s)}) \simeq \prod_{b \in \mathscr{P}(A^G;(P,s))} \operatorname{End}_k V_b.$$

(2) *In particular, if* $(\operatorname{End}_k V_s)_1^{\bar{N}(P,s)}$ *is semisimple, we have*

$$\dim(\operatorname{End}_k V_s)_1^{\bar{N}(P,s)} = \sum_{b \in \mathscr{P}(A^G;(P,s))} m(b)^2.$$

The proof of Lemma (1.22) is immediate, once one notes that the morphism σ_s induces a surjection from A_P^G onto $(\operatorname{End}_k V_s)_1^{\bar{N}(P,s)}$ (see (1.1), for example). Indeed, it follows that A_P^G maps onto the semisimple quotient of $(\operatorname{End}_k V_s)_1^{\bar{N}(P,s)}$, and so the simple factors of this quotient correspond to points of A_P^G. It is then clear by the definitions that they are precisely the elements of $\mathscr{P}(A^G; (P, s))$.

Now, let us return to the proof of (1.21). Since $\bar{N}(P, s)$ is a p'-group, the algebra $(\operatorname{End}_k V_s)^{\bar{N}(P,s)}$ is semisimple and is equal to $(\operatorname{End}_k V_s)_1^{\bar{N}(P,s)}$. Thus (1.21) follows from Lemma (1.22)(2) and from Proposition (1.8)(2), provided we prove that, whenever $b \in \mathscr{P}(A^G; (P, s))$, we have $t(s) \neq 0$ if and only if $t(b) \neq 0$.

Let e be an element of b and i be an element of s (e and i are primitive idempotents of, respectively, A^G and A^P) such that $ei = ie = i$. Since $\ker t$ contains $J(eA^Ge)$ and $J(iA^Pi)$, we have $t(eA^Ge) = kt(e) = kt(b)$ and $t(iA^Pi) = kt(i) = kt(s)$. But we also have $eA^Ge = \operatorname{Tr}_P^G(eA^P \cdot i \cdot A^Pe)$ since s is a source of b. By the assumptions on t we deduce that $t(eA^Ge) = t(iA^Pi)$, whence the desired result (see [Pi–Pu]).

2. On Hecke Algebras

A. *Generalities*

In this section, we set the notation and state or recall some more or less known results on Hecke algebras (see, e.g., [Ca] or [La]).

Let R be a commutative ring, let G be a finite group and let H be a subgroup of G. The permutation RG-module $\mathrm{Ind}_H^G R$ is identified with $RG \cdot \mathscr{S}H$ (as a left submodule of RG). We set $E_R(G/H) = \mathrm{End}_R(\mathrm{Ind}_H^G R)$, and we consider this algebra as endowed with the natural algebra morphism $\sigma_H: RG \to E_R(G/H)$ which defines the structure of RG-module of $\mathrm{Ind}_H^G R$.

The Hecke algebra associated with the triple (G, R, H) is by definition the algebra of RG-endomorphisms of $\mathrm{Ind}_H^G R$, i.e., the set of fixed points of G acting on $E_R(G/H)$ by conjugation through σ_H:

$$\mathscr{H}_R(G, H) = E_R(G/H)^G.$$

The following observations are known (see [Ca, La]) and easy to check: Let $\alpha_H: RG \to E_R(G/H)$ be the R-linear map defined by

$$\alpha_H(x)(g\mathscr{S}H) = \begin{cases} x\mathscr{S}H & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases} \quad \text{for all } x \in RG.$$

We have

(α1) $h\alpha_H(x)h'^{-1} = \alpha_H(hx)$ for $h, h' \in H$, and α_H is a morphism of $H \times H$-modules.

(α2) If $x \in G$ and $h \in {}^xH$, we have $\alpha_H(x) = \alpha_H(hx)$, whence

$$\alpha_H(x) \in E_R(G/H)^{H \cap {}^xH}$$

Let us put $a_H(x) = \mathrm{Tr}_{H \cap {}^xH}^G(\alpha_H(x))$ for all $x \in G$. The following properties of the family $(a_H(x))$ are standard (see [Ca, La]):

(2.1) (1) $a_H(x) = a_H(y)$ *if and only if* $HxH = HyH$.

(2) *The set of distinct* $a_H(x)$'s $(x \in G)$ *is a basis of* $\mathscr{H}_R(G, H)$ *over* R.

(3) *The endomorphism* $a_H(x)$ *of the module* $RG \cdot \mathscr{S}H$ *sends* $\mathscr{S}H$ *onto* $\mathscr{S}(HxH)$.

Proof of (3). We have $a_H(x)(\mathscr{S}H) = \mathrm{Tr}_{H \cap {}^xH}^G(\alpha_H(x))(\mathscr{S}H) = \mathrm{Tr}_{H \cap {}^xH}^H(\alpha_H(x))(\mathscr{S}H) = \mathrm{Tr}_{H \cap {}^xH}^H(x\mathscr{S}H) = \mathscr{S}(HxH)$.

The ordinary trace map $\mathrm{tr}: E_R(G/H) \to R$ induces on $\mathscr{H}_R(G, H)$ the linear map defined by

$$\mathrm{tr}\, a_H(x) = \begin{cases} |G:H| & \text{if } x \in H \\ 0 & \text{if } x \notin H, \end{cases}$$

from which it is easy to deduce

$$\operatorname{tr}(a_H(x)\,a_H(y)) = \begin{cases} |G : H \cap {}^xH| & \text{if} \quad HxH = Hy^{-1}H \\ 0 & \text{if} \quad HxH \neq Hy^{-1}H. \end{cases} \tag{2.2}$$

We introduce the following notation:

For $a = \sum_{g \in G} a(g)\,g \in RG$, we set $a^0 = \sum_{g \in G} a(g^{-1})\,g$. Moreover, we denote by $N_{RG}(\mathscr{S}H)$ the set of all a in RG such that $\mathscr{S}Ha^0 \in RG\mathscr{S}H$. It is clear that $N_{RG}(\mathscr{S}H)$ is a subalgebra of RG.

The following definition can also be found in [La, p. 178].

Definition. We denote by $\omega_H : N_{RG}(\mathscr{S}H) \to \mathscr{H}_R(G, H)$ the algebra morphism defined by

$$\omega_H(a)(g\mathscr{S}H) = g\mathscr{S}Ha^0 \qquad \text{for} \quad a \in N_{RG}(\mathscr{S}H),\ g \in G.$$

We give the first properties of the morphism ω_H in an omnibus proposition.

(2.3) Proposition. (1) *The morphism $\omega_H : N_{RG}(\mathscr{S}H) \to \mathscr{H}_R(G, H)$ is onto, and its kernel is the left annihilator of $\mathscr{S}H$ in RG* [La].

(2) *We have $RN_G(H) \subset N_{RG}(\mathscr{S}H)$, and for $n \in N_G(H)$, we have $\omega_H(n) = a_H(n^{-1})$.*

(3) *We have $(RG)^H \subset N_{RG}(\mathscr{S}H)$, and for $x \in G$ we have*

$$\omega_H(\operatorname{Tr}^H_{C_H(x)}(x)) = |H \cap {}^xH : C_H(x)|\; a_H(x^{-1}).$$

In particular, whenever $a \in R^H$, we have

$$\omega_H(a) = \operatorname{Tr}^G_H(\alpha_H(a^0)).$$

(4) *Whenever $z \in ZRG$, we have $\omega_H(z) = \sigma_H(z^0)$, and if C is a conjugacy class of G, we have*

$$\omega_H(\mathscr{S}C^0) = \sum_x |C \cap xH|\; a_H(x),$$

where x runs over a set of representatives of (H, H)-double cosets of G.

Proof. (1) To prove the surjectivity, it suffices by (2.1)(2) to exhibit, for all $x \in G$, an element of $N_{RG}(\mathscr{S}H)$ whose image is $a_H(x)$. Choose $x \in G$, and choose $\{t_1, ..., t_n\}$ a left transversal to $H \cap {}^xH$ in H, and $\{u_1, ..., u_n\}$ a right transversal to $H \cap {}^xH$ in H. We then set $X = \{t_i x^{-1} u_i;\ 1 \leqslant i \leqslant n\}$. It is not difficult to check that $\mathscr{S}H \cdot \mathscr{S}X = \mathscr{S}X \cdot \mathscr{S}H = \mathscr{S}(Hx^{-1}H)$, whence that

$\mathcal{S}X \in N_{RG}(\mathcal{S}H)$ (in fact $\mathcal{S}X$ commutes with $\mathcal{S}H$) and $\omega_H(\mathcal{S}X) = a_H(x)$ by (2.1)(3), as required.

Since $\omega_H(a)$ is a G-endomorphism of $RG \cdot \mathcal{S}H$, it is defined by its action on $\mathcal{S}H$. Then it is clear that $\ker \omega_H = \{a \in RG \mid a \cdot \mathcal{S}H = 0\}$.

(2) Trivial. Note that ω_H induces an injective morphism from $R\bar{N}_G(H)^0$ (where $\bar{N}_G(H)^0$ denotes the opposite group to $N_G(H)/H$) into $\mathcal{H}_R(G, H)$.

(3) We have $\omega_H(\mathrm{Tr}^H_{C_H(x)}(x))\mathcal{S}H = \mathcal{S}H \cdot \mathrm{Tr}^H_{C_H(x)}(x^{-1}) = \mathrm{Tr}^H_{C_H(x)}(\mathcal{S}Hx^{-1})$ $= |H \cap {}^xH \colon C_H(x)| \, \mathrm{Tr}^H_{H \cap {}^xH}(\mathcal{S}Hx^{-1}) = |H \cap {}^xH \colon C_H(x)| \, \mathcal{S}(Hx^{-1}H)$.

Since $a_H(x) = \mathrm{Tr}^G_{H \cap {}^xH}(\alpha_H(x))$, we see that

$$\begin{aligned}\omega_H(\mathrm{Tr}^H_{C_H(x)}(x^{-1})) &= |H \cap {}^xH : C_H(x)| \, \mathrm{Tr}^G_{H \cap {}^xH}(\alpha_H(x)) \\ &= \mathrm{Tr}^G_{C_H(x)}(\alpha_H(x)) = \mathrm{Tr}^G_H(\mathrm{Tr}^H_{C_H(x)}\alpha_H(x)) \\ &= \mathrm{Tr}^G_H(\alpha_H(\mathrm{Tr}^H_{C_H(x)}(x)), \qquad \text{which proves (3).}\end{aligned}$$

(4) It is clear from the definition that $\omega_H(z) = \sigma_H(z^0)$. To find the coefficient of $\omega_H(\mathcal{S}C^0)$ on $a_H(x)$ we note that we have (applying to $\mathcal{S}H$) $\mathcal{S}H \cdot \mathcal{S}C = \sum \lambda(C, x) \; \mathcal{S}(HxH)$ and $\lambda(C, x)$ is the coefficient of x in the element $\mathcal{S}H \cdot \mathcal{S}C$ (expressed as a combination of the natural basis elements of RG). Thus $\lambda(C, x) = |C \cap xH|$, as required.

B. *In characteristic p*

From now on, we assume that $R = k$, a field with characteristic $p > 0$, and that H is a p-group denoted by P.

It results from (2.3)(3) that

(2.4) *Whenever C is a P-conjugacy class of G, then $\omega_P(\mathcal{S}C) = 0$ unless there exists $x \in G$ with $C_P(x) = P \cap {}^xP$.*

This remark will be crucial in the proof we shall give (see (26)) of the fact that defect groups are Sylow intersections (a result due to Green, and improved in [Ro]). First, let us note that no idempotent is mapped to zero by σ_P or ω_P, because of the following statement.

(2.5) PROPOSITION. (1) $\ker \sigma_P \subset JkG$.

(2) $\ker \omega_P \cap (kG)^P \subset J((kG)^P)$.

Proof. (1) If V is any kG-module, then, $\mathrm{Hom}_{kG}(\mathrm{Ind}^G_P k, V) \simeq V^P$ is not zero. Thus every irreducible kG-module is an image of $\mathrm{Ind}^G_P k$, which proves (1).

(2) It suffices to check that $\ker \omega_P \cap (kG)^P$ contains no idempotent. But if i is an idempotent of $(kG)^P$, then ikG is a projective (whence free) kP-module, and $\mathcal{S}PikG$ is then not zero: its dimension is the kP-rank of ikG. Thus we have $\mathcal{S}Pi \neq 0$, which shows (by (2.3)(1) that $i \notin \ker \omega_P$.

Using (2.4) and (2.5)(1), we can now prove the following result (Green, Robinson).

(2.6) COROLLARY. *Let D be a defect group of a p-block of G, and let P be a Sylow p-subgroup containing D. There exists a p-regular element x of G with the following two properties*:

(a) $D = P \cap {}^{x}P$.

(b) *D is a Sylow p-subgroup of $C_G(x)$.*

Proof. As usual (see [Br1], for example) we reduce to the case where D is normal in G. Let $b = \sum_C b(C) \cdot \mathscr{S}C$ (where C runs over the set of p-regular conjugacy classes of G) be a primitive idempotent of ZkG with defect group D. The image of b through the morphism $\sigma_P : kG \to E_k(G/P)$ is not zero by (2.5)(1), and so there exists C such that $b(C)\, \sigma_P(b\mathscr{S}C) \neq 0$. Standard arguments give now that C has defect D.

Moreover, since $\sigma_P(\mathscr{S}C) \neq 0$ and since $\sigma_P(\mathscr{S}C) = \omega_P(\mathscr{S}C^0)$ (by (2.3)(4)), it follows from (2.4) that there exists $x \in C$ with $C_P(x) = P \cap {}^{x}P$, hence $D = P \cap {}^{x}P$, as required.

We can also notice the following immediate consequence of the preceding formalism.

(2.7) COROLLARY. *Suppose $P \lhd G$. Then the morphism ω_P induces a surjective algebra morphism from $(kG)^P$ onto $k\bar{C}_G(P)^0$ (where $\bar{C}_G(P) = C_G(P)/Z(P)$). In particular, $(kG)^P$ is local if and only if $C_G(P) \subset P$.*

Proof. By (2.3)(3), it is immediate that

$$\omega_P(\mathrm{Tr}^G_{C_P(x)}(x)) = \begin{cases} a_P(x^{-1}) & \text{if } x \in C_G(P) \\ 0 & \text{if } x \notin C_G(P). \end{cases}$$

Now (2.7) is clear. The last assertion follows from (2.5)(2).

Let us end this section with an interesting property of the morphism ω.

We say that the subgroup H of G controls the G-fusion of its p-subgroups if whenever P is a p-subgroup of H and g is an element of G such that $P^g \subset H$, then $g \in C_G(P)H$.

(2.7) PROPOSITION. *Let k be a field of characteristic p, and let H be a subgroup of G. Then the image of $(kG)^H$ under ω_H is all of $\mathscr{H}_k(G, H)$ if and only if H controls the G-fusion of its p-subgroups.*

Proof. By (2.1)(2) and (2.3)(3) we see that $\omega((kG)^H)$ is all of $\mathscr{H}_k(G, H)$ if and only if

(F1) $(\forall g \in G)(\exists g' \in HgH)$ such that $p \nmid |H \cap {}^{g'}H : C_H(g')|$. But if $g' = h'gh$ for $h, h' \in H$, we have

$$|H \cap {}^{g'}H : C_H(g')| = |H \cap {}^{g}H : C_H(ghh')|,$$

and $(F1)$ is equivalent to

(F2) $(\forall g \in C)(\exists h \in H)$ such that $p \nmid |H \cap {}^{g}H : C_H(gh^{-1})|$. We claim that (F2) is equivalent to

(F3) $(\forall h \in G)(\forall P \in \mathrm{Syl}_p(H \cap {}^{g}H))(\exists h \in H)$ such that $P \subset C_H(gh^{-1})$.
Indeed, it is clear that (F3) $\Rightarrow$ (F2). Let us prove that (F2) $\Rightarrow$ (F3). Given g, and $P \in \mathrm{Syl}_p(H \cap {}^{g}H)$, assuming (F2) we see that there exists $h \in H$ such that $C_H(gh^{-1})$ contains $h_1 P h_1^{-1}$ where $h_1 \in H \cap {}^{g}H$. If $h_1 = gh_2 g^{-1}$ with $h_2 \in H$, then

$$P \subset h_1^{-1} C_H(gh^{-1})\, h_1 = C_H(gh_2^{-1}h^{-1}h_1), \qquad \text{which proves (F3).}$$

Now we prove that (F3) holds if and only if H controls the G-fusion of its p-subgroups. Let us assume that (F3) holds, and let P and P^g be contained in H for $g \in G$. Then $P \subset H \cap {}^{g}H$; let Q be a Sylow p-subgroup of $H \cap {}^{g}H$ containing P. By (F3), there exists $h \in H$ such that $Q \subset C_H(gh^{-1})$. Let us set $z = gh^{-1}$. Then $z \in C_G(Q)$, hence $z \in C_G(P)$ and $g = zh$, as required. The converse is as easy.

Let us give an amusing application of Proposition (2.7).

Let H be a subgroup of G which controls the G-fusion of its p-subgroups, so such that

$$\omega_H\colon\quad (kG)^H \to \mathscr{H}_k(G, H) = \mathrm{End}_{kG}(\mathrm{Ind}_H^G k)$$

is onto. By classical facts about lifting idempotents, there exists a decomposition of 1 into a sum of orthogonal idempotents of $(kG)^H$, say $1 = \sum_e e$, such that $\sum \omega_H(e)$ is a decomposition of 1 into a sum of orthogonal primitive idempotents in $\mathscr{H}_k(G, H)$, i.e., such that

$$\mathrm{Ind}_H^G k = kG \cdot \mathscr{S}H = \bigoplus_e kG \cdot \mathscr{S}H \cdot e^0$$

is a decomposition into indecomposable kG-modules. It is easy to recognize the Scott module associated with H in this decomposition: it corresponds to the idempotent e_0 such that $e_0\, \mathscr{S}G \neq 0$.

We can now note that the Loewy series of kH is somewhat reflected in each of the projective modules kGe^0: we have

$$kG \cdot e^0 \supset kG \cdot J(kH)\, e^0 \supset kG \cdot J^2(kH)\, e^0 \supset \cdots.$$

Suppose, for example, that $H = \langle x \rangle$ is a cyclic p-group with order p^m, with no G-fusion on $\langle x \rangle$. Then each of the indecomposable summands of $\mathrm{Ind}_H^G k$ can be extended p^m times by itself to build up a projective kG-module, since

$$kG^0 \supset kGe^0(1-x) \supset \cdots \supset kGe^0(1-x)^i \supset \cdots \supset \{0\},$$

with $kG^0(1-x)^i/kGe^0(1-x)^{i+1} \simeq kGe^0 \mathscr{S}H$.

C. *Application to Bilinear Forms*

Now, we come back to the form $\rho_{P,G}^{kG,\chi_P}$ (see Sect. 1; in particular (1.15)). We shall see that the preceding formalism provides the explicit information on that form which was given in [Ro]. The following result is due to Scott (see also [Ca]).

(2.8) *Let S be a Sylow p-subgroup of G, and P be a p-subgroup of G. Then the algebra $E_k(G/S)_P^G$ has for basis the set of distinct $a_S(x)$ such that $S \cap {}^xS \leqslant_G P$.*

Indeed, 2.8 follows from explicit formulae in [Gr1], since $E_k(G/S)$ is a permutation module under G-conjugation.

From now on, we assume that P is a normal p-subgroup of G.

Let us say that *the element $a_S(x)$ has defect group P* if $S \cap {}^xS = P$. The following fact is clear by (2.8) and (2.2).

(2.9) *If $a_S(x)$ and $a_S(Y)$ have defect P, then*

$$\rho_{P,G}^{E_k(G/S),\mathrm{tr}}(a_S(x), a_S(y^{-1})) = \begin{cases} 1 & \text{if } \ a_S(x) = a_S(y) \\ 0 & \text{if not.} \end{cases}$$

Let us say that a conjugacy class C of G with defect P is P-distinguished if there exists an element $x \in C$ such that $S \cap {}^xS = P$.

By (2.4) and (2.3)(4), we see that

(2.10) *Whenever C is a conjugacy class of G with defect group P such that $\sigma_S(\mathscr{S}C) \neq 0$, then C is P-distinguished.*

Let us recall moreover the following two facts.

(1) The set of all $\mathscr{S}C$'s, where C runs over the set of p-regular conjugacy classes, generates ZkG modulo its radical.

(2) Whenever C is a G-conjugacy class with defect strictly contained in P, we have $\mathscr{S}C \in \operatorname{rad} \rho_{P,G}^{kG,\bar{\chi}_P}$.

By (1) and (2) above, and by (2.10), it is now clear that

(2.11) *The rank of* $\rho_{P,G}^{kG,\bar{\chi}_P}$ *is equal to the rank of the matrix* $(\rho_{P,G}^{kG,\bar{\chi}_P}(\mathscr{S}C', \mathscr{S}C^0))$, *where* C *and* C' *run over the set of* p*-regular* P*-distinguished conjugacy classes of* G.

But, by definition of $\rho_{P,G}^{kG,\bar{\chi}_P}$, we have

$$\rho_{P,G}^{kG,\bar{\chi}_P}(x, y) = \rho_{P,G}^{Ek(G/S),\mathrm{tr}}(\sigma_S(x), \sigma_S(y)).$$

Let M_S denote the matrix of the system of $\sigma_S(\mathscr{S}C)$'s, where C runs over the set of p-regular P-distinguished classes of G, expressed on the $a_S(x)$'s. It is now clear, by (2.9) and (2.11), that

(2.12) *The rank of* $\rho_{P,G}^{kG,\bar{\chi}_P}$ *is equal to the rank of the matrix* $M_S \cdot {}^tM_S$.

Since M_S (see (2.3)(4)) is precisely the matrix denoted by N in [Ro], a fact first noted by Landrock, we see that (2.12), combined with (1.15), is the main result of [Ro].

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