Linear Algebra and its Applications 430 (2009) 865–867

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Note

A note on the span of Hadamard products of vectors

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ARTICLE INFO

Article history:
Received 13 June 2008
Accepted 9 September 2008
Available online 19 October 2008

AMS classification:
15A03
15A04
15A57

Keywords:
Hadamard product
Range

ABSTRACT

We give a new proof of Theorem 6 in [L. Qiu and X. Zhan, On the span of Hadamard products of vectors, Linear Algebra Appl. 422 (2007) 304–307].

The *Hadamard product* $A \circ B$ of two $m \times n$ complex matrices $A = (A_{ij}), B = (B_{ij})$ is their entry-wise product: $(A \circ B)_{ij} = A_{ij}B_{ij}$. Qiu and Zhan [3] studied the range of the Hadamard product of $n \times n$ complex matrices. In this note, we give a new proof of the following theorem:

Theorem 1 [3, Theorem 6]. *Let $B_1, B_2, \ldots, B_k$ be $n \times n$ complex matrices. Then*

$$\mathrm{span}\{(B_1x_1) \circ \cdots \circ (B_kx_k) | x_1, \ldots, x_k \in \mathbb{C}^n\} = \mathrm{range}((B_1B_1^*) \circ \cdots \circ (B_kB_k^*)),$$

where $*$ means conjugate transpose.

Proof. Let e_i be the vector in $\mathbb{C}^n$ with a 1 in the ith entry and 0 in all other entries. Then for $1 \leqslant i \leqslant n$ the ith column of $(B_1B_1^*) \circ \cdots \circ (B_kB_k^*)$ is

$$((B_1B_1^*) \circ \cdots \circ (B_kB_k^*))e_i = (B_1B_1^*e_i) \circ \cdots \circ (B_kB_k^*e_i).$$

So

$$\mathrm{range}((B_1B_1^*) \circ \cdots \circ (B_kB_k^*)) \subseteq \mathrm{span}\{(B_1x_1) \circ \cdots \circ (B_kx_k) | x_1, \ldots, x_k \in \mathbb{C}^n\}.$$

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doi:10.1016/j.laa.2008.09.010

We show

$$\mathrm{span}\{(B_1x_1)\circ\cdots\circ(B_kx_k)|x_1,\ldots,x_k\in\mathbb{C}^n\}\subseteq \mathrm{range}((B_1B_1^*)\circ\cdots\circ(B_kB_k^*)). \tag{1}$$

Pick any $x_1,x_2,\ldots,x_k\in\mathbb{C}^n$. Let E be the orthogonal projection onto $\mathrm{range}((B_1B_1^*)\circ\cdots\circ(B_kB_k^*))^{\perp}$. Then for any $y\in\mathbb{C}^n$ we have

$$\begin{aligned}\langle(B_1x_1)\circ\cdots\circ(B_kx_k),Ey\rangle &= \sum_{i=1}^n\langle B_1x_1,e_i\rangle\cdots\langle B_kx_k,e_i\rangle\overline{\langle Ey,e_i\rangle}\\ &= \sum_{i=1}^n\langle x_1\otimes\cdots\otimes x_k\otimes\bar{y},(B_1^*e_i)\otimes\cdots\otimes(B_k^*e_i)\otimes\bar{E}e_i\rangle\\ &= \left\langle x_1\otimes\cdots\otimes x_k\otimes\bar{y},\sum_{i=1}^n(B_1^*e_i)\otimes\cdots\otimes(B_k^*e_i)\otimes\bar{E}e_i\right\rangle,\end{aligned}$$

where $\bar{}$ means complex conjugate. On the other hand, using $E^*=E^2=E$ we have

$$\begin{aligned}\left\|\sum_{i=1}^n(B_1^*e_i)\otimes\cdots\otimes(B_k^*e_i)\otimes\bar{E}e_i\right\|^2 &= \sum_{i,j=1}^n\langle B_1^*e_i,B_1^*e_j\rangle\cdots\langle B_k^*e_i,B_k^*e_j\rangle\langle\bar{E}e_i,\bar{E}e_j\rangle\\ &= \sum_{i,j=1}^n(B_1B_1^*)_{ji}\ldots(B_kB_k^*)_{ji}E_{ij}\\ &= \sum_{i,j=1}^n E_{ij}((B_1B_1^*)\circ\cdots\circ(B_kB_k^*))_{ji}\\ &= \mathrm{trace}(E((B_1B_1^*)\circ\cdots\circ(B_kB_k^*)))\\ &= 0.\end{aligned}$$

So

$$\sum_{i=1}^n(B_1^*e_i)\otimes\cdots\otimes(B_k^*e_i)\otimes\bar{E}e_i=0,$$

and we find

$$\langle(B_1x_1)\circ\cdots\circ(B_kx_k),Ey\rangle=0.$$

Since y is arbitrary, we conclude

$$(B_1x_1)\circ\cdots\circ(B_kx_k)\in\mathrm{range}((B_1B_1^*)\circ\ldots\circ(B_kB_k^*)).$$

We have now shown (1) and the result follows. □

Corollary 2 [3, Theorem 5]. *Let $A_1,A_2,\ldots,A_k$ be $n\times n$ positive semidefinite matrices. Then*

$$\mathrm{span}\{(A_1x_1)\circ\cdots\circ(A_kx_k)|x_1,\ldots,x_k\in\mathbb{C}^n\}=\mathrm{range}(A_1\circ\cdots\circ A_k).$$

Proof. Set $B_i=A_i^{\frac{1}{2}}$ $(1\leqslant i\leqslant k)$ in Theorem 1 and recall $\mathrm{range}(A_i)=\mathrm{range}(B_i)$ $(1\leqslant i\leqslant k)$. □

Corollary 3 [3, Theorem 4]. *Let $A_1,A_2,\ldots,A_k$ be $n\times n$ positive semidefinite matrices. Then*

$$\mathrm{span}\{(A_1x)\circ\cdots\circ(A_kx)|x\in\mathbb{C}^n\}=\mathrm{range}(A_1\circ\cdots\circ A_k).$$

Proof. For $1\leqslant i\leqslant n$ the ith column of $A_1\circ\cdots\circ A_k$ is

$$(A_1\circ\cdots\circ A_k)e_i=(A_1e_i)\circ\cdots\circ(A_ke_i).$$

So by Corollary 2 we have

$$\begin{aligned}\mathrm{range}(A_1 \circ \cdots \circ A_k) &\subseteq \mathrm{span}\{(A_1x) \circ \ldots \circ (A_kx) | x \in \mathbb{C}^n\}\\ &\subseteq \mathrm{span}\{(A_1x_1) \circ \cdots \circ (A_kx_k) | x_1, \ldots, x_k \in \mathbb{C}^n\}\\ &= \mathrm{range}(A_1 \circ \cdots \circ A_k),\end{aligned}$$

as desired. □

Remark. Corollary 2 is equivalent to Theorem 1 (see [3, Theorem 6]). Qiu and Zhan [3] proved Corollary 2 first and got the other theorems. Their proof makes full use of positive semidefiniteness, and the key lemma there [3, Lemma 3] needs a slight analytic argument [4, p. 15]. The idea of taking tensor products in this note is from [2]. See also [1, Section 2.8].

Acknowledgments

The author would like to thank Akihiro Munemasa and Hiroki Tamura for comments and discussions. Thanks are also due to Xingzhi Zhan for carefully reading the manuscript.

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