On the effects of characteristic lengths in bending and torsion on Mode III crack in couple stress elasticity

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Abstract

The problem of a stationary semi-infinite crack in an elastic solid with microstructures subject to remote classical $K_{III}$ field is investigated in the present work. The material behavior is described by the indeterminate theory of couple stress elasticity developed by Koiter. This constitutive model includes the characteristic lengths in bending and torsion and thus it is able to account for the underlying microstructure of the material as well as for the strong size effects arising at small scales. The stress and displacement fields turn out to be strongly influenced by the ratio between the characteristic lengths. Moreover, the symmetric stress field turns out to be finite at the crack tip, whereas the skew-symmetric stress field displays a strong singularity. Ahead of the crack tip within a zone smaller than the characteristic length in torsion, the total shear stress and reduced tractions occur with the opposite sign with respect to the classical LEFM solution, due to the relative rotation of the microstructural particles currently at the crack tip. The asymptotic fields dominate within this zone, which however has limited physical relevance and becomes vanishing small for a characteristic length in torsion of zero. In this limiting case the full-field solution recovers the classical $K_{III}$ field with square-root stress singularity. Outside the zone where the total shear stress is negative, the full-field solution exhibits a bounded maximum for the total shear stress ahead of the crack tip, whose magnitude can be adopted as a measure of the critical stress level for crack advancing. The corresponding fracture criterion defines a critical stress intensity factor, which increases with the characteristic length in torsion. Moreover, the occurrence of a sharp crack profile denotes that the crack becomes stiffer with respect to the classical elastic response, thus revealing that the presence of microstructures may shield the crack tip from fracture.

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1. Introduction

The main reason motivating the extension of the classical theory of elasticity to couple stress (CS) and strain-gradient (SG) constitutive models is that the former is not able to characterize the constitutive behavior of brittle materials at the micron scale, due to the lack of a length scale. In particular, it cannot predict the size

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effect experimentally observed when the representative field scale of the deformation field becomes comparable to the length scale of the microstructure, such as the grain size in a polycrystalline or granular aggregate. For example, it is well known that in the presence of stress concentration, such as near holes and notches, the macroscopic strength of these materials is higher if the grain size is smaller and that the bending and torsional strengths of beams and wires are greater if their cross-section is thinner (Gauthier and Jahsman, 1975; Fleck et al., 1994). In general, both the size and shape of the microstructures have a strong influence on the mechanical properties of materials. In particular, a fine-grained, elongated microstructure is peculiar of high strength materials, while materials with a coarse grained microstructure display high fracture toughness (Tajima, 1993). The indeterminate theory of CS elasticity developed by Koiter (1964) may be considered as an effort to include material characteristic lengths into the continuum theory. It allows accounting for the size and kind of microstructures since it involves two material parameters, namely the length ℓ and ratio η, which are related to the characteristic lengths in bending and torsion. As a consequence, the CS elastic theory can describe the mechanical behavior of many materials with microstructures, like fibrous composites, cellular materials, and laminates, where moments may be transmitted through fibers, or in the cell ribs or walls. Analytical investigations predict that the characteristic lengths in fibrous composites are on the order of the spacing between fibers (Hlavacek, 1975) or fiber thickness (Fleck and Shu, 1995); in cellular solids they are comparable to the average cell size (Adomeit, 1967; Maiti et al., 1984; Chen et al., 1998); in laminates they are on the order of the lamination thickness (Herrmann and Achenbach, 1967). Also foam materials (Lakes, 1986, 1993), granular materials (Chang et al., 2003), masonry (Casolo, 2006), bone (Yang and Lakes, 1982), glassy and semi-crystalline polymers (Nikolov et al., 2007) can be modeled within the framework of CS elasticity. The characteristic length in bending for many polycrystalline metals and structural ceramics is typically of the same order of the compositional grain size, namely few microns (Fleck et al., 1994), whereas their characteristic length in torsion is vanishing small (Lakes, 1995). In contrast, several particle reinforced composites, rubbers and polymethyl methacrylate (PMMA), an amorphous polymer, exhibit vanishing small characteristic lengths (Gauthier and Jahsman, 1975; Lakes, 1995). In a recent work, Maranganti and Sharma (2007) show that strain-gradient elasticity is irrelevant for most crystalline metals and ceramics, since the associated internal material lengths extracted from the atomic displacement correlation functions are on the order of few nanometers, which is too small for any continuum theory. However, these Authors claim that materials with a microstructure such as foams or composites may be fruitfully modeled using strain-gradient elasticity and specify that their results are only applicable to pure materials that do not contain any artificial structural features. They also assert that strain-gradient elasticity is quite useful even in materials exhibiting small non-local characteristic length scales e.g. in analysis of defects in graphene (Zhang et al., 2006). Therefore, strain-gradient theories can also be considered applicable and meaningful for the analysis of fracture mechanics problems in materials with complex microstructures and many advanced materials and systems, like foams, cellular materials, laminates, and fibrous composites.

If the classical theory of elasticity is adopted for the analysis of the stress and deformation fields near the tip of a crack in a material with microstructures, then the results are expected to be rather inaccurate at distance to the crack tip comparable with the characteristic lengths of the material. Therefore, for the investigations of the crack tip fields at the micron scale it becomes necessary to make use of enhanced constitutive models, which may account for the presence of microstructure. The CS elasticity theory may be considered as sufficiently accurate for the investigation of the crack tip zone and still enough simple to allow obtaining closed form analytical solutions, which can be usefully compared with the results provided by the various SG finite element models proposed (Amanatidou and Aravas, 2002; Chen and Wang, 2002; Li and Xie, 2004; Wei, 2006) for their validation.

The effects of elastic strain gradients on the stress and displacement fields for a stationary Mode III crack were investigated by Vardoulakis et al. (1996), Zhang et al. (1998), Unger and Aifantis (2000), Fannjiang et al. (2002), Paulino et al. (2003), and Georgiadis (2003). While Vardoulakis et al. (1996), Fannjiang et al. (2002), Paulino et al. (2003), and Georgiadis (2003) introduced strain gradients through the second gradient of displacement, Zhang et al. (1998) only considered rotation gradients by assuming a CS theory which is the specialization of the model here adopted when η = 0, namely for a fixed ratio between the characteristic lengths in bending and torsion. The results obtained therein reveal that the skew-symmetric stress components have \( r^{-3/2} \) singularity near the crack tip, where \( r \) is the distance to the crack tip. Although this singularity is much stron-
ger than the conventional square-root singularity, it does not violate the boundness of the energy flux towards the crack tip and leads to a finite energy release rate. A similar trend but a reduced value of the singularity is also observed for Mode III crack propagation in elastic–plastic CS materials with linear strain hardening, both for a single (Radi and Gei, 2004) or two (Radi, 2007) characteristic lengths. In their investigations, Zhang et al. (1998) established that asymptotic fields dominate within a zone of 0.5\( \ell \) near the crack tip, whereas SG effects are observed within a zone of 5\( \ell \). Moreover, they found that both shear stress and couple stress contribute to the crack tip energy release rate, which must coincide with the remote energy release rate. They also found that the symmetric shear stress field vanishes at the crack tip. However, these Authors missed the leading order term in the asymptotic expansion of the out-of-plane displacement, which is responsible for a finite symmetric shear stress field at the crack tip and does not contribute to the energy flux towards the crack tip. The inclusion of this term yields a positive out of plane displacement near to the crack tip, unlike the results of Zhang et al. (1998), which found a negative displacement in a large sector ahead of the crack tip.

It must be remarked that the simple CS theory adopted by Zhang et al. (1998) is not appropriate for modeling an antiplane problem, since it does not properly account for the size of the characteristic length in torsion, crucial for an accurate definition of the material behavior under antiplane loadings. In their work, Zhang et al. (1998) only investigated the effects of the characteristic length in bending on Mode III fracture and they did not explore the role played by the characteristic length in torsion, which affects the ratio \( \eta \) and has no influence on \( \ell \). Since the characteristic length in torsion is expected to have a strong influence on the antiplane fracture problem, the problem of a stationary Mode III crack in a CS elastic solid with two characteristic lengths has been considered in the present work. In particular, the influence of the ratio \( \eta \) on the stress, couple stress and displacement fields and crack tip energy release rate has been analytically investigated. The governing equations are reported in Section 2, together with the definition of the generalized \( J \)-integral for CS elastic materials. An asymptotic analysis is first performed in Section 3 by taking into consideration the first two terms in the asymptotic expansion of the out-of-plane displacement. The results here obtained show that the symmetric shear stresses turn out to be finite at the crack tip and the skew-symmetric stress field dominates the asymptotic fields, producing a remarkable increase of tractions level at the crack tip. However, it occurs with the negative sign ahead of the crack tip, in agreement with the results provided by Zhang et al. (1998). The analytical full-field solution for a semi-infinite crack subject to remote classical \( K_{\text{III}} \) field is obtained in Section 4 by following the approach introduced by Atkinson and Leppington (1977) and also adopted by Zhang et al. (1998), which makes use of Fourier transform and Wiener–Hopf technique (Noble, 1958; Freund, 1990). Results are then presented in Section 5. The knowledge of the full-field solution allows evaluating the size of the zone ahead of the crack tip where the total shear stress has the negative sign, which almost coincides with the zone of dominance of the asymptotic fields. Its size is expected to strongly depend on the characteristic length in torsion and thus on the ratio \( \eta \). Outside this zone, which has limited physical relevance, the total shear stress is found to exhibit a bounded positive maximum. Therefore, a fracture criterion based on the maximum shear stress hypothesis has been introduced in order to obtain the critical value of the stress intensity factor \( K_{\text{IIIC}} \) required for crack initiation, which explicitly depends on the microstructure through the material characteristic lengths. A similar approach has also been used by Eringen (1979), who studied the problem of a finite crack subject to antiplane shear within the context of non-local linear elasticity and found finite stresses at the crack tip.

The inclusion of two material characteristic lengths thus provides more realistic predictions on the tractions level ahead of the crack tip than the classical LEFM solution and gives more accurate results than the CS theory of elasticity with a single characteristic length, allowing evaluating the increase in fracture toughness due to the presence of microstructures.

2. Governing equations

Reference is made to a Cartesian coordinate system \((0, x_1, x_2, x_3)\) centered at the crack tip. Under antiplane shear deformation, the indeterminate theory of CS elasticity (Koiter, 1964) adopted in the present study provides the following kinematical compatibility conditions between the out-of-plane displacement \( w \), rotation vector \( \varphi \), strain tensor \( \varepsilon \), and deformation curvature tensor \( \chi \)
\( \varepsilon_{13} = \frac{1}{2} w_{11}, \quad \varepsilon_{23} = \frac{1}{2} w_{22}, \quad \varphi_1 = \frac{1}{2} w_{12}, \quad \varphi_2 = -\frac{1}{2} w_{33}, \quad (1) \)

\( \chi_{11} = -\chi_{22} = \frac{1}{2} w_{12}, \quad \chi_{21} = -\frac{1}{2} w_{11}, \quad \chi_{12} = \frac{1}{2} w_{22}. \quad (2) \)

Therefore, rotations are derived from displacements and the tensor field \( \chi \) turns out to be irrotational. According to the CS theory (Koiter, 1964) the non-symmetric Cauchy stress tensor \( t \) can be decomposed into a symmetric part \( \sigma \) and a skew-symmetric part \( \tau \), namely \( t = \sigma + \tau \). In addition, the couple stress tensor \( \mu \) is introduced as the work-conjugated quantity of \( \chi^T \). For the antiplane problem within the CS theory \( \varepsilon, \sigma, \tau, \chi, \) and \( \mu \) are purely deviatoric tensors. The reduced tractions vector \( p \) and couple stress tractions vector \( q \) are defined as

\[
p = t^T n + \frac{1}{2} \nabla \mu_{nn} \times n, \quad q = \mu^T n - \mu_{nn} n,
\]

respectively, where \( n \) denotes the outward unit normal and \( \mu_{nn} = n \cdot \mu n \). The conditions of quasistatic equilibrium of forces and moments write

\[
\sigma_{13,1} + \sigma_{23,2} + \tau_{13,1} + \tau_{23,2} = 0, \quad \mu_{11,1} + \mu_{21,2} + 2 \tau_{23} = 0, \quad \mu_{12,1} + \mu_{22,2} - 2 \tau_{13} = 0.
\]

(4)

Within the context of small deformations theory, the total strain \( \varepsilon \) and the deformation curvature \( \chi \) are connected to stress and couple stress through the following isotropic constitutive relations

\[
\sigma = 2 G \varepsilon, \quad \mu = 2 G \ell^2 (\chi^T + \eta \chi),
\]

(5)

where \( G \) is the elastic shear modulus, \( \nu \) the Poisson ratio, \( \ell \) and \( \eta \) is the CS parameters introduced by Koiter (1964), with \( -1 < \eta < 1 \). Both material parameters \( \ell \) and \( \eta \) depend on the microstructure and can be connected to the material characteristic lengths in bending and in torsion, namely

\[
\ell_b = \ell / \sqrt{2}, \quad \ell_t = \ell \sqrt{1 + \eta}.
\]

(6)

Typical values of \( \ell_b \) and \( \ell_t \) for some classes of materials with microstructure can be found in Lakes (1986, 1995). In particular, experimental results reported by Lakes (1986) provide \( \ell_b = 0.032 \) mm and \( \ell_t = 0.065 \) mm for a syntactic foam consisting of hollow glass microbubbles embedded in an epoxy matrix and \( \ell_b = 0.327 \) mm and \( \ell_t = 0.62 \) mm for a high-density rigid polyurethane closed-cell foam. These values correspond to \( \ell = 0.045 \) mm and \( \eta = 1 \) for the first material and to \( \ell = 0.462 \) mm and \( \eta = 0.797 \) for the second material.

A sound micromechanical interpretation of the CS constitutive parameters can also be found in Hu et al. (1999) and Bigoni and Drugan (2007). The variation of \( \ell_t \) with the ratio \( \eta \) is plotted in Fig. 1. The limit value

![Fig. 1. Variation of characteristic lengths in bending \( \ell_b \) and torsion \( \ell_t \) with the ratio \( \eta \).](image-url)
of \( \eta = -1 \) corresponds to a vanishing characteristic length in torsion, which is typical of polycrystalline metals. Moreover, from the definitions (6) it follows that \( \ell_\nu = \ell_b \) for \( \eta = -0.5 \) (see Fig. 1) and \( \ell_\nu = \ell = \sqrt{2} \ell_b \) for \( \eta = 0 \), namely for the particular case studied by Zhang et al. (1998).

The constitutive equations of the indeterminate CS theory do not define the skew-symmetric part \( \tau \) of the total stress tensor \( \mathbf{t} \), which instead is determined by the equilibrium Eq. (4)\textsubscript{2,3}. Constitutive Eq. (5) together with compatibility relations (1) and (2) give stresses and couple stresses in terms of the displacement \( w \):

\[
\begin{align*}
\sigma_{13} &= Gw_{13}, \quad \sigma_{23} = Gw_{23}, \\
\mu_{11} &= -\mu_{22} = G\ell^2(1 + \eta)w_{12}, \quad \mu_{21} = G\ell^2(w_{22} - \eta w_{11}), \quad \mu_{12} = -G\ell^2(w_{11} - \eta w_{22}).
\end{align*}
\]  

(7)

The introduction of (8) into (4)\textsubscript{2,3} yields

\[
\tau_{13} = -\frac{G\ell^2}{2} \Delta w_{1}, \quad \tau_{23} = -\frac{G\ell^2}{2} \Delta w_{2},
\]  

(9)

where \( \Delta \) denotes the Laplace operator. A substitution of (7) and (9) into (4)\textsubscript{1} gives the following PDE for the function \( w \):

\[
\Delta w - \frac{\ell^2}{2} \Delta \Delta w = 0.
\]  

(10)

According to (3), the non-vanishing components of the reduced traction and couple stress traction vectors along the crack line \( x_2 = 0 \), where \( \mathbf{n} = (0, \pm 1, 0) \), can be written as

\[
p_3 = t_{23} + \frac{1}{2} \mu_{22,1}, \quad q_1 = \mu_{21},
\]  

(11)

respectively. By using (7)\textsubscript{2}, (8)\textsubscript{1,2}, (9)\textsubscript{2}, and (11), the conditions of vanishing reduced traction and couple stress traction on the crack surface yield the following boundary conditions for the function \( w \):

\[
w_{2} - \frac{\ell^2}{2} [(2 + \eta)w_{11} + w_{22}]_{2} = 0, \quad w_{22} - \eta w_{11} = 0, \quad \text{for } x_1 < 0, \ x_2 = 0.
\]  

(12)

Ahead of the crack tip the skew-symmetry of the Mode III crack problem requires

\[
w = 0, \quad w_{22} - \eta w_{11} = 0, \quad \text{for } x_1 > 0, \ x_2 = 0.
\]  

(13)

Note that the ratio \( \eta \) enters the boundary conditions (12) and (13), but it does not appear into the governing Eq. (10), which is the same as in Zhang et al. (1998). As \( \eta \) tends to vanish the boundary conditions (12) and (13) recover these considered by Zhang et al. (1998). However, they differ from the boundary conditions assumed by Vardoulakis et al. (1996), Unger and Aifantis (2000), Fannjiang et al. (2002), and Paulino et al. (2003), which do not comply with variational considerations.

2.1. The limiting case \( \eta = -1 \)

In the particular case of \( \eta = -1 \) Eq. (10) and boundary conditions (12) and (13) are satisfied by the classical solution of the problem of a semi-illimitate Mode III crack in an infinite solid displaying linear elastic behavior, defined by the field equation

\[
\Delta w = 0,
\]  

(14)

together with the following boundary conditions:

\[
w_{2} = 0, \quad \text{for } x_1 < 0, \ x_2 = 0,
\]  

(15)

\[
w = 0, \quad \text{for } x_1 > 0, \ x_2 = 0.
\]  

(16)

Moreover, in this case the couple stress and skew symmetric stress fields identically vanish, since from (8) and (9) it follows that

\[
\mu_{11} = -\mu_{22} = 0, \quad \mu_{21} = -\mu_{12} = G\ell^2 \Delta w = 0, \quad \tau_{13} = \tau_{23} = 0.
\]  

(17)
2.2. Strain-energy density and generalized J-integral

The strain-energy density $W$ for CS elastic materials is a function of strain and deformation curvature tensors, which are the work conjugates of symmetric stress and couple stress tensors, respectively. The introduction of the additional kinematic variables occurs through the constitutive CS parameters $\ell$ and $\eta$, namely

$$W = G\dot{e} \cdot \varepsilon + G\dot{e}^2 (\chi \cdot \chi + \eta \chi \cdot \chi^T).$$

(18)

Under antiplane shear deformation by using (1) and (2) the strain-energy density (18) becomes

$$W = \frac{G}{2} \left( w_{11}^2 + w_2^2 \right) + \frac{G\ell^2}{4} \left[ (w_{11} + w_{22})^2 + 2(1 + \eta) \left( w_{12}^2 - w_{11} w_{22} \right) \right].$$

(19)

The generalized $J$-integral for CS elastic materials was derived by Atkinson and Leppington (1974), Atkinson and Leppington (1977), and Lubarda and Markenscoff (2000). For a Mode III crack it reduces to

$$J = \int_{\Gamma} (W n_1 - \tau n \cdot e_3 w_{11} - \mu n \cdot \varphi_1) \, ds = \int_{\Gamma} (W n - p \cdot e_3 \nabla w - \nabla \varphi^T q) \cdot e_1 \, ds,$$

(20)

where $\Gamma$ is an arbitrary contour surrounding the crack tip, $n$ is the unit normal to $\Gamma$ and $e_1$ is the unit vector along the direction of the $x_1$ axis. The generalized $J$-integral (20) can be proved to be path-independent and equal to the energy release rate, as already done by Zhang et al. (1998) for the particular case of $\eta = 0$. Moreover, by choosing a circular contour around the crack tip with $n = (\cos \theta, \sin \theta, 0)$ and letting its radius tends to zero, the integral (20) becomes

$$J = \frac{G}{2} \lim_{r \to 0} \int_{-\pi}^\pi \left\{ \left( w_{11}^2 - w_2^2 \right) \cos \theta - 2 w_{11} w_2 \sin \theta + \ell^2 \left[ \frac{(\Delta w)^2}{2} \cos \theta + w_{11} \Delta w_r - w_{12} \Delta w \right] \right\} r \, d\theta.$$  

(21)

3. Mode III asymptotic crack tip fields

An asymptotic analysis of the crack tip fields is performed in the present section by considering the following expression for the out-of-plane displacement $w$ in separate variables form

$$w(r, \theta) = r^\ell F_q(\theta),$$

(22)

where $r$ and $\theta$ are polar coordinates centered at the crack tip. The exponent $q$ and function $F_q$ define the radial dependence of the displacement as $r \to 0$ and the corresponding angular variation, respectively. By using the following derivative rules which hold for an arbitrary function $f(x_1, x_2) = f(r, \theta)$

$$f_1 = f_r \cos \theta - f_\theta \frac{\sin \theta}{r}, \quad f_2 = f_r \sin \theta + f_\theta \frac{\cos \theta}{r},$$

(23)

and keeping the leading order terms only as $r$ tends to zero, and thus for $r < \ell$, the governing Eq. (10) yields the following ODE for the unknown function $F_q(\theta)$:

$$F_q''(\theta) + 2(q^2 - 2q + 2)F_q'(\theta) + q^2(q - 2)^2 F_q(\theta) = 0.$$  

(24)

The boundary conditions (13) at $\theta = 0$ imply

$$F_q(0) = 0, \quad F_q'(0) = 0,$$

(25)

and boundary conditions (12) at $\theta = \pi$ yield

$$F_q''(\pi) + q(1 + \eta - \eta q)F_q(\pi) = 0,$$

(26)

$$F_q''(\pi) + [(1 + \eta)(q^2 - 3q + 2) + q^2]F_q(\pi) = 0.$$  

(27)

For $q \neq 1$ Eq. (24) admits the following solution:

$$F_q(\theta) = B_1 \sin(q\theta) + B_2 \cos(q\theta) + B_3 \sin((q - 2)\theta) + B_4 \cos((q - 2)\theta).$$

(28)
Conditions (25) necessarily imply that \( B_2 = B_4 = 0 \), whereas conditions (26) and (27) give

\[
[(1 + \eta)qB_1 + (q + \eta q - 4)B_3](q - 1) \sin(q\pi) = 0,
\]

\[
\{(1 + \eta)qB_1 + [(1 + \eta)q + 2(1 - \eta)]B_3\}(q - 1)(q - 2) \cos(q\pi) = 0. \tag{29}
\]

The homogeneous boundary value problem (29) admits a non-trivial solution for the constants \( B_1 \) and \( B_3 \) if and only if

\[
(1 + \eta)(3 - \eta)q(q - 1)^2(q - 2) \sin(2q\pi) = 0, \tag{30}
\]

namely for \( q = k/2 \), where \( k \) is an integer.

The introduction of the asymptotic field (22) with \( F_q(\theta) \) given by (28) in the expression of the generalized \( J \)-integral (21) yields

\[
J = 2G\ell^2 \lim_{r \to 0} r^{2q-3} (q - 1)B_3[(q - 2)B_3 - qB_1] \frac{\sin(2q\pi)}{2q - 3}. \tag{31}
\]

Therefore, the boundness of the flux of energy toward the crack tip (31) requires \( q \geq 3/2 \) and thus the first admissible value for the exponent \( q \) is 3/2.

In the particular case of \( q = 1 \) Eq. (24) admits the following special solution:

\[
F_1(\theta) = (A_1 + A_2\theta) \sin \theta + (A_3 + A_4\theta) \cos \theta. \tag{32}
\]

Then, conditions (25) necessarily imply that \( A_2 = A_3 = 0 \). Moreover, condition (26) is identically satisfied, whereas condition (27) gives \( A_4 = 0 \). The introduction of the asymptotic displacement field (22) with \( q = 1 \) and \( F_1(\theta) = A_1 \sin \theta \) in the generalized \( J \)-integral (21) yields \( J = 0 \) and thus this term does not contribute to the energy flux towards the crack tip as well as to the crack tip opening displacement.

### 3.1. Two terms asymptotic solution

According to the performed asymptotic analysis, the leading order terms in the asymptotic expansion of the out-of-plane displacement turn out to be

\[
w(r, \theta) = Ar \sin \theta + Br^{3/2} \left[ (1 + \eta) \sin \frac{\theta}{2} - \left( \frac{5}{3} - \eta \right) \sin \frac{3}{2} \theta \right], \tag{33}
\]

where \( A = A_1 \) is an amplitude factor for the lowest order asymptotic crack tip fields and

\[
B = \frac{3B_1}{5 - 3\eta} = \frac{B_3}{1 + \eta}, \tag{34}
\]

is an amplitude factor for the term of order 3/2, obtained by using (29).1.

Therefore, the solution of the homogeneous asymptotic problem can be determined up to the amplitude factors \( A \) and \( B \), which depend on far-field loading and specimen geometry. The constant \( B \) can be estimated by matching the asymptotic solution with the far-field conditions by using the path independent generalized \( J \)-integral (21), in agreement with the classical LEFM approach. The introduction of (33) in the integral (21), yields

\[
J = (1 + \eta)(3 - \eta)\pi B^2 G\ell^2. \tag{35}
\]

Since the crack is subject to a remotely imposed classical \( K_{III} \) field then the \( J \)-integral must assume the finite value

\[
J = \frac{K_{III}^2}{2G}. \tag{36}
\]

By comparing (35) and (36), the following relation between the amplitude constant \( B \) and the stress intensity factor \( K_{III} \) can be obtained

\[
B = \frac{K_{III}}{G\ell \sqrt{2\pi(1 + \eta)(3 - \eta)}}. \tag{37}
\]
The positive sign in (37) has been chosen to ensure that the sliding displacement of the crack faces \( w(r, \pi) \) occurs in the same direction of the applied remote loading.

The introduction of (33) in condition (12) gives a non-vanishing finite term \( w_2 = A \). However, in order to satisfy the condition of vanishing reduced traction (12) it couples with the third derivatives of terms of order \( r^3 \) in the asymptotic expansion of \( w \), which enter (12) throughout \( w_{112} \) and \( w_{222} \).

According to (1), (7)–(9), (33), and (37), the asymptotic fields for displacement, rotation, symmetric stress, couple stress, and skew-symmetric stress under remotely imposed classical \( K_{III} \) field then become

\[
\begin{align*}
\omega(r, \theta) &= Ar \sin \theta + \frac{K_{III} r^{\gamma/2}}{G \ell \sqrt{2\pi(1+\eta)(3-\eta)}} 
\left[ (1+\eta) \sin \frac{\theta}{2} - \left( \frac{5}{3} - \eta \right) \sin \frac{3}{2} \theta \right], \\
\varphi_1(r, \theta) &= -\frac{K_{III} \sqrt{r}}{2G \ell \sqrt{2\pi(1+\eta)(3-\eta)}} [1 - 3\eta + (1+\eta) \cos \theta \cos \frac{\theta}{2}], \\
\varphi_2(r, \theta) &= \frac{K_{III} \sqrt{r}}{2G \ell \sqrt{2\pi(1+\eta)(3-\eta)}} [3 - \eta - (1+\eta) \cos \theta \sin \frac{\theta}{2}], \\
\sigma_{13}(r, \theta) &= -\frac{K_{III} \sqrt{r}}{\ell \sqrt{2\pi(1+\eta)(3-\eta)}} [3 - \eta - (1+\eta) \cos \theta \sin \frac{\theta}{2}], \\
\sigma_{23}(r, \theta) &= GA - \frac{K_{III} \sqrt{r}}{\ell \sqrt{2\pi(1+\eta)(3-\eta)}} \left[ 1 - 3\eta + (1+\eta) \cos \theta \cos \frac{\theta}{2} \right], \\
\mu_{11}(r, \theta) &= -\mu_{22} = -\frac{K_{III} \ell}{2\sqrt{2\pi}r} \sqrt{\frac{1+\eta}{3-\eta}} \left[ 2(1-\eta) + (1+\eta)(\cos \theta - \cos 2\theta) \right] \cos \frac{\theta}{2}, \\
\mu_{21}(r, \theta) &= -\frac{K_{III} \ell}{2\sqrt{2\pi}r} \sqrt{\frac{1+\eta}{3-\eta}} \left[ (1+\eta)(\sin \theta - \sin 2\theta) \right] \cos \frac{\theta}{2}, \\
\mu_{12}(r, \theta) &= -\frac{K_{III} \ell}{2\sqrt{2\pi}r} \sqrt{\frac{1+\eta}{3-\eta}} \left[ 4(1-\eta) - (1+\eta)(\cos \theta + \cos 2\theta) \right] \sin \frac{\theta}{2}, \\
\tau_{13}(r, \theta) &= \frac{K_{III} \ell}{2\sqrt{2\pi}r^3} \sqrt{\frac{1+\eta}{3-\eta}} \sin \frac{3}{2} \theta, \\
\tau_{23}(r, \theta) &= \frac{K_{III} \ell}{2\sqrt{2\pi}r^3} \sqrt{\frac{1+\eta}{3-\eta}} \cos \frac{3}{2} \theta,
\end{align*}
\]

where the constant \( A \) cannot be determined by the asymptotic analysis and \( J \)-integral argument. It will be determined from the full-field solution in Section 5. The corresponding strain and curvature fields have not been reported, however, they can be easily found from the constitutive relations (5).

The asymptotic fields (38) confirm that the ratio \( \eta \) has a strong influence on the angular distribution of the crack tip fields. In particular, as \( \eta \) approaches the limit value \(-1\), and thus \( \ell \) tends to zero, the asymptotic fields for displacement, rotations, and symmetric stresses (38) become unbounded. In this case indeed the problem admits the classical \( K_{III} \) solution, but the asymptotic solution (38) does not converge to the classical solution for materials with conventional elasticity.

Results (38) reveal that the strain and symmetric stress fields are finite at the crack tip, the curvature and couple stress fields display the square-root singularity, the skew-symmetric stress field is singular as \( r^{-3/2} \) as \( r \to 0 \) and thus give the most important contribution to the total stress field near to the crack tip. Note that the skew-symmetric stress field does not contribute to the strain-energy density \( W \), which is instead dominated by the couple stress field. It follows that the energy flux toward the crack tip (20) is bounded, even if the skew-symmetric stresses display a strong singularity. As a consequence the tractions level ahead of the crack tip increases with respect to the classical square-root singularity predicted by the LEFM theory. However, the present solution displays a negative shear stress \( \tau_{23} \) ahead of the crack tip. Therefore, the total shear stress \( \tau_{23} \) and reduced traction \( p_3 \) at \( \theta = 0 \) switch their sign from the positive classical \( K_{III} \) fields, which holds at large
radial distance, namely for \( r \gg \ell \), where \( t_{23} = p_3 = \sigma_{23} = K_{III}/(2\pi r)^{1/2} \), to the negative shear stress \( t_{23} = \tau_{23} \) and reduced traction \( p_3 = \tau_{23} + \mu_{22,3}/2 \) near to the crack tip, namely for \( r < \ell \), where

\[
\tau_{23}(r, 0) = -\frac{K_{III}\ell}{2\sqrt{2\pi r^3}} \sqrt{\frac{1 + \eta}{3 - \eta}} < 0, \quad p_3(r, 0) = -\frac{K_{III}\ell}{4\sqrt{2\pi r^3}} \sqrt{(1 + \eta)(3 - \eta)} < 0.
\]

(39)

Negative shear stress \( \tau_{23} \) and reduced traction \( p_3 \) ahead of the crack tip are peculiar of fracture process in CS materials. A similar trend is also observed for Mode III crack in CS elastic materials with \( \eta = 0 \) (Zhang et al., 1998) as well as for CS elastic–plastic behavior (Radi and Gei, 2004; Radi, 2007). This unusual aspect seems to be due to the presence of microstructures, such as compositional grains, cells, and fibers.

By using (38)\(_{1,6}\) the couple stress \( \mu_{22} \) ahead of the crack tip and the sliding displacement on the crack face near to the crack tip, namely for \( r < \ell \), turn out to be

\[
\mu_{22}(r, 0) = \frac{K_{III}\ell}{\sqrt{2\pi r}} \sqrt{\frac{1 + \eta}{3 - \eta}} (1 - \eta) > 0, \quad w(r, \pi) = \frac{8K_{III}r^{3/2}}{3G\ell\sqrt{2\pi(1 + \eta)(3 - \eta)}} > 0,
\]

(40)

respectively. The leading order term in (38)\(_1\) does not contribute to the crack tip opening displacement and thus the crack profile varies as \( r^{3/2} \) as \( r \to 0 \), so that its slope vanishes at \( r = 0 \) and the crack tip profile is sharp and not blunted like the classical \( K_{III} \) field.

4. Full-field solution

In the present section the Wiener–Hopf analytic continuation technique (Noble, 1958; Freund, 1990) is used to obtain the full-field solution for a semi-infinite crack in an infinite medium subject to remote classical \( K_{III} \) field. Only the upper half-plane \((x_2 \geq 0)\) is considered due to the skew-symmetry of the problem. Use of the Fourier transform and inverse transform is made. For the function \( w(x_1, x_2) \) they are

\[
\tilde{w}(s, x_2) = \int_{-\infty}^{\infty} w(x_1, x_2)e^{isx_1} dx_1, \quad w(x_1, x_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{w}(s, x_2)e^{-isx_1} ds,
\]

(41)

respectively, where \( s \) is a real variable. Introduction of (41)\(_2\) into the governing Eq. (10), yields the following ODE for \( \tilde{w}(s, x_2) \)

\[
\tilde{w}_{2222} - 2(s^2 + 1/\ell^2)\tilde{w}_{22} + s^2(s^2 + 2/\ell^2)\tilde{w} = 0,
\]

(42)

which admits as bounded solution in the upper half-plane

\[
\tilde{w}(s, x_2) = C(s)e^{-\sqrt{s^2 + 2/\ell^2}x_2} + D(s)e^{-|s|x_2}, \quad \text{for } x_2 \geq 0,
\]

(43)

where the functions \( C(s) \) and \( D(s) \) can be determined by the boundary conditions (12) and (13). The branch of the square root such that \( \sqrt{1} = 1 \) is chosen. Introduction of (41)\(_2\) and (43) into the boundary conditions (12)\(_2\) and (13)\(_2\), which hold on the entire line \( x_2 = 0 \), yields the following relation between \( C(s) \) and \( D(s) \)

\[
C(s) = -\frac{xs^2}{1 + xs^2}D(s),
\]

(44)

where

\[
x = (1 + \eta)\frac{\ell^2}{2},
\]

(45)

and thus the function \( \tilde{w} \) in (43) can be written as:

\[
\tilde{w}(s, x_2) = D(s) \left[ e^{-|s|x_2} - \frac{xs^2}{1 + xs^2}e^{-\sqrt{s^2 + 2/\ell^2}x_2} \right], \quad \text{for } x_2 \geq 0.
\]

(46)

The Fourier transform of the boundary condition (12)\(_1\), which only applies to \( x_1 < 0 \), yields
\[ G \left\{ \bar{w}_{,2}(s, 0) + \frac{\ell^2}{2} \left[ (2 + \eta) s^2 \bar{w}_{,2}(s, 0) - \bar{w}_{,222}(s, 0) \right] \right\} = \bar{p}_{3+}(s), \]  

(47)

where \( \bar{p}_{3+} \) is the transform of the reduced traction \( p_3 \) ahead of the crack tip, at \( x_1 > 0 \) and \( x_2 = 0 \), introduced in (11)\(_1\), namely

\[ \bar{p}_{3+}(s) = G \int_0^\infty \left\{ w_{,2}(x_1, 0) - \frac{\ell^2}{2} \left[ (2 + \eta) w_{,112}(x_1, 0) + w_{,222}(x_1, 0) \right] \right\} e^{ix_1} \, dx_1, \]  

(48)

which is analytic in the upper half complex \( s \) plane, \( \text{Im}(s) > 0 \). Similarly, the Fourier transform of the boundary condition (13)\(_1\) which applies to \( x_1 > 0 \) only, gives

\[ \bar{w}(s, 0) = \bar{w}_-(s), \]  

(49)

where

\[ \bar{w}_-(s) = \int_{-\infty}^0 w(x_1, 0)e^{ix_1} \, dx_1. \]  

(50)

is analytic in the lower half complex \( s \) plane, \( \text{Im}(s) < 0 \). Introduction of (46) in (47) and (49) gives

\[ \bar{p}_{3+}(s) = GD(s) \left[ \frac{(zs^2)^2}{1 + zs^2} (s^2 + 2/\ell^2)^{1/2} - (1 + zs^2)|s| \right], \]  

(51)

\[ \bar{w}_-(s) = \frac{D(s)}{1 + zs^2}, \]  

(52)

respectively. Elimination of \( D(s) \) from (51) and (52) yields

\[ \bar{p}_{3+}(s) = -G |s| f(s) \bar{w}_-(s), \]  

(53)

where

\[ f(s) = (1 + zs^2)^2 - zs^2(2zs^2)^{1/2}(1 + \eta + zs^2)^{1/2}. \]  

(54)

In order to apply to the Wiener–Hopf technique of analytic continuation, Eq. (53) needs to be factorized into the product of two functions analytic in the upper and lower half \( s \) planes, respectively. Note that the function \( |s| \) can be factorized as

\[ |s| = s_+^{1/2} s_-^{1/2}, \]  

(55)

where \( s_+^{1/2} \) and \( s_-^{1/2} \) have branch cuts from 0 to \( -i\infty \) and from 0 to \( i\infty \), and are therefore analytic in the upper and lower half \( s \) planes, respectively. The branch cuts are chosen such that the square-root functions are positive when \( s \) is real and positive. It can be shown that the function \( f(s) \) has only two roots in the complex \( s \) plane, namely for \( s = \pm ia/\sqrt{\alpha} \), where \( a \) is the unique real and positive, non-dimensional root of the equation (see Appendix A)

\[ (3 - \eta)a^6 - 6a^4 + 4a^2 - 1 = 0. \]  

(56)

The value of \( a \) which satisfies Eq. (56) is given in (A.4) and plotted in Fig. 2 as function of \( \eta \). Note that \( a = 1 \) for \( \eta = 0 \).

As shown in Appendix A, by following the approach proposed by Atkinson and Leppington (1977) the function \( f(s) \) can then be factorized as

\[ f(s) = \frac{3 - \eta}{2} (zs^2 + a^2) k_+(s) k_-(s), \]  

(57)

where the functions

\[ k_+(s) = e^{-R(\eta\sqrt{2})}, \quad k_-(s) = e^{-R(\eta\sqrt{2})}, \]  

(58)

are analytic in the upper and lower half complex \( s \) planes, respectively, and the function \( R \) is given by
\[ R(x) = \frac{1}{\pi} \int_0^1 \arctg \left( \frac{t^3 \sqrt{1 - r^2}}{(1 + r^2 - t^2)} \right) \frac{dt}{t + x}. \]  

(59)

By using (45), (55), and (45), after multiplying both sides of Eq. (53) by \( s \), it can be factorized as

\[ \frac{4}{(3 - \eta)(1 + \eta)G^2} \left( s + ia/\sqrt{2} \right) k_+(s) = -s^{3/2}(s - ia/\sqrt{2}) k_-(s) \hat{w}_-(s). \]  

(60)

The left and right sides of (60) are analytic functions in the upper and lower half \( s \) plane, respectively, and thus define an entire function on the \( s \) plane. The Fourier transform of the asymptotic variation of the reduced traction ahead of the crack tip (39) and sliding displacement on the crack face (40) gives \( p_+ \sim s^{1/2} \) and \( \hat{w}_- \sim s^{-5/2} \) as \( s \to \infty \), and thus both sides of (60) are bounded as \( s \to \infty \) and must equal a constant \( F \) in the entire \( s \) plane according to Liouville’s theorem, so that

\[ \hat{w}_-(s) = -\frac{F}{s^{3/2}(s - ia/\sqrt{2}) k_-(s)}, \]  

(61)

\[ \bar{p}_+(s) = F \frac{G^2}{4} (1 + \eta)(3 - \eta) \left( s + ia/\sqrt{2} \right) k_+(s) \]  

(62)

The constant \( F \) can be determined from the Fourier transform of the remotely applied classical \( K_{III} \) field, namely for \( r \to \infty \), which corresponds to \( s \to 0 \). The Fourier transform of sliding displacement and shear stress in classical \( K_{III} \) field are

\[ \hat{w}_-(s) = -\frac{1 + i}{2Gs^{3/2}} \bar{K}_{III}, \quad \bar{p}_+(s) = \frac{1 + i}{2s^{1/2}} \bar{K}_{III}, \text{ as } s \to 0. \]  

(63)

By using (58) and (A.7), it can be shown that as \( s \to 0 \) then

\[ k_+(0) = k_-(0) = \sqrt{k(0)} = \frac{1}{a} \sqrt{\frac{2}{3 - \eta}}. \]  

(64)

From (61), (62), and (64) it follows that

\[ \lim_{s \to 0} s^{3/2} \hat{w}_-(s) = -i F \frac{\ell}{2} \sqrt{(1 + \eta)(3 - \eta)}, \quad \lim_{s \to 0} s^{1/2} \bar{p}_+(s) = i F G \frac{\ell}{2} \sqrt{(1 + \eta)(3 - \eta)}. \]  

(65)

A comparison of (63) and (65) implies that
\[ F = \frac{K_{III}}{G\ell} \frac{1 - i}{\sqrt{(1 + \eta)(3 - \eta)}}, \]  

so that the introduction of (66) and (58) into (61) and (62) yields

\[
\begin{align*}
\bar{w}_-(s) &= -\frac{K_{III}}{G} \frac{1 - i}{\sqrt{2(3 - \eta)}} \frac{e^{R(\nu / \sqrt{2})}}{s^{3/2}(\sqrt{2}s - ia)}, \\
\bar{p}_{3+}(s) &= \frac{K_{III}}{4} \frac{1 - i}{\sqrt{2(3 - \eta)}} \left[ (2s)^{1/2} + \frac{ia}{s^{1/2}} \right] e^{-R(-\nu / \sqrt{2})}.
\end{align*}
\]

5. Results

Stress, couple stress, and displacement fields can be obtained from (67) and (68) by inverse Fourier transform, according to (41)\(_2\). In particular, the Fourier transforms of the symmetric, skew-symmetric, and total shear stress along the crack line ahead of the crack tip can be obtained from (7)\(_2\) and (9)\(_2\) by using (46), (52)–(54), (67), and (68), namely

\[
\begin{align*}
\bar{\sigma}_{23}(s, 0) &= \frac{K_{III}(1 - i)}{\sqrt{2(3 - \eta)}} \frac{1 + 2s^2 - \sqrt{2s^2} \sqrt{1 + \eta + 2s^2}}{s^{3/2}(\sqrt{2}s - ia)} e^{R(\nu / \sqrt{2})}, \\
\bar{\tau}_{23}(s, 0) &= \frac{K_{III}(1 - i)}{\sqrt{2(3 - \eta)}} \frac{\sqrt{2s^2} \sqrt{1 + \eta + 2s^2}}{s^{3/2}(\sqrt{2}s - ia)} e^{R(\nu / \sqrt{2})}, \\
\bar{t}_{23}(s, 0) &= \frac{K_{III}(1 - i)}{\sqrt{2(3 - \eta)}} \frac{1 + 2s^2}{s^{3/2}(\sqrt{2}s - ia)} e^{R(\nu / \sqrt{2})}.
\end{align*}
\]

The integration path in the inverse Fourier transform may be transformed in the lower half \( s \) plane to two straight lines on the two sides of the negative imaginary axis, where \( s = -iy \), plus two half circles centered at the origin, one with radius approaching infinity and the other with a vanishing small radius. The integration path on both large and small half circles then gives vanishing small contribution, whereas the integration over the two straight lines, after the substitution \( y = \sqrt{2t}/\ell \), yields

\[
\begin{align*}
\sigma_{23}(x_1, 0) &= -\frac{K_{III} \sqrt{2}}{\pi \sqrt{(3 - \eta)\ell}} \int_0^\infty \frac{1 - (1 + \eta) \left[ t^2 - t^4 \sqrt{2t^2 - 1} H(t - 1) \right]}{(a + t\sqrt{1 + \eta})^2} e^{R(t) - 2\sqrt{2}x_1/t} \, dt, \\
\tau_{23}(x_1, 0) &= -\frac{K_{III} \sqrt{2}(1 + \eta)}{\pi \sqrt{(3 - \eta)\ell}} \int_0^\infty \frac{\sqrt{t} t^2 - 1 H(t - 1)}{a + t\sqrt{1 + \eta}} e^{R(t) - 2\sqrt{2}x_1/t} \, dt, \\
t_{23}(x_1, 0) &= \frac{K_{III} \sqrt{2}}{\pi \sqrt{(3 - \eta)\ell}} \int_0^\infty \frac{1 - t^2(1 + \eta)}{(a + t\sqrt{1 + \eta})^2} e^{R(t) - 2\sqrt{2}x_1/t} \, dt,
\end{align*}
\]

for \( x_1 > 0 \), where \( H \) denotes the Heaviside step function. Similarly, the reduced traction ahead of the crack tip, for \( x_1 > 0 \), can be obtained from the inverse Fourier transform of (68) by using (A.17), namely

\[
\begin{align*}
p_3(x_1, 0) &= -\frac{K_{III} \sqrt{2}(1 + \eta)^2}{\pi \sqrt{(3 - \eta)\ell}} \int_0^\infty \frac{(1/\eta - t^2)^2 - t^2 \sqrt{2t^2 - 1} H(t - 1)}{(a + t\sqrt{1 + \eta})^2} e^{R(t) - 2\sqrt{2}x_1/t} \, dt.
\end{align*}
\]

The normalized shear stress ahead of the crack tip at \( x_2 = 0 \), namely \( t_{23} (2\pi\ell)^{1/2}/K_{III} \), versus normalized distance \( x_1/\ell \) is shown in Fig. 3a, together with the shear stress traction in the classical \( K_{III} \) field. The full-field solution displays a smooth transition from the positive remote classical \( K_{III} \) field to the negative near-tip asymptotic field in (38)\(_{10}\), as \( x_1 \) decreases and becomes smaller than the characteristic length in torsion \( \ell_t \). Strain-gradient effects are observed up to a distance of 5\( \ell_t \) to the crack tip. For larger distance to the crack tip the full-field solution essentially coincides with the classical \( K_{III} \) field.
In Fig. 3b the shear stress is normalized with respect to \( t \) instead of \( t_2 \), namely \( t_2(2\pi t)^{1/2}/K_{III} \). In this case, the normalized shear stress displays a small variation with the ratio \( \eta \) and the maximum value is attained at \( x_1 \approx \ell_1 \) for every value of \( \eta \).

As the ratio \( \eta \) and the characteristic length in torsion \( \ell_t \) tend to \(-1\) and 0, respectively (Fig. 1), then the total shear stress \( t_2 \) approaches the classical \( K_{III} \) field from below and it switches to negative values very small distance to the crack tip, denoted by \( x_1^0 \). The variations of the ratios \( x_1^0/\ell_1 \) and \( x_1^0/\ell_t \) with \( \eta \) are plotted in Fig. 4a and b (dash-dotted lines). It can be observed that the zone ahead of the crack tip with negative shear stress extends up to a distance of \( 0.3 \pm 0.4/\ell_t \) to the crack tip, and thus it significantly reduces in size and tends to vanish as \( \ell_t \to 0 \).

The near-tip asymptotic fields (38) approximately dominate within the zone where the shear stress has the negative sign. Since the size of this zone is very small, it can be considered of no physical importance and the asymptotic solution may also be considered physically irrelevant. Outside of this zone the total shear stress exhibits a maximum that is bounded and positive for \(-1 < \eta < 1\). The maximum is attained at a distance \( x_1^{max} \) to the crack tip, which satisfies the condition \( t_{23,1}(x_1^{max}, 0) = 0 \), namely from (70)3

\[
\int_0^\infty \frac{1 - t^2(1 + \eta)}{a + t\sqrt{1 + \eta}} \sqrt{t} e^{t(\eta - \sqrt{2}x_1^{max}/\ell)} \, dt = 0.
\]  

(72)

Fig. 4. Variations with the ratio \( \eta \) of the critical stress intensity factor \( K_{INC} \), maximum shear stress \( t_{23}^{max} \) and its location \( x_1^{max} \) and size \( x_1^0 \) of the zone with negative shear stress ahead of the crack tip.
The variations of the non-dimensional ratios $x_1^{max}/\ell$ and $x_3^{max}/\ell$, with $\eta$ are plotted in Fig. 4a and b (dashed lines). It can be observed that the maximum shear stress occurs at distance $1.0 \div 1.1 \ell$, to the crack tip, and it is finite for $\ell > 0$. The occurrence of a maximum positive value for the total shear stress ahead of the crack tip, $t_3^{max} = t_{33}(x_3^{max}, 0)$, allows formulating a simple fracture criterion by assuming a critical shear stress level $\tau_C$ at which the crack may start propagating. The corresponding fracture criterion may thus be written as

$$
\tau_3^{max} = \tau_C,
$$

where the cohesive shear stress $\tau_C$ of the material can be estimated, e.g., by employing the Griffith’s definition of the surface energy (Eringen, 1979).

The variation of the maximum shear stress $t_3^{max}$ with $\eta$ is plotted in Fig. 4a and b (dotted lines) by adopting two different normalization options. Note that $t_3^{max}$ becomes unbounded as $\ell \to 0$ and $\eta \to -1$, in agreement with the classical $K_{III}$ field. As $\ell$ and $\eta$ increase then $t_3^{max}$ decreases, as already observed in Fig. 3. According to the considered fracture criterion (73), a critical stress intensity factor $K_{III}$ can be defined by using (70), namely

$$
K_{III} = \tau_C \frac{\pi \sqrt{(3-\eta)/\ell}}{\sqrt{2}} \int_0^\infty \frac{1 - t^2(1+\eta)}{(a + t\sqrt{1+\eta})/\ell} e^{R(t) - \sqrt{2\eta^{max}t}} dt.
$$

The variations of the normalized values of $K_{III}$ with $\eta$ are plotted in Fig. 4a and b (solid lines). It can be noted that the fracture toughness explicitly depends on the microstructure through the parameters $\ell$, and $\eta$ and, in particular, it increases with $\ell$ and $\eta$. It follows that materials with larger characteristic length in torsion are expected to exhibit higher fracture toughness under Mode III loading conditions, in agreement with the investigations of Maiti et al. (1984), Tsangarakis (1984), Lakes (1993), Chen et al. (1998), and Tong et al. (2005), which also found that the presence of microstructures increases the fracture toughness of the material. Moreover, as $\ell \to 0$ then the considered fracture criterion (73) yields a vanishing small $K_{III}$. In this case indeed the maximum shear stress is unbounded.

Besides the total shear stress plotted in Fig. 3, the reduced shear traction $p_3$ ahead of the crack tip is plotted in Fig. 5. The latter stress is more appropriate from the point of view of variational considerations. By using (11) and (8) it can be noted that $p_3$ coincides with $t_{33}$ as $\eta$ tends to $-1$ since in this case $\mu_{22} = 0$, whereas $p_3$ is smaller than $t_{33}$ for $\eta > -1$ since $\mu_{22,1} < 0$ ahead of the crack tip.

The asymptotic analysis performed in Section 3 defines the crack tip fields except for the constant $A$, which can be now evaluated by using the result (70) for $x_3 = 0$ and (38). It follows that

$$
A = \frac{K_{III}/2}{G\pi \sqrt{(3-\eta)/\ell}} \int_0^\infty \frac{1 - (1+\eta)}{(a + t\sqrt{1+\eta})/\ell} e^{R(t) - \sqrt{2\eta^{max}t}} dt.
$$

Fig. 5. Variation of reduced shear traction $p_3$ along the $x_1$ axis, ahead of the crack tip.
The variation of $A$ with $\eta$ is plotted in Fig. 6. In particular, it approaches infinity as $\eta$ tends to $-1$. Since $\sigma_{23} = A G$ at the crack tip, then $\sigma_{23}$ is finite and positive therein, whereas it becomes unbounded for $\eta = -1$.

The variations of the normalized symmetric and skew-symmetric shear stresses ahead of the crack tip at $x_2 = 0$, namely $\sigma_{23} (2\pi\ell)^{1/2}/K_{III}$ and $\tau_{23} (2\pi\ell)^{1/2}/K_{III}$, versus normalized distance $x_1/\ell$ are presented in Fig. 7a and b, respectively, together with the classical $K_{III}$ field and the corresponding two terms asymptotic fields (38)5,10, where the constant $A$ has been determined in (75). It can be seen that $\sigma_{23}$ is positive and finite at the crack tip, where it attains a maximum. Moreover, the symmetric stress field $\sigma_{23}$ tends to the singular classical $K_{III}$ field as $\eta$ approaches the limit value $-1$. The skew-symmetric stress field $\tau_{23}$ turns out to be negative and unbounded at the crack tip. The corresponding asymptotic solutions for the symmetric and skew-symmetric stress fields coincide with the results of the full-field analysis for very small values of $x_1$ only. In particular, the radius of validity of the asymptotic solutions becomes vanishing small as $\eta$ approaches $-1$. In this limiting case, it can be seen in Fig. 7a that the two terms asymptotic solution predicts a negative $\sigma_{23}$ at small distance to the crack tip, which is not confirmed by the full-field solution.

The couple stress field $\mu_{22}$ can be obtained from the derivative of the symmetric shear stress component $\sigma_{23}$ by using (8)1 and (7)2, namely

$$\mu_{22} = -\ell^2 (1 + \eta) \sigma_{23,1}.$$  \hfill (76)

Fig. 6. Variation of the constant $A = \sigma_{23}(0,0)/G$ with the ratio $\eta$.

Fig. 7. Variation of symmetric (a) and skew-symmetric (b) shear stresses $\sigma_{23}$ and $\tau_{23}$ along the $x_1$ axis ahead of the crack tip and corresponding two terms asymptotic fields (thin solid lines).
Then, by using \((70)\), Eq. \((76)\) yields the following variation of the couple stress \(\mu_{22}\) ahead of the crack tip

\[
\mu_{22}(x_1, 0) = K_{III} \sqrt{\ell} \frac{2^{3/4}(1 + \eta)}{\pi \sqrt{3 - \eta}} \int_0^\infty \frac{1 - (1 + \eta)[t^2 - t \sqrt{t^2 - 1} H(t - 1)]}{a + t \sqrt{1 + \eta}} \sqrt{t} e^{\delta(t - \sqrt{3}x_1/t)} dt, \quad \text{for} \ x_1 > 0.
\]

(77)

The normalized couple stress ahead of the crack tip at \(x_2 = 0\), namely \(\mu_{22} (2\pi/\ell)^{1/2}/K_{III}\), versus normalized distance \(x_1/\ell\) is shown in Fig. 8, for four distinct values of \(\eta\). As \(x_1\) increases, the couple stress \(\mu_{22}\) decreases, and vanishes for \(x_1 \gg \ell\), in agreement with the classical \(K_{III}\) field. The decrease is very rapid for negative values of the ratio \(\eta\). For example, for \(\eta = -0.9\) then \(\mu_{22}\) vanishes almost completely at \(x_1 = 5\ell\), whereas for positive values of \(\eta\) the effects of couple stress and rotational gradients are felt at larger distance to the crack tip. In fact, \(\mu_{22}\) first displays a rapid decrease for \(x_1 \ll \ell\), then it moderately increases and attains a relative maximum at \(x_\eta \approx \ell, \eta = 2\ell\). After that, it gradually tends to vanish for \(x_1 \gg \ell\). The reason is unclear, but it seems to be due to the effect of the microstructures.

The sliding displacement \(w\) behind the crack tip can be obtained from the inverse Fourier transform of Eq. \((67)\). However, in order to ensure that the inverse Fourier transform converges the function \(w_{s1}\) is first obtained from the inverse Fourier transform of \(-isw\), namely

\[
w_{s1}(x_1, 0) = \frac{K_{III}}{G} \frac{1 + i1}{\sqrt{2(3 - \eta)}} \frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{p[(\sqrt{2}\sqrt{s}) - is_1]} \text{ds}}{\sqrt{s}(\sqrt{2s} - ia)} \text{ds}, \quad \text{for} \ x_1 < 0.
\]

(78)

A conversion of the integration path to the positive imaginary \(s\) axis, where \(s = iy\) with \(y \geq 0\), yields

\[
w_{s1}(x_1, 0) = \frac{K_{III}}{G} \frac{1}{\pi \sqrt{3 - \eta}} \int_0^\infty \frac{e^{p[(-y\sqrt{2}) + iy_1]} \text{dy}}{\sqrt{y}(\sqrt{2y} - a)} \text{dy}, \quad \text{for} \ x_1 < 0.
\]

(79)

Then, by integrating \(w_{s1}\) with respect to \(x_1\), imposing the condition \(w = 0\) at \(x_1 = 0\) and substituting for \(y = \sqrt{2t}/\ell\), the following expression for the displacement on the crack surface can be obtained

\[
w(x_1, 0) = \frac{K_{III}}{\sqrt{2G}} \frac{\sqrt{\ell}}{\pi \sqrt{3 - \eta}} \int_0^\infty \frac{(e^{2x_1/\ell} - 1)e^{\delta(-t)}}{t\sqrt{t(\sqrt{1 + \eta} - a)}} \text{dt}, \quad \text{for} \ x_1 < 0.
\]

(80)

Fig. 8. Variation of normalized couple stress \(\mu_{22}\) along the \(x_1\) axis, ahead of the crack tip.
Finally, the introduction of Eq. (A.19) for $e^{R(i)}$ in the sliding displacement (80) yields

$$w(x_1, 0) = \frac{K_{III} \sqrt{\ell(3 - \eta)}}{2^{3/4} \pi G} \int_0^\infty \left( \frac{e^{2\eta/\ell} - 1}{t^2} \right) \left( \frac{1}{t^3} - t^2 \right) + t^3 \sqrt{t^2 - 1} H(t - 1) \right) e^{-R(i)} dt,$$  
where the Cauchy principal value of the integral must be considered, since the integrand function is singular at $t = a/\sqrt{1 + \eta}$.

The normalized sliding displacement on the crack face, $w G (\pi/2\ell)^{1/2}/K_{III}$ at $x_2 = 0$, versus normalized distance $r/\ell$ to the crack tip is shown in Fig. 9a and plotted in logarithmic scales in Fig. 9b. As well known, the crack tip profile is blunted for the classical $K_{III}$ displacement field (dotted lines)

$$w(r, \theta) = \frac{K_{III}}{G} \sqrt{\frac{2r}{\pi}} \sin \frac{\theta}{2},$$

whereas it evidently becomes sharp for CS elastic materials (Fig. 9a), namely it form a cusp with zero enclosed angle and zero first derivative of the displacement at the crack tip, in agreement with the atomistically sharp crack tip experimentally observed at niobium/sapphire interface for cleavage fracture by Elssner et al. (1994). The magnitude of the sliding displacement between the crack faces remarkably decreases as $\eta$ increases from $-1$ to 1, indicating that the crack becomes stiffer in comparison with the classical elastic response, recovered for $\eta \to -1$. In fact, the strain-energy density $W$ for CS materials in (18) increases with respect to classical elasticity due to the contribution of the strain rotational gradients, thus resulting in a stiffer material. This occurrence confirms that the microstructure may shield the crack tip from fracture, as already observed by Georgiadis (2003) for SG elastic behavior, by Tong et al. (2005) and Wei and Xu (2005) for Mode I and Mode II fracture and by Radi and Gei (2004) and Radi (2007) for Mode III ductile crack propagation in materials with microstructures.

The thin solid lines in Fig. 9a, correspond to the two terms asymptotic solution (38), evaluated at $\theta = 0$ for the three considered values of $\eta$. The asymptotic fields approach the full-field solution just for $x_1 \ll \ell$. Moreover, the radius of validity of the asymptotic solution increases for large values of $\eta$ and $\ell$. Also the thin straight lines with slopes 3/2 in Fig. 9b, correspond to the near-tip asymptotic field (38), and the dotted line with slope 1/2 corresponds to the classical solution of LEFM under $K_{III}$ remote field (82). From Fig. 9b it can be observed that the full-field solution displays a smooth transition between these two fields. Note that the leading order term in (38), which is linear in $r$, does not contribute to the crack tip opening displacement at $\theta = \pi$.

It must be also remarked that the total shear stress (70)3 and the couple stress (77) ahead of the crack tip and the sliding displacement on the crack face (81) recover the corresponding expressions given by Zhang et al. (1998) as $\eta$ tends to vanish and thus $a$ tends to 1.

![Fig. 9. (a) Variation of crack face sliding displacement $w$ along the crack face; (b) same variation plotted in logarithmic scales. The sliding displacements in the near-tip asymptotic field (with slope 3/2) and in the classical $K_{III}$ field (with slope 1/2) are shown for comparison.](image-url)
The out-of-plane displacement \( w \) ahead of the crack tip for \( x_1 > 0 \) can be obtained from the inverse Fourier transform of Eq. (46), by using (52) and (67). A conversion of the integration path to the negative imaginary \( s \) axis, where \( s = -iy \) with \( y \geq 0 \), and the further substitution \( y = \sqrt{2t/\ell} \) yield the following expression for the displacement \( w \)

\[
w(x_1, x_2) = \frac{K_{III} \sqrt{\ell}}{2\pi G \sqrt{3 - \eta}} \int_0^\infty \left\{ (1 + \eta) \sqrt{t} \left[ H(t - 1) \sin \left( \sqrt{2(t - 1)} \frac{x_2}{\ell} \right) - \sin \left( \sqrt{2tx_2/\ell} \right) \right] + \frac{\sin \left( \sqrt{2tx_2/\ell} \right)}{t^{\eta/2}} \right\} a + t \sqrt{1 + \eta} \, dt,
\]

which hold for \( x_1 > 0 \). Contour plots of the normalized displacement ahead of the crack tip, namely \( wG(\pi/2\ell)\sqrt{K_{III}} \), are drawn in Fig. 10a and b for \( x_1 \) and \( x_2 \) ranging between 0 and 2\( \ell \), for \( \eta = -0.9 \) and \( \eta = 0.9 \), respectively. The displacement turns out to be everywhere positive within this region, although the asymptotic analysis performed by Zhang et al. (1998) provided a large angular sector undergoing negative displacement ahead of the crack tip. These authors indeed neglected the lowest order term in their asymptotic investigations, so that they found a negative displacement due to the contribution of the sole second order term in (38.1).

5.1. Angular variation of the asymptotic analysis

The angular variation of the out-of-plane displacement \( w \) at distance 0.1\( \ell \) and 0.4\( \ell \) to the crack tip, predicted by the two term asymptotic solution (38.1) (solid lines) and by the full-field analysis (83) (dashed lines for \( 0 < \theta < 90^\circ \)) are plotted in Fig. 11. The angular variation corresponding to the classical \( K_{III} \) displacement field (82) is also plotted (dotted lines). As \( \eta \) decreases both the full-field and asymptotic solutions approach the classical \( K_{III} \) displacement field, in particular at large distance to the crack tip. Near to the crack tip the asymptotic solution almost coincides with the result of the full-field analysis, especially for large positive values of \( \eta \), whereas the accuracy of the asymptotic approximation decreases at larger distance, as well as \( \eta \rightarrow -1 \). In this case indeed the two terms asymptotic solution predicts a region ahead of the crack tip undergoing negative out-of-plane displacement, sufficiently far from the crack tip, so that the negative contribution of the second order term in (38.1) prevails over the positive leading order term. This region is clearly observable in Fig. 11b for \( \eta = -0.9 \), where a large sector with negative displacement is predicted by the asymptotic solution (38.1) at distance 0.4\( \ell \) to the crack tip, whereas it disappears at distance 0.1\( \ell \) (Fig. 11a). However, the full-field solution plotted in Fig. 10 does not provide negative displacement, thus denoting that the higher order terms neglected in the two terms asymptotic solution, but included in the full-field analysis, essentially give a positive contribution to the displacement ahead of the crack tip.

![Fig. 10. Contour plots of the normalized out-of-plane displacement \( wG(\pi/2\ell)\sqrt{K_{III}} \) ahead of the crack tip, for (a) \( \eta = -0.9 \) and (b) \( \eta = 0.9 \).](image-url)
The negative contribution to the out-of-plane displacement provided by the second order term plotted in Fig. 12a is related to a significant rotation of the material particles currently at the tip of the crack, much more accentuated for negative values of the ratio \( \eta \), as it can be observed from Figs. 12b and 13, where the asymptotic rotation fields are plotted for three different values of \( \eta \).

The normalized angular distributions of the cylindrical components of the asymptotic symmetric stress field are plotted in Fig. 14 for three different values of \( \eta \), together with the corresponding angular variation for the classical \( K_{III} \) displacement field. Note that the symmetric shear stress traction \( \sigma_{03} \) is finite on the crack surfaces at \( \theta = \pi \), whereas the reduced shear traction \( p_3 \) is vanishing small as required by the boundary condition (11). Moreover, ahead of the crack tip \( \sigma_{03} \) is negative at \( r = 0.4\ell \) for \( \eta = -0.9 \), in agreement with the result plotted in Fig. 7a.

The angular variations of the normalized skew-symmetric stress components plotted in Fig. 15 agree with the inversion in the sign of the second order term in the asymptotic displacement field (38), occurring at small \( \theta \). Indeed, the singular shear stress \( \tau_{03} \) is negative ahead of the crack tip, unlike the classical LEFM Mode III solution. The switch in the shear direction has been also noted by Zhang et al. (1998) for \( \eta = 0 \), by Georgiadis

![Fig. 11. Normalized angular variation of the out-of-plane displacement field \( w \) at distance (a) 0.1\( \ell \) and (b) 0.4\( \ell \) to the crack tip, predicted by the two term asymptotic solution (solid lines) and by the full-field analysis for \( 0 < \theta < 90^\circ \) (dashed lines). The angular variation for the classical \( K_{III} \) displacement field is also plotted (dotted lines).](image)

![Fig. 12. Normalized angular variations of (a) second order term of the out-of-plane displacement \( w \) and (b) rotation \( \phi_2 \) asymptotic fields.](image)
Fig. 13. Normalized angular variation of the rotation field $\varphi_1$ at distance (a) 0.1\(\ell\) and (b) 0.4\(\ell\) to the crack tip, predicted by the two term asymptotic solution.

Fig. 14. Normalized angular variation of symmetric stresses $\sigma_{33}$ and $\sigma_{66}$ predicted by the two term asymptotic solution at distance 0.1\(\ell\) (a and c) and 0.4\(\ell\) (b and d) to the crack tip. The angular variation for the classical $K_{III}$ stress field is also plotted (dotted lines).
(2003) for SG elasticity and by Radi and Gei (2004) and Radi (2007) for crack propagation in elastic–plastic CS materials. Moreover, the magnitude of the skew-symmetric stress components decreases as the characteristic lengths ratio $\eta$ decreases from 1 to $-1$.

**Fig. 15.** Normalized angular variation of skew-symmetric stress asymptotic fields.

**Fig. 16.** Normalized angular variation of couple stress asymptotic fields.
The normalized angular variations of the cylindrical components of the asymptotic couple stress field are plotted in Fig. 16 for different values of $\eta$. Note that the component $\mu_{lr}$ vanishes on the crack surfaces at $\theta = \pi$ as required by the boundary condition (12)$_2$.

6. Conclusions

The structure of the asymptotic fields near a Mode III crack tip in CS elastic materials with finite characteristic lengths in bending and torsion has been investigated in the present work. Moreover, the full-field solution for a semi-infinite crack subject to remote classical $K_{III}$ field has been obtained by using Fourier transforms and Wiener–Hopf technique (Noble, 1958).

The performed asymptotic analysis shows that the angular distribution of the crack tip fields results as strongly influenced by the ratio $\eta$ between characteristic lengths. Moreover, it predicts a substantial increase in the singularities of the skew-symmetric stress and couple stress fields, whereas the symmetric stress field turns out to be finite at crack tip. In particular, ahead of the crack tip within a zone smaller than the characteristic lengths, the total shear stress and the reduced tractions switch their sign with respect to the classical LEFM solution, due to the relative rotation of the particles currently at the crack tip.

The obtained full-field solution displays a continuous transition from the classical $K_{III}$ fields, which hold at a distance to the crack tip larger than $5\ell_t$, to the asymptotic fields (38), whose zone of dominance significantly reduces for small characteristic length in torsion $\ell_t$. In particular, the size of this zone tends to vanish as $\eta$ approaches the limit value $-1$, which corresponds to a vanishing small $\ell_t$. In this case indeed the asymptotic crack tip fields (38) have no domain of physical validity because the total shear stress at the crack tip becomes unbounded and thus recovers the classical $K_{III}$ field of linear elasticity, which displays square-root singularity. Outside this extremely small zone, which actually has no physical relevance also for $\ell_t > 0$, the shear stress distribution exhibits a local bounded maximum at a distance to the crack tip comparable to the characteristic length in torsion $\ell_t$. The magnitude of the maximum positive shear stress can be adopted as a measure of the critical stress level for crack initiation. The crack will initiate propagating when the maximum shear stress reaches the cohesive shear stress of the material, which may be estimated by employing the Grittith’s definition of the surface energy (Eringen, 1979). According to the considered fracture criterion based on the maximum shear stress hypothesis, the Mode III fracture toughness of the material is shown to increases with the characteristic length in torsion.

The occurrence of a sharp crack profile also denotes that the crack becomes stiffer with respect to the classical elastic response, thus revealing that the presence of microstructures may shield the crack tip from fracture. Shielding can reasonably be expected since the contribution of strain gradients to the strain-energy density increases the stiffness of the material. The present investigation thus confirms that the microstructure remarkably affects the solutions of fracture mechanics problems.

The use of the CS theory of elasticity developed by Koiter (1964) for the analysis of the stress and couple stress fields near the tip of a Mode III crack gives accurate predictions on the tractions level ahead of the crack tip occurring at small distance to it, but larger than $0.5\ell_t$, and improves the classical solution at distances to the crack tip up to $5\ell_t$, where the classical $K_{III}$ field is then recovered. Moreover, it sheds some light on the shielding mechanisms against fracture originating from the presence of microstructures (compositional grains, cells, fibers).

As a conclusion, the present approach provides a means to link scales in fracture mechanics, namely from atomistic through microscale to macroscopic fracture, which results as basic for the comprehension of the detailed mechanisms by which fracture may occur in brittle materials with complex microstructure, up to the micron scale. Moreover, the analytical results here presented can be used as a benchmark for successive numerical investigations of fracture problems in materials with microstructure.

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Appendix A

In order to factorize the function \( f(s) \) defined in (54), let us introduce the function \( F(s) = f(s/\sqrt{a}) \). According to the Wiener–Hopf technique, \( F(s) \) can be conveniently written in the form

\[
F(s) = (1 + s^2)^2 - s^2(s^2)^{1/2}(1 + \eta + s^2)^{1/2}.
\] (A.1)

According to the argument principle (Freund, 1990), it can be checked that the number of roots of \( F(s) = 0 \) in the entire complex plane is two. In particular, the equation \( F(s) = 0 \) is satisfied by \( s = \pm ia \), where \( a \) is real and positive, and depends on the parameter \( \eta \) according to the following relation:

\[
(1 - a^2)^2 - a^2\sqrt{a^2 - 1} - \eta = 0,
\] (A.2)

and thus \( a \) is the only real and positive root of the Eq. (56), namely

\[
a = \sqrt{\frac{1}{3 - \eta} \left( 2 + \frac{2^{1/3}\eta}{b} + \frac{b}{2^{1/3}} \right)},
\] (A.3)

where

\[
b = 3^{1/2} (1 + \eta^2 - (1 - \eta/3)\sqrt{1 + \frac{14}{3}\eta + 9\eta^2}.
\] (A.4)

The principle cube root is considered in (A.4), such that the cube root of a negative number is a negative number. Rather than factorize the function \( F(s) \), which has two roots in the complex plane, let us define the function

\[
K(s) = \frac{2}{3 - \eta} \frac{F(s)}{s^2 + a^2},
\] (A.5)

which has no roots and behaves at zero and infinite as

\[
K(0) = \frac{2}{a^2(3 - \eta)}, \quad \lim_{s \to \infty} K(s) = 1,
\] (A.6)

respectively. In order to perform the factorization \( K(s) = K_+ (s) K_- (s) \) it is convenient to consider

\[
(s^2)^{1/2} = \lim_{\varepsilon \to 0} (s^2 + \varepsilon^2)^{1/2},
\] (A.7)

so that the function \( K(s) \) becomes

\[
K(s) = \lim_{\varepsilon \to 0} \frac{2}{3 - \eta} \frac{(1 + s^2)^2 - s^2(s^2)^{1/2}(1 + \eta + s^2)^{1/2}}{s^2 + a^2},
\] (A.8)

with branch cuts on the imaginary \( s \) axis, namely from \(-ie \) to \(-i\infty \) and from \( ie \) to \( i\infty \). Following the standard factorization procedure (Noble, 1958) it is possible to write the functions \( K_+ (s) \) and \( K_- (s) \) in the form:

\[
K_{\pm}(s) = \exp \left[ \pm \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln K(z)}{z - s} \, dz \right].
\] (A.9)

By using Cauchy theorem the integration path for \( K_{\pm}(s) \) can be transformed in the upper half \( s \) plane to two straight lines on the two sides of the positive imaginary axis plus an half circle centered at the origin with radius \( R \) approaching infinity plus a vanishing small half circle with radius \( \rho \) centered at \( ie \) (see Fig. A.1). Both contributions from the half circles with radii \( R \) and \( \rho \) tend to vanish as \( R \to \infty \) and \( \rho \to 0 \), respectively. It follows that

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln K(z)}{z - s} \, dz = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\sqrt{1 + \eta}} \operatorname{arctg} \left[ \frac{\Im K(iy)}{\Re K(iy)} \right] \, dy.
\] (A.10)
Since from (A.8)
\[ K(iy) = \frac{2}{3-\eta} \frac{(1-y^2)^2 + iy^3 \sqrt{1+\eta-y^2}}{a^2 - y^2}, \tag{A.11} \]
for \( y > 0 \), then from (A.10) and (A.11) it follows:
\[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln K(z)}{z-s} \, dz = \frac{1}{\pi} \int_{0}^{1} \arctg \left[ \frac{t^2 \sqrt{1-t^2}}{\left( \frac{1}{1+\eta} - t^2 \right)^2} \right] \frac{dt}{t + \frac{\mu}{\sqrt{1+\eta}}} \tag{A.12} \]
Therefore, from (A.9) and (A.12) one obtains
\[ K_-(s) = e^{-R(\mu/\sqrt{1+\eta})}, \tag{A.13} \]
where \( R(x) \) has been defined in (59). Similarly, one may obtain
\[ K_+(s) = e^{-R(-\mu/\sqrt{1+\eta})}. \tag{A.14} \]
Note that the pole \( i \) and the branch points \( i \) and \( i \sqrt{1+\eta} \) do not give a contribution to the integral along the imaginary axis because the integral along a circle with radius \( \rho \to 0 \) centered in each of these points is vanishing, since \( \rho \) in \( \rho \to 0 \) as \( \rho \to 0 \).
From A.5, A.13 and A.14 it follows that
\[ f(s) = F(\sqrt{zs}) = \frac{3-\eta}{2} (zs^2 + a^2) k_+(s) k_-(s), \tag{A.15} \]
where
\[ k_\pm(s) = K_\pm(\sqrt{zs}) = e^{-R(\pm\mu/\sqrt{2})}. \tag{A.16} \]
Eqs. (A.15) and (A.16) imply that
\[ e^{R(-\mu/\sqrt{2})} e^{R(\mu/\sqrt{2})} = \frac{3-\eta}{2} \frac{zs^2 + a^2}{f(s)}. \tag{A.17} \]
The change of variable \( t = is\ell/\sqrt{2} \) and the introduction of (A.1) in (A.17) then yield
\[ e^{R(-\ell)} e^{R(\ell)} = \frac{3-\eta}{2(1+\eta)} \frac{t^2 - t^2}{\left( \frac{1}{1+\eta} - t^2 \right)^2 - t^2 \sqrt{t^2 \sqrt{t^2} - 1}} \]
\[ = \frac{3-\eta}{2} \frac{t^2 - \frac{a^2}{1+\eta}}{\left( 3-\eta \right) t^6 - \frac{6}{(1+\eta)} t^4 + \frac{4}{(1+\eta)^2} t^2 - \frac{1}{(1+\eta)^3}}. \tag{A.18} \]
Finally, by using (56) Eq. (A.18) may be written as:
\[ \mathcal{E}^{(i-j)} = \frac{3 - \eta}{2} \left( \frac{1}{1 + \eta - i^2} \right)^2 + \frac{i^2 \sqrt{i^2 - 1}}{(3 - \eta) i^4} + \frac{1}{\sigma (1 + \eta)} \mathcal{E}^{-\delta(i)} \]  

(A.19)

References


