Available online at www.sciencedirect.com
ScienceDirect
Theoretical Computer Science 374 (2007) 66–73
Theoretical Computer Science
www.elsevier.com/locate/tcs

Lattice grids and prisms are antimagic☆

Yongxi Cheng*

Department of Computer Science, Tsinghua University, Beijing 100084, China

Received 10 January 2006; received in revised form 23 November 2006; accepted 7 December 2006

Communicated by D.-Z. Du

Abstract

An *antimagic labelling* of a finite undirected simple graph with m edges and n vertices is a bijection from the set of edges to the integers $1, \ldots, m$ such that all n vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with the same vertex. A graph is called *antimagic* if it has an antimagic labelling. In 1990, Hartsfield and Ringel conjectured that every connected graph, but K_2, is antimagic. In [T.-M. Wang, Toroidal grids are antimagic, in: Proc. 11th Annual International Computing and Combinatorics Conference, COCOON'2005, in: LNCS, vol. 3595, Springer, 2005, pp. 671–679], Wang showed that the toroidal grids (the Cartesian products of two or more cycles) are antimagic. Two open problems left in Wang's paper are about the antimagicness of lattice grid graphs and prism graphs, which are the Cartesian products of two paths, and of a cycle and a path, respectively. In this article, we prove that these two classes of graphs are antimagic, by constructing such antimagic labellings.

Keywords: Antimagic; Labelling; Lattice grid; Prism

1. Introduction

All graphs in this paper are finite, undirected and simple. In 1990, Hartsfield and Ringel [3] introduced the concept of *antimagic* graph. An *antimagic labelling* of a graph with m edges and n vertices is a bijection from the set of edges to the integers $1, \ldots, m$ such that all n vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with that vertex. A graph is called antimagic if it has an antimagic labelling. Hartsfield and Ringel showed that paths P_n $(n \geq 3)$, cycles, wheels and complete graphs K_n $(n \geq 3)$ are antimagic. They conjectured that all trees except K_2 are antimagic. Moreover, all connected graphs except K_2 are antimagic. These two conjectures are unsettled. In 2004, Alon et al. [1] showed that the latter conjecture is true for all graphs with n vertices and minimum degree $\Omega(\log n)$. They also proved that a graph G with n (≥ 4) vertices and maximum degree $\Delta(G) \geq n - 2$ is antimagic, and all complete partite graphs except K_2 are antimagic. In [4], Hefetz proved several special cases and variants of the latter conjecture. In particular, he proved that for integers $k > 0$ a graph with 3^k vertices is antimagic

☆ This work was supported in part by National Natural Science Foundation of China Grant 60553001 and National Basic Research Programme of China Grant 2007CB807900, 2007CB807901.

* Tel.: +86 10 62797304.
E-mail address: cyx@mails.tsinghua.edu.cn.

0304-3975/$ - see front matter
doi:10.1016/j.tcs.2006.12.003

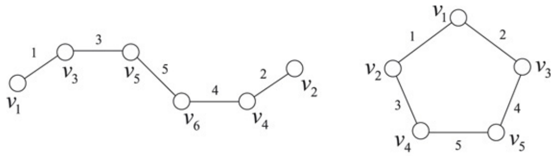

Fig. 1. Antimagic labellings of $P[n+1]$ and $C[m]$, for $n=5$, $m=5$.

if it admits a K_3-factor. In [5], Wang showed that the toroidal grids (the Cartesian products of two or more cycles) are antimagic. Two open problems left in [5] are about the antimagicness of lattice grid graphs and prism graphs, which are the Cartesian products of two paths, and of a cycle and a path, respectively.

In this paper, we prove that these two classes of graphs are antimagic, by constructing such antimagic labellings. In contrast to toroidal grids, lattice grids and prisms have more different local structures, we will use new strategies in the construction of the labellings. Our main results are the following two theorems, which are proved in Sections 3 and 4 respectively.

Theorem 1.1. *All lattice grid graphs* $P_1[m+1] \times P_2[n+1]$ *are antimagic, for integers* $m, n \geq 1$.

Theorem 1.2. *All prism graphs* $C[m] \times P[n+1]$ *are antimagic, for integers* $m \geq 3, n \geq 1$.

For more results, open problems and conjectures on antimagic graphs and various graph labelling problems, please see [2].

2. Preliminaries

The *Cartesian product* $G_1 \times G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph with vertex set $V_1 \times V_2$, and (u_1, u_2) is adjacent to (v_1, v_2) in $G_1 \times G_2$ if and only if $u_1 = v_1$ and $u_2v_2 \in E_2$, or, $u_2 = v_2$ and $u_1v_1 \in E_1$. The Cartesian product of two paths is a lattice grid graph, and the Cartesian product of a path and a cycle is a prism grid graph.

Before proving our main results, we first describe antimagic labelling on paths and cycles respectively (see Fig. 1). The labelling methods are the same as in [5], here we rephrase them for the sake of completeness.

Lemma 2.1. *All paths* $P[m+1]$ *are antimagic for integers* $m \geq 2$.

Proof. Suppose the vertex set is $\{v_1, \ldots, v_{m+1}\}$ and the edge set is arranged to be $\{v_iv_{i+2} | i = 1, \ldots, m-1\} \cup \{v_mv_{m+1}\}$. The following labelling $f(v_iv_{i+2}) = i$, for $1 \leq i \leq m-1$, and $f(v_mv_{m+1}) = m$ is antimagic, since we have

$$f^+(v_i) = \begin{cases} i & i = 1, 2; \\ 2i-2 & i = 3, \ldots, m; \\ 2m-1 & i = m+1. \end{cases}$$

Therefore,

$$f^+(v_1) < f^+(v_2) < \cdots < f^+(v_{m+1}). \quad \blacksquare$$

Lemma 2.2. *All cycles* $C[m]$ *are antimagic for integers* $m \geq 3$.

Proof. Suppose the vertex set is $\{v_1, \ldots, v_m\}$ and the edge set is arranged to be $\{v_1v_2\} \cup \{v_iv_{i+2} | i = 1, \ldots, m-2\} \cup \{v_{m-1}v_m\}$. The following labeling $f(v_1v_2) = 1$, $f(v_iv_{i+2}) = i+1$, for $1 \leq i \leq m-2$, and $f(v_{m-1}v_m) = m$ is antimagic, since we have

$$f^+(v_i) = \begin{cases} 3 & i = 1; \\ 2i & i = 2, \ldots, m-1; \\ 2m-1 & i = m. \end{cases}$$

Therefore,

$$f^+(v_1) < f^+(v_2) < \cdots < f^+(v_m). \quad \blacksquare$$

3. Proof of Theorem 1.1

Let $f : E(P_1[m+1] \times P_2[n+1]) \to \{1, 2, \ldots, 2mn+m+n\}$ be an edge labelling of $P_1[m+1] \times P_2[n+1]$, and denote the induced sum at vertex (u, v) by $f^+(u, v) = \sum f((u, v), (y, z))$, where the sum runs over all vertices (y, z) adjacent to (u, v) in $P_1[m+1] \times P_2[n+1]$. To prove Theorem 1.1, first, we construct a labeling that is antimagic on product graphs of two paths $P_1[m+1]$ and $P_2[n+1]$, for $n \geq m \geq 2$. Then, we give an antimagic labelling of graphs $P_1[2] \times P_2[n+1]$, for $n \geq 1$.

3.1. $P_1[m+1] \times P_2[n+1]$ is antimagic, for $n \geq m \geq 2$

Assume that $P_1[m+1]$ has edge set $\{u_i u_{i+2} | i = 1, \ldots, m-1\} \cup \{u_m u_{m+1}\}$, and $P_2[n+1]$ has edge set $\{v_i v_{i+1} | i = 1, \ldots, n\}$. We will construct an antimagic labelling of $P_1[m+1] \times P_2[n+1]$ for $n \geq m \geq 2$, which contains two phases.

Phase 1: For the $mn+m$ edges contained in copies of $P_1[m+1]$ component (i.e. the edges $((u_i, v_j), (u_{i+2}, v_j))$ and $((u_m, v_j), (u_{m+1}, v_j))$, for $1 \leq i \leq m-1$, $1 \leq j \leq n+1$), label them with even numbers $2, 4, \ldots, 2mn+2m$ (notice $n \geq m$).

Specifically, first label the edges of $P_1[m+1]$ with U and R such that $u_1 u_3$ is labelled with U, and two edges are labelled with different letters if they are incident to a same vertex. Obviously, there is one unique such labelling. For each edge $u_i u_j \in E(P_1[m+1])$ labelled with U, label the edges $((u_i, v_1), (u_j, v_1)), ((u_i, v_2), (u_j, v_2)), \ldots, ((u_i, v_{n+1}), (u_j, v_{n+1}))$ in usual order; for each edge $u_i u_j \in E(P_1[m+1])$ labelled with R, label the edges $((u_i, v_1), (u_j, v_1)), ((u_i, v_2), (u_j, v_2)), \ldots, ((u_i, v_{n+1}), (u_j, v_{n+1}))$ in reversed order, and

$$2,\ 4, \ldots, 2n+2, \quad \text{(labels for } ((u_1, v_i), (u_3, v_i)), i = 1, 2, \ldots, n+1)$$
$$2n+4,\ 2n+6, \ldots, 4n+4, \quad \text{(labels for } ((u_2, v_i), (u_4, v_i)), i = 1, 2, \ldots, n+1)$$
$$\cdots\cdots\cdots$$
$$2mn+2m-2n, \ldots, 2mn+2m, \quad \text{(labels for } ((u_m, v_i), (u_{m+1}, v_i)), i = 1, 2, \ldots, n+1)$$

Phase 2: Denote by $A : a_1 < a_2 < \cdots < a_s$ the sequence of all odd numbers in $\{1, 2, \ldots, 2mn+m+n\}$, and denote by $B : b_1 < \cdots < b_t$ the sequence of all even numbers in $\{2mn+2m+1, \ldots, 2mn+m+n\}$, i.e. the even numbers that are not used in Phase 1. Notice that $t \leq \frac{1}{2}(2mn+m+n) - (mn+m) = \frac{1}{2}(n-m)$. We merge A and B into a sequence $C : a_1, a_2, \ldots, a_{s-t}, b_1, a_{s-t+1}, b_2, \ldots, b_t, a_s$ of $s+t$ terms ($s+t = mn+n$), and denote the sequence C by $c_1, c_2, \ldots, c_{mn+n}$, which are the labels for the other $mn+n$ edges contained in copies of $P_2[n+1]$ component.

For the i-th $P_2[n+1]$ component (with vertices $(u_i, v_1), (u_i, v_2), \ldots, (u_i, v_{n+1})$), label its edges in usual order according to the indices in the sequence C, $i = 1, 2, \ldots, m+1$, and:

$$c_1,\ c_2, \ldots, c_n, \quad \text{(labels for the 1st } P_2[n+1] \text{ component)}$$
$$c_{n+1},\ c_{n+2}, \ldots, c_{2n}, \quad \text{(labels for the 2nd } P_2[n+1] \text{ component)}$$
$$\cdots\cdots\cdots$$
$$c_{mn+1}, c_{mn+2}, \ldots, c_{mn+n}, \quad \text{(labels for the } (m+1)\text{-th } P_2[n+1] \text{ component)}$$

Notice that $2t \leq n-m$, hence only the edges in the $(m+1)$-th $P_2[n+1]$ component may be labelled with even numbers (see Fig. 2).

In what follows, we will show that the above labelling is antimagic. In the product graph $P_1[m+1] \times P_2[n+1]$, at each vertex (u, v), the edges incident to this vertex can be partitioned into two parts, one part is contained in a copy of $P_1[m+1]$ component, and the other part is contained in a copy of $P_2[n+1]$ component. Let $f_1^+(u, v)$ and $f_2^+(u, v)$ denote the sum at vertex (u, v) restricted to $P_1[m+1]$ component and $P_2[n+1]$ component respectively, i.e. $f_1^+(u, v) = \sum f((u, v), (y, v))$, where the sum runs over all vertices y adjacent to u in $P_1[m+1]$, and $f_2^+(u, v) = \sum f((u, v), (u, z))$, where the sum runs over all vertices z adjacent to v in $P_2[n+1]$. Therefore, $f^+(u, v) = f_1^+(u, v) + f_2^+(u, v)$. The following two claims imply the antimagicness of the above labelling.

Claim 3.1. *For the above labelling of $P_1[m+1] \times P_2[n+1]$, $n \geq m \geq 2$, we have:*

$$f^+(u_1, v_2) < f^+(u_1, v_3) < \cdots < f^+(u_1, v_n) <$$

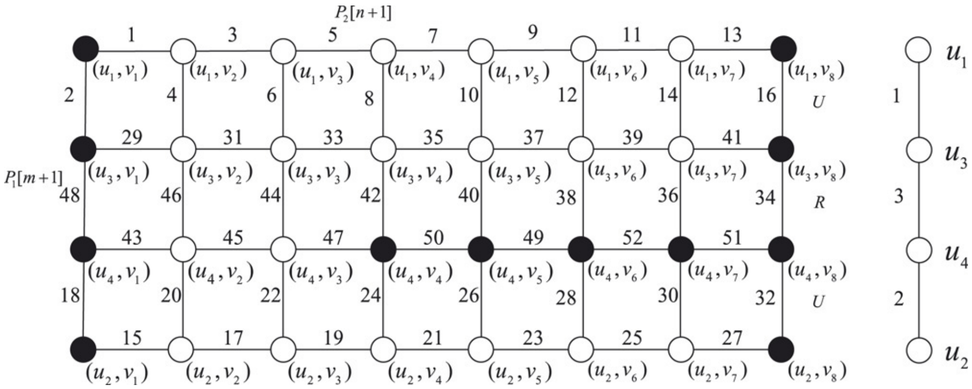

Fig. 2. Antimagic labelling of $P_1[m+1] \times P_2[n+1]$, for $m = 3$, $n = 7$.

$$
\begin{aligned}
&f^+(u_2, v_2) < f^+(u_2, v_3) < \cdots < f^+(u_2, v_n) < \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&f^+(u_m, v_2) < f^+(u_m, v_3) < \cdots < f^+(u_m, v_n) < \\
&f^+(u_{m+1}, v_2) < \cdots < f^+(u_{m+1}, v_{n-2t}),
\end{aligned}
$$

where t ($\le \frac{1}{2}(n-m)$) is the number of even numbers in $\{2mn+2m+1, \ldots, 2mn+m+n\}$. In addition, all the above sums are even numbers.

Proof. Since $f_1^+(u_1, v_2) < f_1^+(u_1, v_3) < \cdots < f_1^+(u_1, v_n)$ and $f_2^+(u_1, v_2) < f_2^+(u_1, v_3) < \cdots < f_2^+(u_1, v_n)$, we have $f^+(u_1, v_2) < f^+(u_1, v_3) < \cdots < f^+(u_1, v_n)$. $f^+(u_1, v_n) < f^+(u_2, v_2)$ since $f_1^+(u_1, v_n) < f_1^+(u_2, v_2)$ and $f_2^+(u_1, v_n) < f_2^+(u_2, v_2)$. $f^+(u_2, v_2) < f^+(u_2, v_3) < \cdots < f^+(u_2, v_n)$ since $f_2^+(u_2, v_{i+1}) - f_2^+(u_2, v_i) \ge 4$ and $f_1^+(u_2, v_{i+1}) - f_1^+(u_2, v_i) \ge -2$, it follows that $f^+(u_2, v_{i+1}) - f^+(u_2, v_i) \ge 2$, for $i = 2, \ldots, n-1$. If $m = 2$, $f_1^+(u_3, v_2) = f_1^+(u_3, v_n) > f_1^+(u_2, v_n)$; if $m > 2$, $f_1^+(u_3, v_2) > f((u_3, v_2), (u_j, v_2)) > f((u_2, v_n), (u_4, v_n)) = f_1^+(u_2, v_n)$, where $j = 4$ or 5. Thus, in either case we have $f_1^+(u_2, v_n) < f_1^+(u_3, v_2)$. Clearly, $f_2^+(u_2, v_n) < f_2^+(u_3, v_2)$. It follows that $f^+(u_2, v_n) < f^+(u_3, v_2)$.

For the vertices of degree 4, clearly, $f_1^+(u_i, v_2) = f_1^+(u_i, v_3) = \cdots = f_1^+(u_i, v_n)$ for $i = 3, \ldots, m+1$. Moreover, $f_1^+(u_3, v_2) < f_1^+(u_4, v_2) < \cdots < f_1^+(u_{m+1}, v_2)$ since $f((u_1, v_2), (u_3, v_2)) < f((u_2, v_2), (u_4, v_2)) < \cdots < f((u_{m-1}, v_2), (u_{m+1}, v_2)) < f((u_m, v_2), (u_{m+1}, v_2))$. It follows that

$$
\begin{aligned}
&f_1^+(u_3, v_2) = f_1^+(u_3, v_3) = \cdots = f_1^+(u_3, v_n) < \\
&f_1^+(u_4, v_2) = f_1^+(u_4, v_3) = \cdots = f_1^+(u_4, v_n) < \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&f_1^+(u_m, v_2) = f_1^+(u_m, v_3) = \cdots = f_1^+(u_m, v_n) < \\
&f_1^+(u_{m+1}, v_2) = \cdots = f_1^+(u_{m+1}, v_{n-2t}).
\end{aligned}
$$

On the other hand, since $c_1 < c_2 < \cdots < c_{mn+n-2t}$, we have that:

$$
\begin{aligned}
&f_2^+(u_3, v_2) < f_2^+(u_3, v_3) < \cdots < f_2^+(u_3, v_n) < \\
&f_2^+(u_4, v_2) < f_2^+(u_4, v_3) < \cdots < f_2^+(u_4, v_n) < \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&f_2^+(u_m, v_2) < f_2^+(u_m, v_3) < \cdots < f_2^+(u_m, v_n) < \\
&f_2^+(u_{m+1}, v_2) < \cdots < f_2^+(u_{m+1}, v_{n-2t}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&f^+(u_3, v_2) < f^+(u_3, v_3) < \cdots < f^+(u_3, v_n) < \\
&f^+(u_4, v_2) < f^+(u_4, v_3) < \cdots < f^+(u_4, v_n) <
\end{aligned}
$$

. . . .

$f^+(u_m, v_2) < f^+(u_m, v_3) < \cdots < f^+(u_m, v_n) <$
$f^+(u_{m+1}, v_2) < \cdots < f^+(u_{m+1}, v_{n-2t})$.

All the above sums are even because each of them contains exactly two odd labels. ■

Claim 3.2. *The remaining* $2m+2+2t$ *sums* $f^+(u_1, v_1)$, $f^+(u_1, v_{n+1})$, $f^+(u_2, v_1)$, $f^+(u_2, v_{n+1}), \ldots, f^+(u_{m+1}, v_1)$, $f^+(u_{m+1}, v_{n+1})$, *and* $f^+(u_{m+1}, v_{n+1-2t})$, $f^+(u_{m+1}, v_{n+2-2t}), \ldots, f^+(u_{m+1}, v_n)$ *are pairwise distinct. In addition, they are all odd numbers.*

Proof. Let us first consider the $2m+2$ sums $f^+(u_1, v_1)$, $f^+(u_1, v_{n+1})$, $f^+(u_2, v_1)$, $f^+(u_2, v_{n+1}), \ldots, f^+(u_{m+1}, v_1)$, $f^+(u_{m+1}, v_{n+1})$, there are two natural cases:

Case 1. m is odd. In this case $u_2u_4 \in E(P_1[m+1])$ is labelled with U, from the way we do the labelling, we have $f_1^+(u_1, v_1) \le f_1^+(u_1, v_{n+1}) \le f_1^+(u_2, v_1) \le f_1^+(u_2, v_{n+1}) \le \cdots \le f_1^+(u_{m+1}, v_1) \le f_1^+(u_{m+1}, v_{n+1})$ and $f_2^+(u_1, v_1) < f_2^+(u_1, v_{n+1}) < f_2^+(u_2, v_1) < f_2^+(u_2, v_{n+1}) < \cdots < f_2^+(u_{m+1}, v_1) < f_2^+(u_{m+1}, v_{n+1})$. Therefore, $f^+(u_1, v_1) < f^+(u_1, v_{n+1}) < f^+(u_2, v_1) < f^+(u_2, v_{n+1}) < \cdots < f^+(u_{m+1}, v_1) < f^+(u_{m+1}, v_{n+1})$.

Case 2. m is even. In this case $u_2u_j \in E(P_1[m+1])$ is labeled with R (where $j = 3$ if $m = 2$, $j = 4$ if $m > 2$), the ordering of the $2m+2$ sums $f^+(u_1, v_1)$, $f^+(u_1, v_{n+1})$, $f^+(u_2, v_1)$, $f^+(u_2, v_{n+1}), \ldots, f^+(u_{m+1}, v_1)$, $f^+(u_{m+1}, v_{n+1})$ is the same as in case 1, but between vertices (u_2, v_1) and (u_2, v_{n+1}). Specifically, we have $f_1^+(u_1, v_1) \le f_1^+(u_1, v_{n+1}) \le f_1^+(u_2, v_1)$, $f_1^+(u_2, v_{n+1}) \le f_1^+(u_3, v_1) \le \cdots \le f_1^+(u_{m+1}, v_{n+1})$ and $f_2^+(u_1, v_1) < f_2^+(u_1, v_{n+1}) < f_2^+(u_2, v_1) < f_2^+(u_2, v_{n+1}) < \cdots < f_2^+(u_{m+1}, v_1) < f_2^+(u_{m+1}, v_{n+1})$. Therefore,

$$f^+(u_1, v_1) < f^+(u_1, v_{n+1}) < f^+(u_2, v_1),\ f^+(u_2, v_{n+1}) < \cdots < f^+(u_{m+1}, v_1) < f^+(u_{m+1}, v_{n+1}).$$

Since $f^+(u_2, v_1) = f_1^+(u_2, v_1) + f_2^+(u_2, v_1) = (4n+4) + (2n+1) = 6n+5$, and $f^+(u_2, v_{n+1}) = f_1^+(u_2, v_{n+1}) + f_2^+(u_2, v_{n+1}) = (2n+4) + (4n-1) = 6n+3$, it follows that $f^+(u_1, v_1) < f^+(u_1, v_{n+1}) < f^+(u_2, v_{n+1}) < f^+(u_2, v_1) < \cdots < f^+(u_{m+1}, v_1) < f^+(u_{m+1}, v_{n+1})$.

Thus, in any of the above two cases, the $2m+2$ sums $f^+(u_1, v_1)$, $f^+(u_1, v_{n+1})$, $f^+(u_2, v_1)$, $f^+(u_2, v_{n+1}), \ldots, f^+(u_{m+1}, v_1)$, $f^+(u_{m+1}, v_{n+1})$ are pairwise distinct, and $f^+(u_{m+1}, v_{n+1})$ is the largest among them. For the other $2t$ sums $f^+(u_{m+1}, v_{n+1-2t})$, $f^+(u_{m+1}, v_{n+2-2t}), \ldots, f^+(u_{m+1}, v_n)$, they are in strict increasing order $f^+(u_{m+1}, v_{n+1-2t}) < f^+(u_{m+1}, v_{n+2-2t}) < \cdots < f^+(u_{m+1}, v_n)$, since: $f_1^+(u_{m+1}, v_{n+1-2t}) = f_1^+(u_{m+1}, v_{n+2-2t}) = \cdots = f_1^+(u_{m+1}, v_n)$ and $f_2^+(u_{m+1}, v_{n+1-2t}) < f_2^+(u_{m+1}, v_{n+2-2t}) < \cdots < f_2^+(u_{m+1}, v_n)$.

At this point, the only remaining issue is to notice that $f^+(u_{m+1}, v_{n+1-2t}) > f^+(u_{m+1}, v_{n+1})$, since $f_1^+(u_{m+1}, v_{n+1-2t}) = f_1^+(u_{m+1}, v_{n+1})$ and $f_2^+(u_{m+1}, v_{n+1-2t}) = a_{s-t} + b_1 \ge (2mn+m+n-1-2t) + (2mn+2m+2) \ge 2mn+m+n-1-(n-m)+2mn+2m+2 = 4mn+4m+1 > 2mn+m+n \ge a_s = f_2^+(u_{m+1}, v_{n+1})$. Hence, the $2m+2t+2$ sums are pairwise distinct. They are all odd numbers since each of them contains exactly one odd label. ■

Combining Claims 3.1 and 3.2, we have proved that the above labelling of $P_1[m+1] \times P_2[n+1]$ is antimagic, for $n \ge m \ge 2$. Please see Fig. 2 as an example of antimagic labelling of $P_1[m+1] \times P_2[n+1]$, for $m = 3, n = 7$.

3.2. $P_1[2] \times P_2[n+1]$ *is antimagic, for* $n \ge 1$

Assume that $P_2[n+1]$ has edge set $\{v_iv_{i+2} | i = 1, \ldots, n-1\} \cup \{v_nv_{n+1}\}$. For $n = 1$, $P_1[2] \times P_2[2]$ is isomorphic to $C[4]$, hence by Lemma 2.2, it is antimagic. For $n > 1$, label $1, 3, \ldots, 2n-1$ to the edges $((u_1, v_1), (u_1, v_3))$, $((u_1, v_2), (u_1, v_4)), \ldots, ((u_1, v_{n-1}), (u_1, v_{n+1}))$, $((u_1, v_n), (u_1, v_{n+1}))$, label $2, 4, \ldots, 2n$ to the edges $((u_2, v_1), (u_2, v_3))$, $((u_2, v_2), (u_2, v_4)), \ldots, ((u_2, v_{n-1}), (u_2, v_{n+1}))$, $((u_2, v_n), (u_2, v_{n+1}))$, and label $2n+1, 2n+2, \ldots, 3n+1$ to $((u_1, v_1), (u_2, v_1))$, $((u_1, v_2), (u_2, v_2)), \ldots, ((u_1, v_{n+1}), (u_2, v_{n+1}))$ (see Fig. 3).

We will show that the above labelling (for $n > 1$) is antimagic. Since the vertex sums restricted to $P_1[2]$ component satisfy that $f_1^+(u_1, v_1) = f_1^+(u_2, v_1) < f_1^+(u_1, v_2) = f_1^+(u_2, v_2) < \cdots < f_1^+(u_1, v_{n+1}) = f_1^+(u_2, v_{n+1})$ ('=' and '<' alternate), and the vertex sums restricted to $P_2[n+1]$ component are

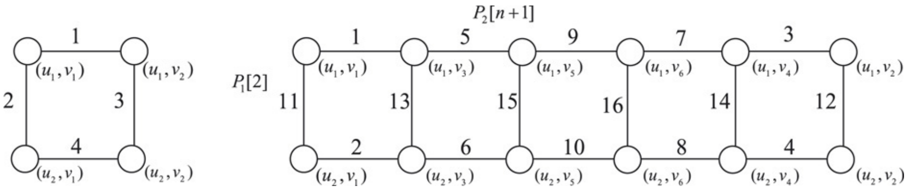

Fig. 3. Antimagic labellings of $P_1[2] \times P_2[2]$ and $P_1[2] \times P_2[n+1]$, for $n = 5$.

$$f_2^+(u_1, v_i) = \begin{cases} 1 & i = 1; \\ 3 & i = 2; \\ 4i - 6 & i = 3, \ldots, n; \\ 4n - 4 & i = n + 1; \end{cases} \qquad f_2^+(u_2, v_i) = \begin{cases} 2 & i = 1; \\ 4 & i = 2; \\ 4i - 4 & i = 3, \ldots, n; \\ 4n - 2 & i = n + 1. \end{cases}$$

It follows that $f_2^+(u_1, v_1) < f_2^+(u_2, v_1) < f_2^+(u_1, v_2) < \cdots < f_2^+(u_2, v_n) = f_2^+(u_1, v_{n+1}) < f_2^+(u_2, v_{n+1})$ (there is one equality). Therefore, $f^+(u_1, v_1) < f^+(u_2, v_1) < f^+(u_1, v_2) < f^+(u_2, v_2) < \cdots < f^+(u_1, v_{n+1}) < f^+(u_2, v_{n+1})$, implying the antimagicness of the above labelling.

Combining the above two cases, we have proved Theorem 1.1.

4. Proof of Theorem 1.2

Assume that in the product graph $C[m] \times P[n+1]$, $C[m]$ has edge set $\{u_1u_2\} \cup \{u_iu_{i+2} | i = 1, \ldots, m-2\} \cup \{u_{m-1}u_m\}$, and $P[n+1]$ has edge set $\{v_iv_{i+2} | i = 1, \ldots, n-1\} \cup \{v_nv_{n+1}\}$. To prove Theorem 1.2, first, we construct a labelling that is antimagic on product graphs $C[m] \times P[n+1]$ for $m \geq 3, n \geq 2$. Then, we give an antimagic labelling of graphs $C[m] \times P[2]$ for $m \geq 3$.

Lemma 4.1. $C[m] \times P[n+1]$ *is antimagic for* $m \geq 3, n \geq 2$.

Proof. The labelling contains two phases.

Phase 1: Using the same way as in the antimagic labelling of cycles in Lemma 2.2, label the edges on the i-th $C[m]$ component (with vertices $(u_1, v_i), (u_2, v_i), \ldots, (u_m, v_i)$), for $i = 1, 2, \ldots, n+1$, and:

$1,\ 2, \ldots, m,$ (labels for the 1st $C[m]$ component)

$m+1,\ m+2, \ldots, 2m,$ (labels for the 2nd $C[m]$ component)

. . . .

$mn+1,\ mn+2, \ldots, mn+m,$ (labels for the $(n+1)$-th $C[m]$ component)

Phase 2: Similarly, label the edges of $P[n+1]$ with U and R such that v_1v_3 is labelled with U, and two edges are labelled with different letters if they are incident to a same vertex. For each edge $v_iv_j \in E(P[n+1])$ labelled with U, the edges $((u_1, v_i), (u_1, v_j)), ((u_2, v_i), (u_2, v_j)), \ldots, ((u_m, v_i), (u_m, v_j))$ will be labeled in usual order; for each edge $v_iv_j \in E(P[n+1])$ labeled with R, the edges $((u_1, v_i), (u_1, v_j)), ((u_2, v_i), (u_2, v_j)), \ldots, ((u_m, v_i), (u_m, v_j))$ will be labelled in reversed order, and:

$mn+m+1,\ mn+m+2, \ldots, mn+2m,$ (labels for $((u_i, v_1), (u_i, v_3)), i = 1, 2, \ldots, m$)

$mn+2m+1, mn+2m+2, \ldots, mn+3m,$ (labels for $((u_i, v_2), (u_i, v_4)), i = 1, 2, \ldots, m$)

. . . .

$2mn+1, 2mn+2, \ldots, 2mn+m,$ (labels for $((u_i, v_n), (u_i, v_{n+1})), i = 1, 2, \ldots, m$)

If $v_2v_j \in E(P[n+1])$ ($j = 3$ if $n = 2$, $j = 4$ if $n > 2$) is labeled with R (i.e. when n is even), we will take a *modification* process on the 2nd $C[m]$ component (with vertices $(u_1, v_2), (u_2, v_2), \ldots, (u_m, v_2)$), which goes as follows. For each $u_iu_j \in E(C[m])$, the edge $((u_i, v_2), (u_j, v_2))$ will be relabelled with $(3m+1) - l_0(i, j)$, where $l_0(i, j)$ is the original label assigned to $((u_i, v_2), (u_j, v_2))$ in Phase 1 (i.e. we 'reverse' the labelling on the 2nd $C[m]$ component, whose edges will still be labelled with the same set of numbers $\{m+1, m+2, \ldots, 2m\}$). Then, we rename each vertex (u_i, v_2) as (u_{m+1-i}, v_2), for $i = 1, 2, \ldots, m$ (see Fig. 4).

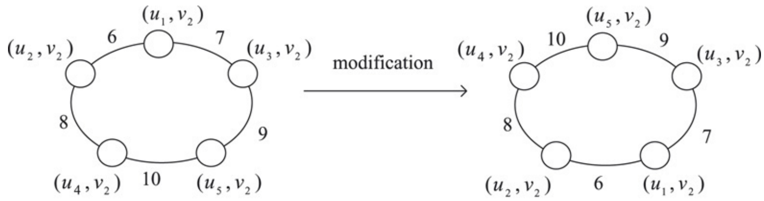

Fig. 4. Modification on the 2nd $C[m]$ component in case n is even, for $m = 5$.

Let $f_1^+(u, v)$ and $f_2^+(u, v)$ be the vertex sum at $(u, v) \in V(C[m] \times P[n+1])$ restricted to $C[m]$ component and $P[n+1]$ component, respectively. Then, $f^+(u, v) = f_1^+(u, v) + f_2^+(u, v)$ is the vertex sum at (u, v). It is easy to see that, for the above labelling, independent of the parity of n (i.e., no matter whether there is a modification process or not), the orderings $f_1^+(u_1, v_2) < f_1^+(u_2, v_2) < \cdots < f_1^+(u_m, v_2)$ and $f_2^+(u_1, v_2) < f_2^+(u_2, v_2) < \cdots < f_2^+(u_m, v_2)$ hold.

Using similar arguments, it is straightforward to prove that for the above labelling we have:

$$\begin{aligned}
&f_1^+(u_1, v_1) < f_1^+(u_2, v_1) < \cdots < f_1^+(u_m, v_1) < \\
&f_1^+(u_1, v_2) < f_1^+(u_2, v_2) < \cdots < f_1^+(u_m, v_2) < \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&f_1^+(u_1, v_{n+1}) < f_1^+(u_2, v_{n+1}) < \cdots < f_1^+(u_m, v_{n+1}),
\end{aligned}$$

and

$$\begin{aligned}
&f_2^+(u_1, v_1) \le f_2^+(u_2, v_1) \le \cdots \le f_2^+(u_m, v_1) \le \\
&f_2^+(u_1, v_2) \le f_2^+(u_2, v_2) \le \cdots \le f_2^+(u_m, v_2) \le \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&f_2^+(u_1, v_{n+1}) \le f_2^+(u_2, v_{n+1}) \le \cdots \le f_2^+(u_m, v_{n+1}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&f^+(u_1, v_1) < f^+(u_2, v_1) < \cdots < f^+(u_m, v_1) < \\
&f^+(u_1, v_2) < f^+(u_2, v_2) < \cdots < f^+(u_m, v_2) < \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&f^+(u_1, v_{n+1}) < f^+(u_2, v_{n+1}) < \cdots < f^+(u_m, v_{n+1}),
\end{aligned}$$

which implies that the above labelling is antimagic. Please see Fig. 5 as an example of antimagic labelling of $C[m] \times P[n+1]$, for $m = 5, n = 3$. ■

Lemma 4.2. *$C[m] \times P[2]$ is antimagic for $m \ge 3$.*

Proof. Assume that $C[m]$ has edge set $\{u_1u_2\} \cup \{u_iu_{i+2} | i = 1, \ldots, m-2\} \cup \{u_{m-1}u_m\}$. Label $1, 3, \ldots, 2m-1$ to the edges $((u_1, v_1), (u_2, v_1))$, $((u_1, v_1), (u_3, v_1)), \ldots, ((u_{m-2}, v_1), (u_m, v_1))$, $((u_{m-1}, v_1), (u_m, v_1))$, label $2, 4, \ldots, 2m$ to the edges $((u_1, v_2), (u_2, v_2))$, $((u_1, v_2), (u_3, v_2)), \ldots, ((u_{m-2}, v_2), (u_m, v_2))$, $((u_{m-1}, v_2), (u_m, v_2))$, and label $2m+1, 2m+2, \ldots, 3m$ to the edges $((u_1, v_1), (u_1, v_2))$, $((u_2, v_1), (u_2, v_2)), \ldots, ((u_m, v_1), (u_m, v_2))$ (see Fig. 6).

We will show that the above labelling ($m \ge 3$) is antimagic. Since the vertex sums restricted to $C[m]$ component are:

$$f_1^+(u_i, v_1) = \begin{cases} 4 & i = 1; \\ 4i-2 & i = 2, \ldots, m-1; \\ 4m-4 & i = m; \end{cases} \qquad f_1^+(u_i, v_2) = \begin{cases} 6 & i = 1; \\ 4i & i = 2, \ldots, m-1; \\ 4m-2 & i = m. \end{cases}$$

It follows that $f_1^+(u_1, v_1) < f_1^+(u_1, v_2) = f_1^+(u_2, v_1) < \cdots < f_1^+(u_{m-1}, v_2) = f_1^+(u_m, v_1) < f_1^+(u_m, v_2)$ (there are two equalities). In addition, $f_2^+(u_1, v_1) = f_2^+(u_1, v_2) < f_2^+(u_2, v_1) = f_2^+(u_2, v_2) < \cdots < f_2^+(u_m, v_1) =$

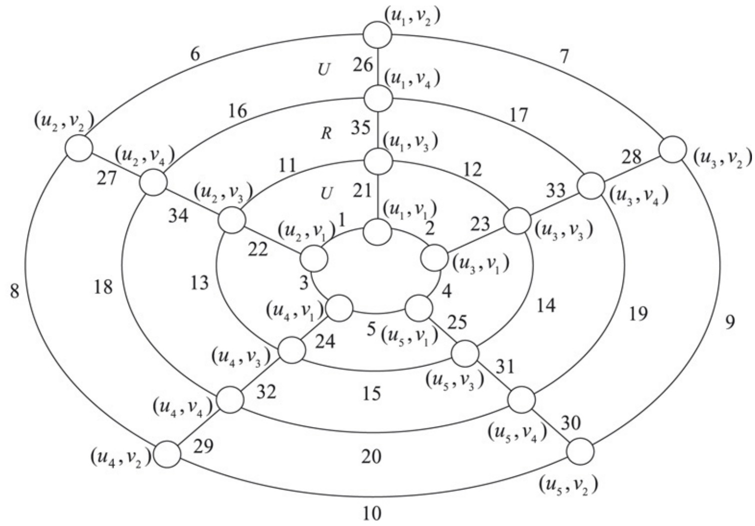

Fig. 5. Antimagic labeling of $C[m] \times P[n+1]$, for $m = 5$, $n = 3$.

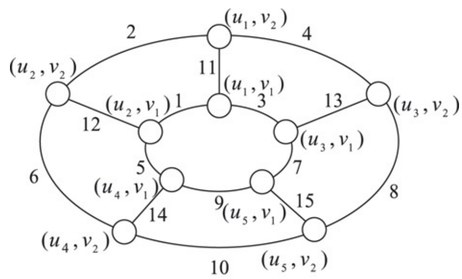

Fig. 6. Antimagic labelling of $C[m] \times P[2]$, for $m = 5$.

$f_2^+(u_m, v_2)$ ('=' and '<' alternate). Therefore, $f^+(u_1, v_1) < f^+(u_1, v_2) < f^+(u_2, v_1) < f^+(u_2, v_2) < \cdots < f^+(u_m, v_1) < f^+(u_m, v_2)$, implying the antimagicness of the above labeling. ■

Combining Lemmas 4.1 and 4.2, we have proved Theorem 1.2.

5. Open problems

In contrast to toroidal grids, it still seems challenging to prove the antimagicness for lattice grid graphs with dimensions higher than two. For example, it may be interesting to construct antimagic labellings for the cubic lattice grids $P[n+1] \times P[n+1] \times P[n+1]$, for $n \geq 1$.

References

[1] N. Alon, G. Kaplan, A. Lev, Y. Roditty, R. Yuster, Dense graphs are antimagic, Journal of Graph Theory 47 (2004) 297–309.
[2] J.A. Gallian, A dynamic survey of graph labeling, The Electronic Journal of Combinatorics 5 (DS6) (2005) 1–148 (ninth edition).
[3] N. Hartsfield, G. Ringel, Pearls in Graph Theory, Academic Press, Inc., Boston, 1990, pp. 108–109 (revised version, 1994).
[4] D. Hefetz, Anti-magic graphs via the Combinatorial NullStellenSatz, Journal of Graph Theory 50 (2005) 263–272.
[5] T.-M. Wang, Toroidal grids are anti-magic, in: Proc. 11th Annual International Computing and Combinatorics Conference, COCOON'2005, in: LNCS, vol. 3595, Springer, 2005, pp. 671–679.