



Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa


Perturbed sampling formulas and local reconstruction in shift invariant spaces

Nikolaos D. Atreas

Department of Mathematics, Physics and Computational Sciences, Faculty of Engineering, Aristotle University of Thessaloniki, 54-124, Thessaloniki, Greece

ARTICLE INFO

Article history:

Received 12 September 2010

Available online 7 December 2010

Submitted by Steven G. Krantz

Keywords:

Non-uniform sampling

Perturbations

Local reconstruction

ABSTRACT

Let V_ϕ be a closed subspace of $L_2(\mathbb{R})$ generated from the integer shifts of a continuous function ϕ with a certain decay at infinity and a non-vanishing property for the function $\Phi^\dagger(\gamma) = \sum_{n \in \mathbb{Z}} \phi(n) e^{-in\gamma}$ on $[-\pi, \pi]$. In this paper we determine a positive number δ_ϕ so that the set $\{n + \delta_n\}_{n \in \mathbb{Z}}$ is a set of stable sampling for the space V_ϕ for any selection of the elements δ_n within the ranges $\pm\delta_\phi$. We demonstrate the resulting sampling formula (called perturbation formula) for functions $f \in V_\phi$ and also we establish a finite reconstruction formula approximating f on bounded intervals. We compute the corresponding error and we provide estimates for the jitter error as well.

© 2010 Elsevier Inc. All rights reserved.

1. Introduction

Let $V_{\phi,0}$ be a subspace of $L_2(\mathbb{R})$ (the space of all measurable square integrable functions on \mathbb{R} with usual inner product $\langle \cdot, \cdot \rangle_{L_2}$ and norm $\| \cdot \|_{L_2}$) containing all finite linear combinations of the integer shifts of a generator function $\phi \in L_2(\mathbb{R})$. We say that the set $\{\phi(\cdot - n)\}_{n \in \mathbb{Z}}$ forms a Riesz basis for the L_2 -closure of $V_{\phi,0}$ defined by

$$V_\phi = \left\{ \sum_{n \in \mathbb{Z}} c_n \phi(\cdot - n) : \{c_n\}_{n \in \mathbb{Z}} \in \ell_2(\mathbb{Z}) \right\}, \quad (1.1)$$

if there exist two positive constants A and B such that

$$A \|c\|_{\ell_2(\mathbb{Z})}^2 \leq \left\| \sum_{n \in \mathbb{Z}} c_n \phi(\cdot - n) \right\|_{L_2}^2 \leq B \|c\|_{\ell_2(\mathbb{Z})}^2 \quad \text{for all } c \in \ell_2(\mathbb{Z}). \quad (1.2)$$

Here and hereafter, $\ell_p(X)$ ($p \geq 1$) is the space of all p -summable sequences over the index set X with usual norm $\| \cdot \|_{\ell_p(X)}$. Whenever $X = \mathbb{Z}$ we write ℓ_p for brevity.

By imposing regularity requirements on V_ϕ (for example the function ϕ is continuous on \mathbb{R} with some decay at infinity) and by assuming that the infinite matrix (or operator) $\Phi = \{\Phi_{k,n} = \phi(k - n)\}_{k,n \in \mathbb{Z}}$ is bounded and has bounded inverse $\Phi^{-1} = \{\Phi_{k,n}^{-1}\}_{k,n \in \mathbb{Z}}$ on ℓ_2 we can prove that V_ϕ is a *sampling space*, i.e. any function $f \in V_\phi$ is stably reconstructed from the sample set $\mathcal{L}(f) = \{f(n)\}_{n \in \mathbb{Z}}$ by the formula

$$f(x) = \sum_{n \in \mathbb{Z}} f(n) S(x - n) \quad (1.3)$$

E-mail address: natreas@gen.auth.gr.

pointwise on \mathbb{R} , where $S(x) = \sum_{k \in \mathbb{Z}} \Phi_{k,0}^{-1} \phi(x - k)$ [18,19]. Eq. (1.3) is an example of a *regular sampling expansion* including classical Shannon-type sampling theorems, wavelet sampling expansions and sampling expansions in shift invariant spaces [13,23,24].

In a variety of applications the sampling process becomes more efficient if the uniform sampling set $\tau = \mathbb{Z}$ is shifted or perturbed by a bounded sequence $\Delta = \{\delta_n\}_{n \in \mathbb{Z}}$ (called *perturbation* sequence) due to fluctuations of the signal or possible delay due to channel cognition. Δ may be also unknown if it is caused from disturbances of the acquisition device or jitter. In both cases we are led to a non-uniform sampling scheme [7,16] and a basic problem is to examine whether the resulting irregular sampling set $\{n + \delta_n\}_{n \in \mathbb{Z}}$ continues to be a *set of stable sampling* for the space V_ϕ in the sense that there exist positive constants C, D such that

$$C \|f\|_{L_2}^2 \leq \| \{f(n + \delta_n)\} \|_{\ell_2}^2 \leq D \|f\|_{L_2}^2 \quad \text{for all } f \in V_\phi. \tag{1.4}$$

If this double inequality holds then there exists another Riesz basis $\{\psi_n^\Delta(\cdot)\}_{n \in \mathbb{Z}}$ for the space V_ϕ providing a stable reconstruction formula for elements $f \in V_\phi$ of the form

$$f(x) = \sum_{n \in \mathbb{Z}} f(n + \delta_n) \psi_n^\Delta(x) \tag{1.5}$$

and a perturbation of regular sampling formula (1.3) is obtained. Notice that the largest bound of the perturbation set Δ for which (1.4) holds is called *maximum perturbation* of Δ .

The existence of stable perturbed sampling sets and formulas has been studied in spaces of band-limited functions (see [12] and references therein), in wavelet spaces [5,15] and in shift invariant spaces [6,14,25] but the resulting sampling formulas are complicated. We mention that certain estimates on the maximum perturbation have been established, based on decay assumptions on the generator function ϕ [6,25]. In [19, Theorem 3.2] a perturbation formula for non-necessarily shift-invariant spaces was established. Also in [19, Theorem 6.2] a partial reconstruction formula suitable for numerical implementation was derived and error estimates were obtained based on Wiener’s lemma for a suitable Gramian matrix.

Our motivation for this work originates from [19]. More precisely, the first tenet of this work is to derive a class of perturbed sampling expansions (1.5) for the space V_ϕ under a certain decay assumption on ϕ and a non-vanishing property for the function $\Phi^\dagger(\gamma) = \sum_{n \in \mathbb{Z}} \phi(n) e^{-in\gamma}$ on $[-\pi, \pi]$.

In Section 2 we determine a maximum perturbation δ_ϕ so that the set $\tau_\delta = \{n + \delta_n : |\delta_n| \leq \delta\}_{n \in \mathbb{Z}}$ is a set of stable sampling for the space V_ϕ for any $0 \leq \delta < \delta_\phi$. Then in Theorem 1 of Section 2 we demonstrate the corresponding reconstruction formula and we present certain examples where we compare the ranges of the perturbations with the ranges obtained in [1, 17,25]. Notice that our sampling formula is different from the reconstruction formula obtained in [19], however it is complicated because it requires evaluation of the inverse of an infinite matrix.

In Section 3 we state our second main result. In Theorem 2 we establish a partial reconstruction formula for V_ϕ suitable for numerical implementation and we provide estimates to the corresponding error based on Wiener’s lemma for infinite matrices and the finite section method [10]. Notice that the resulting decay rate estimates are smaller compared to the estimates obtained in [19].

Finally in Section 4 we deal with the case where the uniform sampling set $\tau = \mathbb{Z}$ is distorted without our knowledge. In this case when we reconstruct f using (1.3) we are facing jitter error [2–4]. In Proposition 1 we address this problem and we determine a number $\delta_{\phi,\epsilon}$ so that for any perturbation Δ bounded by the number $\delta_{\phi,\epsilon}$ the jitter error is less than a pre-determined error ϵ .

2. Perturbed reconstruction formulas for V_ϕ

In this section we establish stable perturbation sampling formulas for the space V_ϕ under the following assumptions on ϕ (notice that we have already assumed that the set $\{\phi(\cdot - n)\}_{n \in \mathbb{Z}}$ forms a Riesz basis for V_ϕ):

(P₁) $\phi : \mathbb{R} \rightarrow \mathbb{C}$ is a continuous function on \mathbb{R} satisfying $\phi(x) = \overline{\phi(-x)}$ for every $x \in \mathbb{R}$ and ϕ belongs in the weighted Wiener amalgam space

$$W_p(L_\infty, u_\alpha) = \{f : \|f\|_{W_p(L_\infty, u_\alpha)} = \| \{ \|u_\alpha(\cdot - n)\phi(\cdot - n)\|_{L_\infty[-1/2, 1/2]} \}_{n \in \mathbb{Z}} \|_{\ell_p} < \infty \},$$

where $1 \leq p \leq +\infty$ and $u_\alpha(x) = (1 + |x|)^\alpha$ for some $\alpha > 1 - \frac{1}{p}$.

(P₂) The 2π -periodic function $\Phi^\dagger(\gamma) = \sum_{n \in \mathbb{Z}} \phi(n) e^{-in\gamma}$ has no real zeros on $[-\pi, \pi]$.

The assumption $\alpha > 1 - \frac{1}{p}$ on the exponent of the polynomial weight u_α ensures that if $\phi \in W_p(L_\infty, u_\alpha)$ then $\phi \in W_1(L_\infty, u_0)$, i.e. $\phi \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$. Indeed if q is the conjugate exponent of p (i.e. $\frac{1}{p} + \frac{1}{q} = 1$), then

$$\begin{aligned} \|\phi\|_{W_1(L_\infty, u_0)} &= \|(\phi u_\alpha) u_\alpha^{-1}\|_{W_1(L_\infty, u_0)} \leq \|\phi\|_{W_p(L_\infty, u_\alpha)} \left(\sum_{n \in \mathbb{Z}} \frac{1}{(|n| + 1/2)^{\alpha q}} \right)^{1/q} \\ &\leq C \|\phi\|_{W_p(L_\infty, u_\alpha)} < \infty. \end{aligned} \tag{2.1}$$

Notice also that if ϕ satisfies condition (P₁) then the 2π -periodic function $\Phi^\dagger(\gamma)$ is real-valued and continuous and so if condition (P₂) holds then $\Phi^\dagger(\gamma)$ is a positive (or negative) function on $[-\pi, \pi]$.

Under the above assumptions on ϕ the infinite matrix

$$\Phi = \{\Phi_{m,n} = \phi(m - n)\}_{m,n \in \mathbb{Z}} \tag{2.2}$$

as an operator on ℓ_2 is self-adjoint, bounded and has bounded inverse on ℓ_2 . More precisely, it is easy to see that the operator Φ satisfies the following double inequality

$$\|\Phi^\dagger\|_0 \|c\|_{\ell_2} \leq \|\Phi c\|_{\ell_2} \leq \|\Phi^\dagger\|_\infty \|c\|_{\ell_2} \quad \text{for all } c \in \ell_2, \tag{2.3}$$

where

$$\|\Phi^\dagger\|_0 = \inf_{\gamma \in [-\pi, \pi]} |\Phi^\dagger(\gamma)| > 0 \quad \text{and} \quad \|\Phi^\dagger\|_\infty = \sup_{\gamma \in [-\pi, \pi]} |\Phi^\dagger(\gamma)| < \infty.$$

Let us define a distortion of the above matrix Φ by

$$\Phi_{\tau_\delta} = \{(\Phi_{\tau_\delta})_{m,n} = \phi(\tau_m - n)\}_{m,n \in \mathbb{Z}},$$

where $\tau_\delta = \{\tau_n = n + \delta_n : |\delta_n| \leq \delta\}_{n \in \mathbb{Z}}$ is a sampling set on \mathbb{R} for some $\delta > 0$ and τ_δ is also an ordered and ε -separated sampling set, i.e.,

$$\tau_{m+1} - \tau_m \geq \varepsilon > 0 \quad \text{for all } m \in \mathbb{Z} \text{ and for some } \varepsilon > 0. \tag{2.4}$$

In addition let us determine a positive real number

$$\delta_\phi = \inf\{x > 0 : G_\phi(x) \geq \|\Phi^\dagger\|_0\}, \tag{2.5}$$

where

$$G_\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ : G_\phi(x) = \sum_{n \in \mathbb{Z}} \sup_{|y| \leq x} |\phi(y + n) - \phi(n)| \tag{2.6}$$

and $\|\Phi^\dagger\|_0 > 0$ by assumption (P₂). Taking into account (2.1) we see that the function $G_\phi(x)$ is well defined on \mathbb{R}^+ and moreover it is continuous, increasing and unbounded on \mathbb{R}^+ with $G(0) = 0$. Therefore $0 < \delta_\phi < +\infty$.

Definition 1. Let ϕ , τ_δ , Φ_{τ_δ} and δ_ϕ be as above. Given ϕ we say that the infinite matrix Φ_{τ_δ} belongs in the class $\mathcal{F}_{\delta_\phi}$ if the matrix Φ_{τ_δ} is produced from a sampling set τ_δ such that $0 \leq \delta < \delta_\phi$.

Lemma 1. If the infinite matrix (or operator) Φ_{τ_δ} belongs in the above class $\mathcal{F}_{\delta_\phi}$, then there exist two positive constants C and D such that

$$C \|c\|_{\ell_2}^2 \leq |\langle \Phi_{\tau_\delta} c, c \rangle_{\ell_2}| \leq D \|c\|_{\ell_2}^2 \quad \text{for all } c \in \ell_2 \tag{2.7}$$

and this double inequality holds for the adjoint operator $\Phi_{\tau_\delta}^*$ as well.

Proof. Let Φ be as in (2.2). Then for any $c \in \ell_2$ we have

$$|\langle \Phi c, c \rangle_{\ell_2}| = \left| \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \phi(m - n) c_n \bar{c}_m \right| = \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} |f_c(\gamma)|^2 \Phi^\dagger(\gamma) d\gamma \right| \geq \|\Phi^\dagger\|_0 \|c\|_{\ell_2}^2, \tag{2.8}$$

where $f_c(\gamma) = \sum_{n \in \mathbb{Z}} c_n e^{-in\gamma}$ and $\Phi^\dagger(\gamma) = \sum_{n \in \mathbb{Z}} \phi(n) e^{-in\gamma}$. Since the function $\Phi^\dagger(\gamma)$ is positive (or negative) on $[-\pi, \pi]$ as we mentioned above, the last inequality in (2.8) is immediately obtained.

Let $\Phi_{\tau_\delta} \in \mathcal{F}_{\delta_\phi}$ where $\mathcal{F}_{\delta_\phi}$ as in Definition 1. Then

$$|\langle \Phi_{\tau_\delta} c, c \rangle_{\ell_2}| \geq \left| |\langle \Phi c, c \rangle_{\ell_2}| - |\langle (\Phi_{\tau_\delta} - \Phi) c, c \rangle_{\ell_2}| \right|. \tag{2.9}$$

First we compute

$$\begin{aligned}
 | \langle (\Phi_{\tau_\delta} - \Phi)c, c \rangle_{\ell_2} | &\leq \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} |\phi(\tau_m - n) - \phi(m - n)| |c_n| |c_m| \\
 &\leq \sum_{m \in \mathbb{Z}} |c_m| \left(\sum_{n \in \mathbb{Z}} |c_n|^2 |\phi(\tau_m - n) - \phi(m - n)| \right)^{1/2} \left(\sum_{n \in \mathbb{Z}} |\phi(\tau_m - n) - \phi(m - n)| \right)^{1/2} \\
 &\leq \|c\|_{\ell_2}^2 \left(\sup_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} |\phi(\tau_m - n) - \phi(m - n)| \right)^{1/2} \left(\sup_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} |\phi(\tau_m - n) - \phi(m - n)| \right)^{1/2}. \tag{2.10}
 \end{aligned}$$

But $\Phi_{\tau_\delta} \in \mathcal{F}_{\delta_\phi}$ and $\tau_m = m + \delta_m$ where $|\delta_m| \leq \delta < \delta_\phi < +\infty$, so

$$\begin{aligned}
 \sup_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} |\phi(\tau_m - n) - \phi(m - n)| &= \sup_{n \in \mathbb{Z}} \sum_{r \in \mathbb{Z}} |\phi(\delta_{n+r} + r) - \phi(r)| \\
 &\leq \sum_{r \in \mathbb{Z}} \sup_{|x| \leq \delta} |\phi(x + r) - \phi(r)| = G_\phi(\delta), \tag{2.11}
 \end{aligned}$$

where the function $G_\phi(x)$ is defined in (2.6). Similarly, $\sup_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} |\phi(\tau_m - n) - \phi(m - n)| \leq G_\phi(\delta)$. We omit the proof here. Substituting the bound (2.11) into (2.10) we obtain

$$| \langle (\Phi_{\tau_\delta} - \Phi)c, c \rangle_{\ell_2} | \leq G_\phi(\delta) \|c\|_{\ell_2}^2. \tag{2.12}$$

Substituting the bounds (2.12) and (2.8) into (2.9) we obtain the lower inequality of (2.7) with $C = \|\Phi^\dagger\|_0 - G_\phi(\delta)$. Clearly $C > 0$ because $\Phi_{\tau_\delta} \in \mathcal{F}_{\delta_\phi}$ and so $0 \leq \delta < \delta_\phi$ with δ_ϕ as in (2.5).

To derive the upper bound of (2.7) we observe that

$$\begin{aligned}
 \|(\Phi_{\tau_\delta} - \Phi)c\|_{\ell_2}^2 &\leq \sum_{m \in \mathbb{Z}} \left(\sum_{n \in \mathbb{Z}} |c_n|^2 (|\phi(\tau_m - n) - \phi(m - n)|) \sum_{n \in \mathbb{Z}} |\phi(\tau_m - n) - \phi(m - n)| \right) \\
 &\leq G_\phi^2(\delta) \|c\|_{\ell_2}^2 \tag{2.13}
 \end{aligned}$$

as we showed above. Using (2.13) and (2.3) and recalling (2.1) we obtain

$$\begin{aligned}
 \|\Phi_{\tau_\delta} c\|_{\ell_2} &\leq \|\Phi c\|_{\ell_2} + \|(\Phi_{\tau_\delta} - \Phi)c\|_{\ell_2} \leq (\|\Phi^\dagger\|_\infty + G_\phi(\delta)) \|c\|_{\ell_2} \\
 &\leq (\|\Phi^\dagger\|_\infty + 2\|\phi\|_{W_1(L_\infty, u_0)}(2\lceil \delta \rceil + 1)) \|c\|_{\ell_2}, \tag{2.14}
 \end{aligned}$$

where $\lceil x \rceil$ is the ceiling of a real number x . Indeed

$$\begin{aligned}
 G_\phi(\delta) &\leq \sum_{n \in \mathbb{Z}} \sum_{k=\lceil -\delta \rceil + 1}^{\lceil \delta \rceil} \sup_{|y-k| \leq 1/2} |\phi(y+n) - \phi(n)| \\
 &\leq (\lceil -\delta \rceil + \lceil \delta \rceil + 2) \sum_{n \in \mathbb{Z}} \sup_{|y| \leq 1/2} (|\phi(y+n)| + |\phi(n)|) \\
 &\leq 2\|\phi\|_{W_1(L_\infty, u_0)}(2\lceil \delta \rceil + 1).
 \end{aligned}$$

Therefore the upper bound of (2.7) is obtained with $D = 2\|\phi\|_{W_1(L_\infty, u_0)}(2\lceil \delta \rceil + 1)$ and $D < +\infty$ because $\delta < \delta_\phi < +\infty$.

Following the same proof as above it is easy to see that (2.7) holds for the adjoint operator $\Phi_{\tau_\delta}^*$ as well. We omit the proof here. \square

Corollary 1. Every infinite matrix $\Phi_{\tau_\delta} \in \mathcal{F}_{\delta_\phi}$ as an operator on ℓ_2 is bounded and has bounded inverse on ℓ_2 . More precisely, there exist two positive constants C, D as in Lemma 1 such that

$$\|c\|_{\ell_2} \leq \|\Phi_{\tau_\delta} c\|_{\ell_2} \leq D \|c\|_{\ell_2} \quad \text{for all } c \in \ell_2 \tag{2.15}$$

and the same double inequality holds for the adjoint operator $\Phi_{\tau_\delta}^*$ as well.

Proof. From the lower inequality of (2.7) we get

$$\|c\|_{\ell_2}^2 \leq | \langle \Phi_{\tau_\delta} c, c \rangle_{\ell_2} | \leq \|\Phi_{\tau_\delta} c\|_{\ell_2} \|c\|_{\ell_2} \quad \text{for all } c \in \ell_2$$

and so the lower inequality of (2.15) is obtained. The upper inequality of (2.15) is obtained directly from (2.14). Therefore the operator Φ_{τ_δ} is injective and the inverse operator $\Phi_{\tau_\delta}^{-1}$ is bounded on $\mathcal{R}_{\Phi_{\tau_\delta}}$, the range of Φ_{τ_δ} . Since the adjoint operator $\Phi_{\tau_\delta}^*$ satisfies (2.7), by using similar arguments as above we deduce that $\Phi_{\tau_\delta}^*$ satisfies (2.15) as well and so $\mathcal{R}_{\Phi_{\tau_\delta}} = \ell_2$, i.e. the operator Φ_{τ_δ} is onto. \square

Theorem 1. Let V_ϕ be a subspace of $L_2(\mathbb{R})$ as in (1.1) and let the generator function ϕ satisfy the aforementioned conditions (P₁) and (P₂). Furthermore let $\tau_\delta = \{\tau_n = n + \delta_n : |\delta_n| \leq \delta\}_{n \in \mathbb{Z}}$ be an ordered and ε -separated sampling set on \mathbb{R} for some $\delta \geq 0$. If δ_ϕ is a positive real number as in (2.5), then for any $0 \leq \delta < \delta_\phi$ the set τ_δ is a set of stable sampling for the space V_ϕ , i.e. (1.4) holds. Moreover every function $f \in V_\phi$ is uniquely reconstructed from the set of samples $\mathcal{L}_{\tau_\delta}(f) = \{f(\tau_n)\}_{n \in \mathbb{Z}}$ (for every selection of a stable sampling set τ_δ) by the formula

$$f(x) = \sum_{n \in \mathbb{Z}} f(\tau_n) \psi_n^{\tau_\delta}(x) \tag{2.16}$$

pointwise on \mathbb{R} , where

$$\psi_n^{\tau_\delta}(x) = \sum_{k \in \mathbb{Z}} (\Phi_{\tau_\delta})_{k,n}^{-1} \phi(x - k) \tag{2.17}$$

and $\Phi_{\tau_\delta}^{-1}$ is the inverse of an infinite matrix $\Phi_{\tau_\delta} \in \mathcal{F}_{\delta_\phi}$ as above. Furthermore the set $\{\psi_n^{\tau_\delta}(\cdot)\}_{n \in \mathbb{Z}}$ is a Riesz basis for the space V_ϕ , i.e. the reconstruction is stable.

Proof. Let $\phi, V_\phi, \tau_\delta, \delta_\phi, \Phi_{\tau_\delta}$ and $\mathcal{F}_{\delta_\phi}$ be as above, $\Phi_{\tau_\delta} \in \mathcal{F}_{\delta_\phi}$ and $f \in V_\phi$. Then there exists a unique sequence $c \in \ell_2$ such that $f(x) = \sum_{k \in \mathbb{Z}} c_k \phi(x - k)$ and

$$\mathcal{L}_{\tau_\delta}(f) = \Phi_{\tau_\delta} c$$

where $\mathcal{L}_{\tau_\delta}(f) = \{f(\tau_n)\}_{n \in \mathbb{Z}}$. Substituting the above equality into (2.15) and using (1.2) it is easy to prove that (1.4) holds for any $f \in V_\phi$, i.e. the set τ_δ is a set of stable sampling for V_ϕ . In addition, since the matrix Φ_{τ_δ} is invertible on ℓ_2 as a result of Corollary 1 we have

$$\begin{aligned} f(x) &= \sum_{k \in \mathbb{Z}} c_k \phi(x - k) = \sum_{k \in \mathbb{Z}} (\Phi_{\tau_\delta}^{-1} \mathcal{L}_{\tau_\delta}(f))_k \phi(x - k) \\ &= \sum_{n \in \mathbb{Z}} f(\tau_n) \left(\sum_{k \in \mathbb{Z}} (\Phi_{\tau_\delta})_{k,n}^{-1} \phi(x - k) \right) = \sum_{n \in \mathbb{Z}} f(\tau_n) \psi_n^{\tau_\delta}(x) \end{aligned}$$

and so Eqs. (2.16) and (2.17) are obtained. By assumption the set $\{\phi(\cdot - n)\}_{n \in \mathbb{Z}}$ is a Riesz basis for V_ϕ and by (2.17) the set $\{\psi_n^{\tau_\delta}(\cdot)\}_{n \in \mathbb{Z}}$ is the image of $\{\phi(\cdot - n)\}_{n \in \mathbb{Z}}$ by means of the bijective operator $\Phi_{\tau_\delta}^{-1}$. Therefore the set $\{\psi_n^{\tau_\delta}(\cdot)\}_{n \in \mathbb{Z}}$ is a Riesz basis for V_ϕ and we are done. \square

Below we present some examples.

Example 1. All sampling functions $\phi(x)$ as above produce a maximum perturbation $0 < \delta_\phi < +\infty$ as in (2.5) (not optimal in general). Recall that $\phi(x)$ is a sampling function if $\phi(n) = \delta_{0,n}$, where $\delta_{0,n}$ is the Kronecker delta symbol. Then $\Phi^\dagger(\gamma) = \sum_{n \in \mathbb{Z}} \phi(n) e^{-in\gamma} = 1$ for all γ in $[-\pi, \pi]$ and so

$$\delta_\phi = \inf\{x \in \mathbb{R}^+ : G_\phi(x) \geq 1\} \tag{2.18}$$

where $G_\phi(x)$ as in (2.6). For example we consider the B_2 -spline $\phi(x) = 1 - |x|$ if $|x| \leq 1$ and $\phi(x) = 0$ elsewhere. For any $0 \leq x \leq 1$ we observe that

$$G_\phi(x) = \sup_{|y| \leq x} (1 - |\phi(y)|) + \sup_{|y| \leq x} |\phi(y - 1)| + \sup_{|y| \leq x} |\phi(y + 1)| = 3x. \tag{2.19}$$

Then we use the above Eqs. (2.18) and (2.19) to obtain a maximum perturbation

$$\delta_\phi = \inf\{x \in [0, 1] : 3x \geq 1\} = \frac{1}{3}.$$

Notice that if $\Delta = \{\delta_n\}_{n \in \mathbb{Z}}$ is a positive (or negative) sequence then $\delta_\phi = \frac{1}{2}$ is an estimate obtained in [6,15] as well. Moreover this estimate is optimal in the sense that for $\delta_\phi = \frac{1}{2}$ the resulting sampling set is not stable [1]. In [17] the authors provided the estimate $\delta_\phi = \frac{1}{3} - \frac{2}{9N}$, $N \geq 1$ for stable sampling sets when some suitable kernels K_N are used to reconstruct the linear spline space.

Let us present another sampling function. Consider the function $\phi(x) = (\frac{\sin(\pi x)}{\pi x})^4$, $x \in \mathbb{R}$. In this case for any $0 \leq x \leq 1/2$ we have

$$G_\phi(x) = 1 - \phi(x) + 2 \sum_{n=1}^{\infty} \phi(n - x)$$

and from this equality we obtain numerically a maximum perturbation

$$\delta_\phi = \inf\{x \in \mathbb{R}^+ : G_\phi(x) \geq 1\} \approx 0.455.$$

Example 2. Consider functions of the form $\phi_c(x) = e^{-c|x|}$, $c > 0$. In this case we use the Poisson summation formula to obtain

$$\Phi^\dagger(\gamma) = 2c \sum_{n \in \mathbb{Z}} \frac{1}{c^2 + (\gamma + 2n\pi)^2}, \quad \gamma \in [-\pi, \pi].$$

Since $\|\Phi^\dagger\|_0 = \Phi^\dagger(\pi)$ we have $\delta_\phi = \inf\{x \in \mathbb{R}^+ : G_\phi(x) \geq \Phi^\dagger(\pi)\}$ where

$$G_\phi(x) = 1 - \phi(x) + 2 \sum_{n=1}^{\infty} (\phi(n-x) - \phi(n))$$

for $0 \leq x \leq 1/2$. For example if $c = 1$ we find numerically that $\delta_\phi \approx 0.21$. We work similarly for functions of the form $\phi(x) = e^{-cx^2}$, $c > 0$.

3. A finite reconstruction formula for V_ϕ

The exact reconstruction formula (2.16) is difficult to be implemented numerically because we must know an infinite number of sampled data and we need to compute the inverse of the infinite matrix $\Phi_{\tau_\delta}^{-1}$. In this section we establish a finite reconstruction formula approximating elements of the space V_ϕ on bounded intervals.

In the previous section we considered infinite matrices Φ_{τ_δ} produced from a function $\phi \in W_p(L_\infty, u_\alpha)$, where $u_\alpha(x) = (1 + |x|)^\alpha$ is a polynomial weight with exponent $\alpha > 1 - \frac{1}{p}$. It turns out that all these matrices belong in the Gröchenig–Shur class $\mathcal{A}_{p, u_\alpha}$ which contains infinite matrices $A = \{a_{m,n}\}_{m,n \in \mathbb{Z}}$ with norm

$$\|A\|_{\mathcal{A}_{p, u_\alpha}} = \sup_{n \in \mathbb{Z}} \|\{u_\alpha(m-n)a_{m,n}\}_{m \in \mathbb{Z}}\|_{\ell_p} + \sup_{m \in \mathbb{Z}} \|\{u_\alpha(m-n)a_{m,n}\}_{n \in \mathbb{Z}}\|_{\ell_p} < \infty.$$

Furthermore if we assume that $\Phi_{\tau_\delta} \in \mathcal{F}_{\delta_\phi}$ (recall Definition 1 of the previous section), then every matrix (or operator) in this class belongs also in the space \mathcal{B}^2 containing all bounded operators on ℓ_2 with usual norm $\|\cdot\|_{\mathcal{B}^2}$ and it has a bounded inverse $\Phi_{\tau_\delta}^{-1} \in \mathcal{B}^2$. Therefore Wiener’s lemma for infinite matrices can be applied on elements of the class $\mathcal{F}_{\delta_\phi}$, see Lemma 2 below. For more details about Wiener’s lemma for infinite matrices we refer to [8,9,11,20–22] and references therein.

Lemma 2. *If $\Phi_{\tau_\delta} \in \mathcal{F}_{\delta_\phi}$, then both matrices Φ_{τ_δ} and $\Phi_{\tau_\delta}^{-1}$ belong in the Gröchenig–Shur class $\mathcal{A}_{p, u_\alpha}$.*

Proof. Let $\Phi_{\tau_\delta} \in \mathcal{F}_{\delta_\phi}$ and $\tau_\delta = \{\tau_n = n + \delta_n : |\delta_n| \leq \delta\}_{n \in \mathbb{Z}}$ for some number δ bounded by the number δ_ϕ as in (2.5). Fix an integer i . Then for any $j \in \mathbb{Z}$ we have

$$1 + \delta + |\tau_i - j| \geq \begin{cases} 1 + \delta + |i - j| - |\delta_i| \geq 1 + |i - j|, & |\delta_i| \leq |i - j|, \\ 1 + \delta + |\delta_i| - |i - j| \geq 1 + |\delta_i| \geq 1 + |i - j|, & |\delta_i| \geq |i - j|. \end{cases}$$

Therefore if $u_\alpha(x)$ is the polynomial weight related to the decay of ϕ (recall condition (P₁)), then for any $1 \leq p < +\infty$ we have

$$\begin{aligned} \|\{u_\alpha(i-j)(\Phi_{\tau_\delta})_{i,j}\}_{j \in \mathbb{Z}}\|_{\ell_p}^p &= \sum_{j \in \mathbb{Z}} ((1 + |i-j|)^\alpha |\phi(\tau_i - j)|)^p \\ &\leq \sum_{j \in \mathbb{Z}} ((1 + \delta + |\tau_i - j|)^\alpha |\phi(\tau_i - j)|)^p = (1 + \delta)^{\alpha p} \sum_{j \in \mathbb{Z}} \left(1 + \frac{|\tau_i - j|}{1 + \delta}\right)^{\alpha p} |\phi(\tau_i - j)|^p \\ &< (1 + \delta)^{\alpha p} \sum_{j \in \mathbb{Z}} ((1 + |\tau_i - j|)^\alpha |\phi(\tau_i - j)|)^p \leq (1 + \delta)^{\alpha p} \|\phi\|_{W_p(L_\infty, u_\alpha)}^p < \infty \end{aligned}$$

and for $p = +\infty$ we obtain a similar estimate. Notice that the same bound holds for $\sup_{j \in \mathbb{Z}} \|\{u_\alpha(i-j)(\Phi_{\tau_\delta})_{i,j}\}_{i \in \mathbb{Z}}\|_{\ell_p}^p$ as well. We omit the proof here. Therefore the matrix Φ_{τ_δ} belongs in the Gröchenig–Shur class $\mathcal{A}_{p, u_\alpha}$. Since $\Phi_{\tau_\delta} \in \mathcal{F}_{\delta_\phi}$ we have $\Phi_{\tau_\delta}, \Phi_{\tau_\delta}^{-1} \in \mathcal{B}^2$ as a result of Corollary 1 and so Wiener’s lemma is applied on Φ_{τ_δ} stating that the inverse matrix $\Phi_{\tau_\delta}^{-1}$ belongs in the Gröchenig–Shur class $\mathcal{A}_{p, u_\alpha}$ as well, i.e. there exists a constant C' such that

$$\|\Phi_{\tau_\delta}^{-1}\|_{\mathcal{A}_{p, u_\alpha}} \leq C' < \infty \tag{3.1}$$

(see [20, Theorem 4.1] with $X = \mathbb{Z}$, $w = u_\alpha$ (u_α is a (1, 2) admissible weight for any $\alpha > 1 - \frac{1}{p}$), ρ is the usual metric on \mathbb{R} and μ_c the usual counting measure). \square

Remark 1. According to [20, Theorem 4.1], the constant C' in (3.1) depends on the norms $\|\Phi_{\tau_\delta}\|_{\mathcal{A}_{1,u_\alpha}}$ and $\|\Phi_{\tau_\delta}^{-1}\|_{\mathcal{B}^2}$ and on some other constants which are affected only from the weight u_α .

Corollary 2. If $\Phi_{\tau_\delta} \in \mathcal{F}_{\delta,\phi} \subset \mathcal{A}_{p,u_\alpha}$ for some selection of $\tau_\delta = \{\tau_n\}_{n \in \mathbb{Z}}$ as above and if $\{\psi_n^{\tau_\delta}(\cdot)\}_{n \in \mathbb{Z}}$ is the Riesz basis related to the reconstruction formula (2.16), then

$$\left\| \left\{ u_\alpha(\cdot - \tau_n) \psi_n^{\tau_\delta}(\cdot) \right\}_{n \in \mathbb{Z}} \right\|_{\ell_p} \|_{L_\infty} < +\infty. \tag{3.2}$$

Proof. Let $\{\psi_n^{\tau_\delta}(\cdot)\}_{n \in \mathbb{Z}}$ be as in (2.17), $n \in \mathbb{Z}$, $x \in \mathbb{R}$ and $Y_{n,x} = \{k \in \mathbb{Z} : |k - x| \leq \frac{|n-x|}{2}\}$. Then for $1 \leq p < +\infty$ we have

$$\begin{aligned} \left\| \left\{ u_\alpha(\cdot - \tau_n) \psi_n^{\tau_\delta}(\cdot) \right\}_{n \in \mathbb{Z}} \right\|_{\ell_p}^p &= \sum_{n \in \mathbb{Z}} \left| \left(\sum_{k \in Y_{n,x}} + \sum_{k \notin Y_{n,x}} \right) u_\alpha(x - \tau_n) (\Phi_{\tau_\delta}^{-1})_{k,n} \phi(x - k) \right|^p \\ &\leq 2^{p-1} \sum_{n \in \mathbb{Z}} \left(\sum_{k \in Y_{n,x}} |u_\alpha(x - \tau_n)| |(\Phi_{\tau_\delta}^{-1})_{k,n}| |\phi(x - k)| \right)^p \\ &\quad + 2^{p-1} \sum_{n \in \mathbb{Z}} \left(\sum_{k \notin Y_{n,x}} |u_\alpha(x - \tau_n)| |(\Phi_{\tau_\delta}^{-1})_{k,n}| |\phi(x - k)| \right)^p \\ &\leq 2^{p-1} \sum_{n \in \mathbb{Z}} \sum_{k \in Y_{n,x}} |u_\alpha(x - \tau_n)|^p |(\Phi_{\tau_\delta}^{-1})_{k,n}|^p |\phi(x - k)| \left(\sum_{k \in Y_{n,x}} |\phi(x - k)| \right)^{p-1} \\ &\quad + 2^{p-1} \sum_{n \in \mathbb{Z}} \sum_{k \notin Y_{n,x}} |u_\alpha(x - \tau_n)|^p |\phi(x - k)|^p |(\Phi_{\tau_\delta}^{-1})_{k,n}| \left(\sum_{k \notin Y_{n,x}} |(\Phi_{\tau_\delta}^{-1})_{k,n}| \right)^{p-1} \\ &\leq 2^{p-1} (1 + \delta)^{\alpha p} \|\phi\|_{W_\infty(L_{1,u_0})}^{p-1} \sum_{n \in \mathbb{Z}} \sum_{k \in Y_{n,x}} |u_\alpha(x - n)|^p |(\Phi_{\tau_\delta}^{-1})_{k,n}|^p |\phi(x - k)| \\ &\quad + 2^{p-1} (1 + \delta)^{\alpha p} \|\Phi_{\tau_\delta}^{-1}\|_{\mathcal{A}_{1,u_0}}^{p-1} \sum_{n \in \mathbb{Z}} \sum_{k \notin Y_{n,x}} |u_\alpha(x - n)|^p |\phi(x - k)|^p |(\Phi_{\tau_\delta}^{-1})_{k,n}| \end{aligned} \tag{3.3}$$

because $\phi \in W_\infty(L_{1,u_0})$ (see (2.1)), $\Phi_{\tau_\delta}^{-1} \in \mathcal{A}_{p,u_\alpha} \subset \mathcal{A}_{1,u_0}$ by Lemma 2 and

$$\begin{aligned} u_\alpha(x - \tau_n) &= (1 + |(x - n) + (n - \tau_n)|)^\alpha \leq (1 + \delta + |x - n|)^\alpha \\ &= (1 + \delta)^\alpha \left(1 + \frac{|x - n|}{1 + \delta} \right)^\alpha \leq (1 + \delta)^\alpha u_\alpha(x - n). \end{aligned}$$

We work with the first term in the right-hand side of (3.3). We observe that for any $k \in Y_{n,x}$ we have

$$u_\alpha(x - n) \leq 2^\alpha \left(1 + \frac{|x - n|}{2} \right)^\alpha \leq 2^\alpha (1 + |n - x| - |k - x|)^\alpha \leq 2^\alpha u_\alpha(k - n),$$

and by Lemma 2 we obtain the bound $2^{p-1} 2^{\alpha p} (1 + \delta)^{\alpha p} \|\phi\|_{W_\infty(L_{1,u_0})}^p \|\Phi_{\tau_\delta}^{-1}\|_{\mathcal{A}_{p,u_\alpha}}^p$. For the second term in the right-hand side of (3.3) we observe that for $k \notin Y_{n,x}$ we have

$$u_\alpha(x - n) \leq 2^\alpha \left(1 + \frac{|x - n|}{2} \right)^\alpha \leq 2^\alpha u_\alpha(x - k)$$

and so we obtain a bound $2^{p-1} 2^{\alpha p} (1 + \delta)^{\alpha p} \|\Phi_{\tau_\delta}^{-1}\|_{\mathcal{A}_{1,u_0}}^p \|\phi\|_{W_\infty(L_{p,u_\alpha})}^p$.

If $p = +\infty$ we easily obtain

$$\left\| \left\{ u_\alpha(\cdot - \tau_n) \psi_n^{\tau_\delta}(\cdot) \right\}_{n \in \mathbb{Z}} \right\|_{\ell_\infty} \leq 2^\alpha (1 + \delta)^\alpha (\|\phi\|_{W_\infty(L_{1,u_0})} \|\Phi_{\tau_\delta}^{-1}\|_{\mathcal{A}_{\infty,u_\alpha}} + \|\Phi_{\tau_\delta}^{-1}\|_{\mathcal{A}_{1,u_0}} \|\phi\|_{W_\infty(L_{\infty,u_\alpha})})$$

and the proof is complete. \square

In order to produce a numerically implementable reconstruction formula for the space V_ϕ approximating the sampling formula (2.16) we replace the infinite matrix $\Phi_{\tau_\delta}^{-1}$ appearing in the representation of the basis functions $\psi_n^{\tau_\delta}$ (see (2.17)) with the inverse (if it exists) of a finite square matrix of the form $\{\phi(\tau_m - n)\}$ and examine if the new formula approximates the original sampling formula in some sense. The finite section method [10] provides answers to these questions. First we give some definitions.

We consider a finite set X containing successive integers and for any positive integer R we define the R -neighborhood of X by

$$X_R = \{\inf X - R, \dots, \sup X + R\}.$$

Let

$$P_{X_R} : \ell_2(\mathbb{Z}) \rightarrow \mathcal{H}_{X_R} : P_{X_R} c = \begin{cases} c_n, & n \in X_R, \\ 0, & \text{elsewhere} \end{cases} \tag{3.4}$$

be the projection of a sequence $c \in \ell_2$ onto a finite dimensional subspace \mathcal{H}_{X_R} and let

$$\Phi_{\tau_\delta, Y} = \begin{cases} (\Phi_{\tau_\delta})_{m,n}, & m, n \in Y, \\ 0, & \text{elsewhere} \end{cases} \tag{3.5}$$

be the finite section of a matrix $\Phi_{\tau_\delta} \in \mathcal{F}_{\delta_\phi}$ on $Y \times Y \subset \mathbb{Z}^2$. Then we have

Lemma 3. *Let X_R be the R -neighborhood of the set X as above and P_{X_R} be a projection operator as in (3.4). If Φ_{τ_δ, X_R} is the finite section of a matrix $\Phi_{\tau_\delta} \in \mathcal{F}_{\delta_\phi}$ as in (3.5), then there exist two positive constants C, D as in Lemma 1 such that*

$$C \|P_{X_R} c\|_{\ell_2} \leq \|(\Phi_{\tau_\delta, X_R})c\|_{\ell_2} \leq D \|P_{X_R} c\|_{\ell_2} \quad \text{for all } c \in \ell_2. \tag{3.6}$$

Proof. Since $\Phi_{\tau_\delta} \in \mathcal{F}_{\delta_\phi}$ the inequality (2.7) holds for any element of the space \mathcal{H}_{X_R} (the range of the operator P_{X_R}), so

$$C \|P_{X_R} c\|_{\ell_2}^2 \leq |(\Phi_{\tau_\delta} P_{X_R} c, P_{X_R} c)_{\ell_2}| \leq D \|P_{X_R} c\|_{\ell_2}^2.$$

From this inequality we easily obtain $C \|P_{X_R} c\|_{\ell_2}^2 \leq |(\Phi_{\tau_\delta, X_R} c, c)_{\ell_2}| \leq D \|P_{X_R} c\|_{\ell_2}^2$ and then (3.6). \square

Lemma 4. *Let Φ_{τ_δ, X_R} be the finite section of a matrix $\Phi_{\tau_\delta} \in \mathcal{F}_{\delta_\phi} \subset \mathcal{A}_{p, u_\alpha}$. Then the matrices Φ_{τ_δ, X_R} and $\Phi_{\tau_\delta, X_R}^{-1}$ belong in $\mathcal{A}_{p, u_\alpha}$. Furthermore there exists a positive constant C_0 independent of the selection of the set X and the positive integer R such that*

$$\sup_{m, n \in X_R} \{ \| \{ u_\alpha(m-n)(\Phi_{\tau_\delta})_{m,n}^{-1} \}_m \|_{\ell_p(X_R)}, \| \{ u_\alpha(m-n)(\Phi_{\tau_\delta})_{m,n}^{-1} \}_n \|_{\ell_p(X_R)} \} \leq C_0.$$

Proof. For any finite set $Y \subset \mathbb{Z}$ there holds

$$\| \Phi_{\tau_\delta, Y} \|_{\mathcal{A}_{p, u_\alpha}} \leq \| \Phi_{\tau_\delta} \|_{\mathcal{A}_{p, u_\alpha}} < \infty. \tag{3.7}$$

Let $|X_R|$ be the cardinality of a set X_R as above and let

$$\Phi_{X_R}^\dagger = \sum_{\lambda \in \mathbb{Z}} \Phi_{\tau_\delta, Y_{X_R, \lambda}}$$

be a block-diagonal infinite matrix where $Y_{X_R, \lambda} = \{s + \lambda | X_R\} : s \in X_R$. Then for any $c \in \ell_2$ we have

$$\begin{aligned} \| \Phi_{X_R}^\dagger c \|_{\ell_2}^2 &= \sum_{m \in \mathbb{Z}} |(\Phi_{X_R}^\dagger c)_m|^2 = \sum_{s \in X_R} \sum_{l \in \mathbb{Z}} |(\Phi_{X_R}^\dagger c)_{s+l|X_R}|^2 \\ &= \sum_{s \in X_R} \sum_{l \in \mathbb{Z}} |(\Phi_{\tau_\delta, Y_{X_R, l}} c)_s|^2 = \sum_{l \in \mathbb{Z}} \| \Phi_{\tau_\delta, Y_{X_R, l}} c \|_{\ell_2}^2. \end{aligned}$$

By (3.6) we obtain

$$C^2 \|c\|_{\ell_2}^2 = C^2 \sum_{l \in \mathbb{Z}} \|P_{Y_{X_R, l}} c\|_{\ell_2}^2 \leq \sum_{l \in \mathbb{Z}} \| \Phi_{\tau_\delta, Y_{X_R, l}} c \|_{\ell_2}^2 \leq D^2 \sum_{l \in \mathbb{Z}} \|P_{Y_{X_R, l}} c\|_{\ell_2}^2 = D^2 \|c\|_{\ell_2}^2$$

and so

$$C \|c\|_2 \leq \| \Phi_{X_R}^\dagger c \|_2 \leq D \|c\|_2 \quad \text{for all } c \in \ell_2 \tag{3.8}$$

for some positive constants C, D as in Lemma 3. As a result the operator $\Phi_{X_R}^\dagger$ is bounded on ℓ_2 and has bounded inverse given by

$$(\Phi_{X_R}^\dagger)^{-1} = \sum_{\lambda \in \mathbb{Z}} (\Phi_{\tau_\delta, Y_{X_R, \lambda}})^{-1} \tag{3.9}$$

and thus by applying Wiener’s lemma for infinite matrices we obtain

$$\|(\Phi_{X_R}^\dagger)^{-1}\|_{\mathcal{A}^{p,u_\alpha}} \leq C_0$$

for some constant C_0 independent of the set X and the positive integer R (see Remark 1 and combine with Eqs. (3.7) and (3.8)). Since from (3.9) there holds

$$(\Phi_{X_R}^\dagger)^{-1}_{i,j} = (\Phi_{\tau_\delta, X_R})^{-1}_{i,j}$$

for any $i, j \in X_R$ the result is proved. \square

Now we can prove the main result of this section.

Theorem 2. Consider a space V_ϕ as in (1.1) generated from the integer shifts of a function ϕ as above and assume that V_ϕ admits a stable reconstruction formula (2.16) with respect to an ordered and ε -separated perturbed sampling set $\tau_\delta = \{\tau_n\}_{n \in \mathbb{Z}}$. For any $f \in V_\phi$ and for any bounded interval \mathcal{X} define by

$$f^*(x) = \sum_{n \in X_R} f(\tau_n) \varphi_n^{\tau_\delta}(x) \tag{3.10}$$

the finite reconstruction approximation of f on \mathcal{X} , where the set X_R is the R -neighborhood of the set $X = \{n \in \mathbb{Z} : \tau_n \in \mathcal{X}\}$,

$$\varphi_n^{\tau_\delta}(x) = \sum_{m \in X_{3R}} (\Phi_{\tau_\delta, X_{3R}})^{-1}_{m,n} \phi(x - m)$$

and $\Phi_{\tau_\delta, X_{3R}}^{-1}$ is the inverse of a square matrix $\Phi_{\tau_\delta, X_{3R}}$ as in (3.5). Then there exists a positive constant C independent of the bounded interval \mathcal{X} , the set X , the positive integer R and the function f such that the error when we reconstruct f on \mathcal{X} using the finite reconstruction approximation $f^*(x)$ is bounded by

$$\sup_{x \in \mathcal{X}} |f(x) - f^*(x)| < C \left(\frac{\|f(\tau_n)\|_{\ell_2(X_R)}}{R^{2\alpha - \frac{3}{2q}}} + \frac{\|f(\tau_n)\|_{\ell_2(\mathbb{Z} - X_R)}}{R^{\alpha - \frac{1}{2q}}} \right).$$

Here, the number $\alpha > 1 - \frac{1}{p}$ ($p \geq 1$) is the exponent of the polynomial weight $u_\alpha(x) = (1 + |x|)^\alpha$ related to the decay rate of ϕ and q is the conjugate exponent of p .

Proof. For any $f \in V_\phi$ and for any $x \in \mathcal{X}$ we have

$$f(x) - f^*(x) = \sum_{n \in X_R} f(\tau_n) (\psi_n^{\tau_\delta}(x) - \varphi_n^{\tau_\delta}(x)) + \sum_{n \notin X_R} f(\tau_n) \psi_n^{\tau_\delta}(x), \tag{3.11}$$

where the functions $\psi_n^{\tau_\delta}(x)$ are as in (2.17). First we deal with the second term in the right-hand side of (3.11) and we have

$$\left| \sum_{n \notin X_R} f(\tau_n) \psi_n^{\tau_\delta}(x) \right| \leq \|f(\tau_n)\|_{\ell_2(\mathbb{Z} - X_R)} \left(\sum_{n \notin X_R} |\psi_n^{\tau_\delta}(x)|^2 \right)^{1/2}.$$

Let $X = \{n \in \mathbb{Z} : \tau_n \in \mathcal{X}\}$ and let X_R be the R -neighborhood of X . By assumption the set τ_δ is an ordered and ε -separated set satisfying (2.4), so for any $x \in \mathcal{X}$ we have

$$|x - \tau_n| = \begin{cases} x - \tau_n \geq \tau_{\inf X - 1} - \tau_n \geq \varepsilon(\inf X - 1 - n), & \inf X > n, \\ \tau_n - x \geq \tau_n - \tau_{\sup X + 1} \geq \varepsilon(n - \sup X - 1), & \sup X < n. \end{cases}$$

Taking into account the above relation we compute

$$\begin{aligned} \sup_{x \in \mathcal{X}} \sum_{n \notin X_R} |\psi_n^{\tau_\delta}(x)|^2 &= \sup_{x \in \mathcal{X}} \sum_{n \notin X_R} |u_\alpha(x - \tau_n) \psi_n^{\tau_\delta}(x)|^2 u_\alpha^{-2}(x - \tau_n) \\ &\leq \sup_{x \in \mathcal{X}} \| \{u_\alpha(x - \tau_n) \psi_n^{\tau_\delta}(x)\}_n \|_{\ell_p(\mathbb{Z} - X_R)}^2 \| \{u_\alpha(x - \tau_n)^{-2}\}_n \|_{\ell_q(\mathbb{Z} - X_R)} \\ &\leq \sup_{x \in \mathcal{X}} \| \{u_\alpha(x - \tau_n) \psi_n^{\tau_\delta}(x)\}_n \|_{\ell_p(\mathbb{Z} - X_R)}^2 \| \{u_\alpha(\varepsilon n)^{-2}\}_n \|_{\ell_q(\mathbb{Z} - \{-R, \dots, R\})} \\ &\leq C_0 \| \{u_\alpha(\varepsilon n)^{-2}\}_n \|_{\ell_q(\mathbb{Z} - \{-R, \dots, R\})} < \frac{C_1}{R^{2\alpha - \frac{1}{q}}} \end{aligned} \tag{3.12}$$

for some positive constant C_1 depending on $\alpha, \varepsilon, q, C_0$. Here the constant C_0 is as in (3.2), $\alpha > 1 - \frac{1}{p}$ for some $p \geq 1$ is the exponent of a polynomial weight $\alpha(x) = (1 + |x|)^\alpha$ related with the decay of the generator ϕ (see condition (P₁) at the beginning of Section 2 and compare with Corollary 2), q is the conjugate exponent of p and the number ε is as in (2.4).

In order to compute an upper bound for the first term in (3.11) we need estimates for $\sup_{x \in \mathcal{X}} \sum_{n \in X_R} |\psi_n^{\tau_\delta}(x) - \varphi_n^{\tau_\delta}(x)|^2$. By definition $\psi_n^{\tau_\delta}(x) = \sum_{m \in \mathbb{Z}} (\Phi_{\tau_\delta})_{m,n}^{-1} \phi(x - m)$ or equivalently

$$\phi(x - l) = \sum_{n \in \mathbb{Z}} (\Phi_{\tau_\delta})_{n,l} \psi_n^{\tau_\delta}(x) = \sum_{n \in X_{3R}} \phi(\tau_n - l) \psi_n^{\tau_\delta}(x) + \sum_{n \notin X_{3R}} \phi(\tau_n - l) \psi_n^{\tau_\delta}(x).$$

Since the projection matrix $\Phi_{\tau_\delta, X_{3R}} = \{ \phi(\tau_n - l) : n, l \in X_{3R} \}$ is invertible as a result of Lemma 3, we multiply both sides of the above equality with the inverse matrix $\Phi_{\tau_\delta, X_{3R}}^{-1}$ and we obtain

$$\begin{aligned} \sum_{k \in X_R} |\psi_k^{\tau_\delta}(x) - \varphi_k^{\tau_\delta}(x)|^2 &= \sum_{k \in X_R} \left| \sum_{l \in X_{3R}} (\Phi_{\tau_\delta, X_{3R}})_{l,k}^{-1} \sum_{n \notin X_{3R}} (\Phi_{\tau_\delta})_{n,l} \psi_n^{\tau_\delta}(x) \right|^2 \\ &\leq 2 \sum_{k \in X_R} \left| \sum_{l \in X_{2R}} (\Phi_{\tau_\delta, X_{3R}})_{l,k}^{-1} \sum_{n \notin X_{3R}} (\Phi_{\tau_\delta})_{n,l} \psi_n^{\tau_\delta}(x) \right|^2 \\ &\quad + 2 \sum_{k \in X_R} \left| \sum_{l \in X_{3R} - X_{2R}} (\Phi_{\tau_\delta, X_{3R}})_{l,k}^{-1} \sum_{n \notin X_{3R}} (\Phi_{\tau_\delta})_{n,l} \psi_n^{\tau_\delta}(x) \right|^2. \end{aligned} \tag{3.13}$$

First we deal with the first term of the right-hand side of (3.13). Taking into account the decay estimates obtained in Lemmas 2 and 4 we compute

$$\begin{aligned} &\sup_{x \in \mathcal{X}} \sum_{k \in X_R} \left| \sum_{l \in X_{2R}} (\Phi_{\tau_\delta, X_{3R}})_{l,k}^{-1} \sum_{n \notin X_{3R}} (\Phi_{\tau_\delta})_{n,l} \psi_n^{\tau_\delta}(x) \right|^2 \\ &\leq \sup_{x \in \mathcal{X}} \sum_{k \in X_R} \left(\sum_{l \in X_{2R}} |(\Phi_{\tau_\delta, X_{3R}})_{l,k}^{-1}| \right) \sum_{l \in X_{2R}} \left| \sum_{n \notin X_{3R}} (\Phi_{\tau_\delta})_{n,l} \psi_n^{\tau_\delta}(x) \right|^2 |(\Phi_{\tau_\delta, X_{3R}})_{l,k}^{-1}| \\ &\leq \|\Phi_{\tau_\delta, X_{3R}}^{-1}\|_{\mathcal{A}_{1,u_0}}^2 \sup_{x \in \mathcal{X}} \sum_{l \in X_{2R}} \left| \sum_{n \notin X_{3R}} (\Phi_{\tau_\delta})_{n,l} \psi_n^{\tau_\delta}(x) \right|^2 \\ &\leq \|\Phi_{\tau_\delta, X_{3R}}^{-1}\|_{\mathcal{A}_{1,u_0}}^2 \sup_{x \in \mathcal{X}} \sum_{l \in X_{2R}} \left(\sum_{n \notin X_{3R}} |(\Phi_{\tau_\delta})_{n,l}| \right) \sum_{n \notin X_{3R}} |\psi_n^{\tau_\delta}(x)|^2 |(\Phi_{\tau_\delta})_{n,l}| \\ &\leq \|\Phi_{\tau_\delta, X_{3R}}^{-1}\|_{\mathcal{A}_{1,u_0}}^2 \left(\sup_{l \in X_{2R}} \sum_{n \notin X_{3R}} |(\Phi_{\tau_\delta})_{n,l}| \right) \left(\sup_{n \notin X_{3R}} \sum_{l \in X_{2R}} |(\Phi_{\tau_\delta})_{n,l}| \right) \left(\sup_{x \in \mathcal{X}} \sum_{n \notin X_{3R}} |\psi_n^{\tau_\delta}(x)|^2 \right) \\ &\leq \|\Phi_{\tau_\delta, X_{3R}}^{-1}\|_{\mathcal{A}_{1,u_0}}^2 \|\Phi_{\tau_\delta}\|_{\mathcal{A}_{p,u_\alpha}}^2 \left(\sup_{l \in X_{2R}} \|\{u_\alpha(n-l)\}_n\|_{\ell_q(\mathbb{Z} - X_{3R})} \right) \\ &\quad \times \left(\sup_{n \notin X_{3R}} \|\{u_\alpha(n-l)\}_l\|_{\ell_q(\mathbb{Z} - X_{2R})} \right) \left(\sup_{x \in \mathcal{X}} \sum_{n \notin X_{3R}} |\psi_n^{\tau_\delta}(x)|^2 \right) \\ &< \|\Phi_{\tau_\delta, X_{3R}}^{-1}\|_{\mathcal{A}_{1,u_0}}^2 \|\Phi_{\tau_\delta}\|_{\mathcal{A}_{p,u_\alpha}}^2 \frac{C_2}{R^{2\alpha-2/q}} \frac{C_1}{R^{2\alpha-1/q}} = \frac{C'}{R^{4\alpha-\frac{3}{q}}}, \end{aligned} \tag{3.14}$$

where the constant C_1 is as in (3.12) and the constant C_2 depends on α and q . We notice that the overall constant C' does not depend on the set X or the positive integer R because the norm $\|\Phi_{\tau_\delta, X_{3R}}^{-1}\|_{\mathcal{A}_{1,u_0}}$ is independent of the set X and the positive integer R as we showed in Lemma 4.

We work similarly for the second term of (3.13). In this case we have

$$\begin{aligned} &\sup_{x \in \mathcal{X}} \sum_{k \in X_R} \left| \sum_{l \in X_{3R} - X_{2R}} (\Phi_{\tau_\delta, X_{3R}})_{l,k}^{-1} \sum_{n \notin X_{3R}} (\Phi_{\tau_\delta})_{n,l} \psi_n^{\tau_\delta}(x) \right|^2 \\ &\leq \sup_{x \in \mathcal{X}} \sum_{k \in X_R} \left(\sum_{l \in X_{3R} - X_{2R}} |(\Phi_{\tau_\delta, X_{3R}})_{l,k}^{-1}| \right) \left(\sum_{l \in X_{3R} - X_{2R}} \left| \sum_{n \notin X_{3R}} (\Phi_{\tau_\delta})_{n,l} \psi_n^{\tau_\delta}(x) \right|^2 |(\Phi_{\tau_\delta, X_{3R}})_{l,k}^{-1}| \right) \end{aligned}$$

$$\begin{aligned}
 &\leq \left(\sup_{k \in X_R} \sum_{l \in X_{3R} - X_{2R}} |(\Phi_{\tau_\delta, X_{3R}}^{-1})_{l,k}| \right) \left(\sup_{l \in X_{3R} - X_{2R}} \sum_{k \in X_R} |(\Phi_{\tau_\delta, X_{3R}}^{-1})_{l,k}| \right) \left(\sup_{x \in \mathcal{X}} \sum_{l \in X_{3R} - X_{2R}} \left| \sum_{n \notin X_{3R}} (\Phi_{\tau_\delta})_{n,l} \psi_n^{\tau_\delta}(x) \right|^2 \right) \\
 &\leq \|\Phi_{\tau_\delta, X_{3R}}^{-1}\|_{\mathcal{A}_{p,u_\alpha}}^2 \left(\sup_{k \in X_R} \| \{u_\alpha^{-1}(l-k)\}_l \|_{\ell_q(X_{3R} - X_{2R})} \right) \left(\sup_{l \in X_{3R} - X_{2R}} \| \{u_\alpha^{-1}(l-k)\}_k \|_{\ell_q(X_R)} \right) \\
 &\quad \times \left(\sum_{l \in X_{3R} - X_{2R}} |(\Phi_{\tau_\delta})_{n,l}| \right) \sup_{x \in \mathcal{X}} \left(\sum_{l \in X_{3R} - X_{2R}} \sum_{n \notin X_{3R}} |\psi_n^{\tau_\delta}(x)|^2 |(\Phi_{\tau_\delta})_{n,l}| \right) \\
 &< \|\Phi_{\tau_\delta, X_{3R}}^{-1}\|_{\mathcal{A}_{p,u_\alpha}}^2 \|\Phi_{\tau_\delta}\|_{\mathcal{A}_{1,u_0}}^2 \frac{C_3}{R^{2\alpha-2/q}} \sup_{x \in \mathcal{X}} \sum_{n \notin X_{3R}} |\psi_n^{\tau_\delta}(x)|^2 \\
 &< \|\Phi_{\tau_\delta, X_{3R}}^{-1}\|_{\mathcal{A}_{p,u_\alpha}}^2 \|\Phi_{\tau_\delta}\|_{\mathcal{A}_{1,u_0}}^2 \frac{C_3}{R^{2\alpha-2/q}} \frac{C_1}{R^{2\alpha-1/q}} = \frac{C''}{R^{4\alpha-\frac{3}{q}}}, \tag{3.15}
 \end{aligned}$$

where the constant C_1 is as in (3.12) and the constant C_3 depends on α and q . The overall constant C'' does not depend on the set X or the positive integer R for the same reasons as above. The bound (3.15) and the bound (3.14) are applied to (3.13). The resulting bound together with the bound (3.12) are applied to (3.11) and for $C = \max\{C_1, C' + C''\}$ the result is obtained. \square

Remark 2. We notice that the decay rate obtained in Theorem 2 is smaller than the decay rate obtained in [19, Theorem 6.2].

4. Jitter error

The sampling reconstruction formula (2.16) and the local reconstruction formula (3.10) can be applied if we are aware of the sampling values $\tau_n = n + \delta_n, n \in \mathbb{Z}$. In some cases sampled data $f(n)$ are perturbed without our knowledge (i.e. $\Delta = \{\delta_n\}$ is unknown). Then when we reconstruct the signal using (1.3) we are facing jitter. In this section we deal with the following problem:

Let V_ϕ be a subspace of $L_2(\mathbb{R})$ as above and let $\{n + \delta_n\}_{n \in \mathbb{Z}}$ be a set of stable sampling for V_ϕ for any perturbation $\Delta = \{\delta_n\}_{n \in \mathbb{Z}}$ bounded by a positive number δ_ϕ as in (2.5). Given a function $f \in V_\phi$, a set of sampled data $\mathcal{L}(f) = \{f(\tau_n)\}_{n \in \mathbb{Z}}$ on a sampling set $\tau_\delta = \{n + \delta_n\}_{n \in \mathbb{Z}}$ (where the elements δ_n are unknown) and a pre-determined error $\epsilon > 0$ we want to find a number $\delta_{\phi,\epsilon}$ such that for any perturbation Δ bounded by $\delta_{\phi,\epsilon}$ there holds

$$\sup_{x \in \mathbb{R}} \left| f(x) - \sum_{n \in \mathbb{Z}} f(\tau_n) \psi_n^\tau(x) \right| < \epsilon$$

where $\tau = \mathbb{Z}$ and $\psi_n^\tau(x), n \in \mathbb{Z}$ as in (2.17). We have

Proposition 1. *If the space V_ϕ admits a stable reconstruction formula as in (2.16) with respect to a sampling set $\tau_\delta = \{\tau_n\}_{n \in \mathbb{Z}}$, then for any $f \in V_\phi$ with norm $\|f\|_{L_2} \leq c < \infty$ we have*

$$\sup_{x \in \mathbb{R}} \left| f(x) - \sum_{n \in \mathbb{Z}} f(\tau_n) \psi_n^\tau(x) \right| \leq \frac{c \|\phi\|_{W_2(L_\infty, u_0)}}{A \|\Phi^\dagger\|_0} G_\phi(\delta),$$

where $\tau = \mathbb{Z}$ and ψ_n^τ as in (2.17). Here the functions $\Phi^\dagger(\gamma)$ and $G_\phi(x)$ are as in (2.3) and (2.6) respectively and A is the lower Riesz bound of the set $\{\phi(\cdot - n)\}_{n \in \mathbb{Z}}$ as in (1.2).

Proof. We have

$$\begin{aligned}
 \sup_{x \in \mathbb{R}} \left| f(x) - \sum_{n \in \mathbb{Z}} f(\tau_n) \psi_n^\tau(x) \right| &\leq \sup_{x \in \mathbb{R}} \sum_{n \in \mathbb{Z}} |f(\tau_n) - f(n)| |\psi_n^\tau(x)| \\
 &\leq \left(\sum_{n \in \mathbb{Z}} |f(\tau_n) - f(n)|^2 \right)^{1/2} \sup_{x \in \mathbb{R}} \left(\sum_{n \in \mathbb{Z}} |\psi_n^\tau(x)|^2 \right)^{1/2}. \tag{4.1}
 \end{aligned}$$

Let $F(x) = \{\phi(x - m)\}_{m \in \mathbb{Z}}$ and Φ be as in (2.2). Recalling (2.17) and using (2.3) we have

$$\begin{aligned}
 \sum_{n \in \mathbb{Z}} |\psi_n^\tau(x)|^2 &= \sum_{n \in \mathbb{Z}} \left| \sum_{m \in \mathbb{Z}} \Phi_{m,n}^{-1} \phi(x - m) \right|^2 = \|\Phi^{-1} F(x)\|_{\ell_2}^2 \\
 &\leq \|\Phi^{-1}\|_{\mathcal{B}^2}^2 \|F(x)\|_{\ell_2}^2 \leq \|\Phi^\dagger\|_0^{-2} \|\phi\|_{W_2(L_\infty, u_0)}^2. \tag{4.2}
 \end{aligned}$$

On the other hand if $f(\tau_n) = \sum_{m \in \mathbb{Z}} c_m \phi(\tau_n - m)$ for some unique $c \in \ell_2$ and if Φ_{τ_δ} is an infinite matrix as in Definition 1, then from (3.12) and (1.2) we compute

$$\sum_{n \in \mathbb{Z}} |f(\tau_n) - f(n)|^2 = \|(\Phi_{\tau_\delta} - \Phi)c\|_{\ell_2}^2 \leq G_\phi^2(\delta) \|c\|_{\ell_2}^2 \leq \frac{G_\phi^2(\delta)}{A^2} \|f\|_{L_2}^2, \quad (4.3)$$

where the function $G_\phi(x)$ is as in (2.6). Substituting (4.2) and (4.3) into (4.1) we get the result. \square

Corollary 3. *Under the assumptions of Proposition 1 there exists a positive real number*

$$\delta_{\phi, \epsilon} = \inf \left\{ \delta \in \mathbb{R}^+ : G_\phi(\delta) \geq \frac{A \|\Phi^\dagger\|_0}{c \|\phi\|_{W_2(L_\infty, u_0)}} \right\}$$

such that for any selection of a function $f \in V_\phi$ with norm $\|f\|_{L_2} \leq c$ and for any selection of an ordered and ϵ -separated perturbed sampling set τ_δ with $\delta < \min\{\delta_{\phi, \epsilon}, \delta_\phi\}$ we have

$$\sup_{x \in \mathbb{R}} \left| f(x) - \sum_{n \in \mathbb{Z}} f(\tau_n) \psi_n^\tau(x) \right| < \epsilon.$$

Proof. Immediate consequence of Proposition 1. \square

References

- [1] A. Aldroubi, K. Gröchenig, Beurling–Landau-type theorems for non-uniform sampling in shift invariant spline spaces, *J. Fourier Anal. Appl.* 6 (1) (2000) 93–103.
- [2] A. Aldroubi, C. Leonetti, Non uniform sampling and reconstruction from sampling sets with unknown jitter, *Sampl. Theory Signal Image Process.* 7 (2) (2008) 187–195.
- [3] N. Atreas, N. Bagis, C. Karanikas, The information loss error and the jitter error for regular sampling expansions, *Sampl. Theory Signal Image Process.* 1 (3) (2002) 261–276.
- [4] P.L. Butzer, W. Splettstößer, On quantization, truncation and jitter errors in the sampling theorem and its generalizations, *Signal Process.* 2 (2) (1980) 101–112.
- [5] W. Chen, S. Itoh, J. Shiki, Irregular sampling theorems on wavelet subspaces, *IEEE Trans. Inform. Theory* 44 (3) (1998) 1131–1142.
- [6] W. Chen, S. Itoh, J. Shiki, On sampling in shift invariant spaces, *IEEE Trans. Inform. Theory* 48 (10) (2002) 2802–2810.
- [7] H.G. Feichtinger, K. Gröchenig, Theory and practice of irregular sampling, in: J. Benedetto, M. Frazier (Eds.), *Wavelets: Mathematics and Applications*, CRC Press, 1994, pp. 305–365.
- [8] S. Jaffard, Propriétés des matrices bien localisées près de leur diagonale et quelques applications, *Ann. Inst. H. Poincaré* 7 (1990) 461–476.
- [9] K. Gröchenig, M. Leinert, Symmetry and inverse-closedness of matrix algebras and functional calculus for infinite matrices, *Trans. Amer. Math. Soc.* 358 (6) (2006) 2695–2711.
- [10] K. Gröchenig, Z. Rzeszutnik, T. Strohmer, Convergence analysis of the finite section method and Banach algebras of matrices, *Integral Equations Operator Theory* 67 (2010) 183–202.
- [11] K. Gröchenig, Wiener’s lemma: Theme and variations. An introduction to spectral invariance and its applications, in: B. Forster, P. Massopust (Eds.), *Four Short Courses on Harmonic Analysis*, in: *Appl. Numer. Harmon. Anal.*, Birkhäuser, 2010, pp. 175–234.
- [12] H.C. Hak, C.E. Shin, Perturbation of non harmonic Fourier series and non-uniform sampling theorem, *Bull. Korean Math. Soc.* 44 (2) (2007) 351–358.
- [13] J.R. Higgins, *Sampling Theory in Fourier and Signal Analysis: Foundations*, Oxford Univ. Press, Oxford, 1996.
- [14] C. Leonetti, *Reconstruction from error-affecting sampled data in shift invariant spaces*, PhD thesis, Department of Mathematics, Vanderbilt University, Nashville, TN, 2007.
- [15] Y. Liu, G. Walter, Irregular sampling in wavelet spaces, *J. Fourier Anal. Appl.* 2 (2) (1995) 181–189.
- [16] F. Marvasti, *Non-Uniform Sampling: Theory and Practice*, Kluwer Acad. Publ., New York, 2001.
- [17] M.Z. Nashed, Q. Sun, Sampling and reconstruction of signals in a reproducing kernel subspace of $L_p(\mathbb{R})^d$, *J. Funct. Anal.* 258 (7) (2010) 2422–2452.
- [18] M.Z. Nashed, G.G. Walter, General sampling theorems for functions in reproducing kernel Hilbert spaces, *Math. Control Signals Systems* 4 (1991) 363–390.
- [19] Q. Sun, Non-uniform average sampling and reconstruction of signals with finite rate of innovation, *SIAM J. Math. Anal.* 38 (2006) 1389–1422.
- [20] Q. Sun, Wiener’s lemma for infinite matrices, *Trans. Amer. Math. Soc.* 359 (2007) 3099–3123.
- [21] Q. Sun, Wiener’s lemma for infinite matrices with polynomial off diagonal decay, *C. R. Acad. Sci. Paris Ser. 1* 340 (2005) 567–570.
- [22] Q. Sun, Wiener’s lemma for infinite matrices II, *Constr. Approx.*, doi:10.1007/s00365-010-9121-8.
- [23] G.G. Walter, *Wavelets and Other Orthogonal Systems with Applications*, CRC Press, 1994.
- [24] A. Zayed, *Advances in Shannon’s Sampling Theory*, CRC Press, Boca Raton, FL, 1993.
- [25] P. Zhao, C. Zhao, P.G. Casazza, Perturbation of regular sampling in shift-invariant spaces for frames, *IEEE Trans. Inform. Theory* 52 (10) (2006) 4643–4648.