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Fuzzy Mappings and Fixed Point Theorem

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1. INTRODUCTION

In this paper, our purpose is twofold. First, we introduce a concept of a fuzzy mapping, i.e., mapping from an arbitrary set to one subfamily of fuzzy sets in a metric linear space X. Each element of this family is interpreted as an approximate quantity. We also introduce a notion of distance between such quantities and give some properties of it.

Then we prove the fixed point theorem for fuzzy mappings. This theorem is a generalization of the fixed point theorem for point-to-set maps [1, 2] arising from the set-representation of fuzzy sets [3].

2. FUZZY MAPPINGS

Let X be any metric linear space and d be any metric in X. A fuzzy set in X is a function with domain X and values in [0, 1]. If A is a fuzzy set and $x \in X$, the function-value $A(x)$ is called the grade of membership of x in A. The collection of all fuzzy sets in X is denoted by $\mathscr{F}(X)$.

Let $A \in \mathscr{F}(X)$ and $\alpha \in [0, 1]$. The α-level set of A, denoted A_α, is defined by

$$A_\alpha = \{x : A(x) \geqslant \alpha\} \qquad \text{if} \quad \alpha \in (0, 1],$$
$$A_0 = \overline{\{x : A(x) > 0\}},$$

whenever $\overline{B}$ is the closure of set (nonfuzzy) B.

Now we distinguish from the collection $\mathscr{F}(X)$ a subcollection of approximate quantities, denoted $\mathscr{W}(X)$.

DEFINITION 2.1. A fuzzy subset A of X is an approximate quantity iff its α-level set is a compact convex subset (nonfuzzy) of X for each $\alpha \in [0, 1]$, and $\sup_{x \in X} A(x) = 1$.

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When $A \in \mathscr{W}(X)$ and $A(x_0) = 1$ for some $x_0 \in X$, we will identify A with an approximation of x_0.

Then we shall define a distance between two approximate quantities.

DEFINITION 2.2. Let $A, B \in \mathscr{W}(X)$, $\alpha \in [0, 1]$. Define

$$p_\alpha(A, B) = \inf_{x \in A_\alpha, y \in B_\alpha} d(x, y),$$
$$D_\alpha(A, B) = \mathrm{dist}(A_\alpha, B_\alpha),$$
$$D(A, B) = \sup_\alpha D_\alpha(A, B),$$

whenever dist is Hausdorf distance. The function p_α is called a α-space, D_α a α-distance, and D a distance between A and B.

It is easy to see that p_α is nondecreasing function of α.

We shall also define an order on the family $\mathscr{W}(X)$, which characterizes accuracy of a given quantity.

DEFINITION 2.3. Let $A, B \in \mathscr{W}(X)$. An approximate quantity A is more accurate than B, denoted $A \subset B$, iff $A(x) \leqslant B(x)$, for each $x \in X$.

It is easy to see that relation $\subset$ is a partial order determined on the family $\mathscr{W}(X)$.

Now we introduce a notion of fuzzy mapping, i.e., a mapping with value in the family of approximate quantities.

DEFINITION 2.4. Let X be an arbitrary set and Y any metric linear space. F is called a fuzzy maping iff F is mapping from the set X into $\mathscr{W}(Y)$, i.e., $F(x) \in \mathscr{W}(Y)$ for each $x \in X$.

A fuzzy mapping F is a fuzzy subset on $X \times Y$ with membership function $F(x, y)$. The function-value $F(x, y)$ is the grade of membership of y in $F(x)$.

Let $A \in \mathscr{F}(X)$, $B \in \mathscr{F}(Y)$. The fuzzy set $F(A)$ in $\mathscr{F}(Y)$ is defined by

$$F(A)(y) = \sup_{x \in X} (F(x, y) \wedge A(y)), \qquad y \in Y,$$

and the fuzzy set $F^{-1}(B)$ in $\mathscr{F}(X)$ is defined by

$$F^{-1}(B)(x) = \sup_{y \in Y} (F(x, y) \wedge B(y)), \qquad x \in X.$$

3. FIXED POINT THEOREM

First of all, we shall give here the basic properties of α-space and α-distance between some approximate quantities.

LEMMA 3.1. *Let $x \in X$, $A \in \mathscr{W}(X)$, and $\{x\}$ be a fuzzy set with membership function equal a characteristic function of set $\{x\}$. If $\{x\} \subset A$, then $p_\alpha(x, A) = 0$ for each $\alpha \in [0, 1]$.*

Proof. If $\{x\} \subset A$, then $x \in A_\alpha$ for each $\alpha \in [0, 1]$.

$$p_\alpha(x, A) = \inf_{y \in A_\alpha} d(x, y) = 0.$$

LEMMA 3.2.

$$p_\alpha(x, A) \leqslant d(x, y) + p_\alpha(y, A) \qquad \text{for any} \quad x, y \in X.$$

Proof.

$$p_\alpha(x, A) = \inf_{z \in A_\alpha} d(x, z) \leqslant \inf_{z \in A_\alpha} (d(x, y) + d(y, z))$$
$$= d(x, y) + p_\alpha(y, A).$$

LEMMA 3.3. *If $\{x_0\} \subset A$, then $p_\alpha(x_0, B) \leqslant D_\alpha(A, B)$ for each $B \in \mathscr{W}(X)$.*

Proof.

$$p_\alpha(x_0, B) = \inf_{y \in B_\alpha} d(x_0, y) \leqslant \sup_{x \in A_\alpha} \inf_{y \in B_\alpha} d(x, y)$$
$$\leqslant D_\alpha(A, B).$$

Now we prove a generalization to fuzzy sets of the fixed point theorem for the contraction mappings.

THEOREM 3.1. *Let X be a complete metric linear space and F be a fuzzy mapping from X to $\mathscr{W}(X)$ satisfying the following condition: there exists $q \in (0, 1)$ such that*

$$D(F(x), F(y)) \leqslant q d(x, y) \qquad \textit{for each } x, y \in X.$$

Then there exists $x^* \in X$ such that $\{x^*\} \subset F(x^*)$.

Proof. Let $x_0 \in X$ and $\{x_1\} \subset F(x_0)$. Then there exists $x_2 \in X$ such that $\{x_2\} \subset F(x_1)$ and $d(x_2, x_1) \leqslant D_1(F(x_1), F(x_0))$. Continuing in this way we produce a sequence (x_n) in X such that $\{x_n\} \subset F(x_{n-1})$ and $d(x_n, x_{n+1}) \leqslant D_1(F(x_{n-1}), F(x_n))$ for each $n \in N$. We shall now show that (x_n) is a Cauchy sequence.

$$d(x_{k+1}, x_k) \leqslant D_1(F(x_k), F(x_{k-1})) \leqslant D(F(x_k), F(x_{k-1}))$$
$$\leqslant q d(x_k, x_{k-1}) \qquad \text{whenever } q \in (0, 1).$$

$$d(x_{k+m}, x_k) \leqslant \sum_{j=k}^{k+m-1} d(x_{j+1}, x_j) \leqslant \sum_{j=k}^{k+m-1} q^j d(x_1, x_0)$$
$$\leqslant q^k/(1-q) \cdot d(x_1, x_0).$$

q^k converges to 0 as $k \to \infty$. Then, since X is a complete space and (x_n) is Cauchy sequence, there exists a limit of sequence (x_n). Let $\lim_{n\to\infty} x_n = x^*$.

$$p_0(x^*, F(x^*)) \leqslant d(x^*, x_n) + p_0(x_n, F(x^*)) \tag{3.1}$$
$$\leqslant d(x^*, x_n) + D_0(F(x_{n-1}), F(x^*)) \tag{3.2}$$
$$\leqslant d(x^*, x_n) + qd(x_{n-1}, x^*).$$

$d(x^*, x_n)$ converges to 0 as $n \to \infty$. Hence, by Lemma 3.1 we conclude that $\{x^*\} \subset F(x^*)$. Inequality (3.1) follows from Lemma 3.2 and (3.2) from Lemma 3.3.

References

1. M. P. Chen and M. H. Shin, Fixed point theorems for point-to-point and point-to-set maps, *J. Math. Anal. Appl.* **71** (1979), 515–524.
2. S. B. Nadler, Jr., Multi-valued contraction mappings, *Pacific J. Math.* **30** (1969), 475–488.
3. H. T. Nguyen, A note on the extension principle for fuzzy sets, *J. Math. Anal. Appl.* **64** (1978), 369–380.