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Invariant Subspaces of Certain Subnormal Operators*

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Let T be a subnormal, nonnormal operator on a Hilbert space and suppose that the point spectrum of T^* is empty. Then there exist vectors $x \neq 0$ for which $(T^* - zI)^{-1}x$ exists and is weakly continuous for all z. It is shown that under certain conditions, the Cauchy integral of this vector function taken around an appropriate contour, not necessarily lying in the resolvent set of T^*, leads to a proper (nontrivial) invariant subspace of T^*.

1. INTRODUCTION

Let T be a subnormal operator on an infinite-dimensional Hilbert space $\mathfrak{H}$ with the minimal normal extension N on $\mathfrak{K} \supset \mathfrak{H}$. It was shown by Bram [1] that sp(T) is obtained by combining sp(N) with some of the holes of this latter set (cf. [8], p. 102). (A hole in a compact set X is a bounded component of its complement $\mathbf{C} - X$.) Thus, if $z \in \text{sp}(T) - \text{sp}(N)$, then z belongs to one of these holes. Moreover, if x ranges over the unit vectors in $\mathfrak{H}$ then $\inf \|(T^* - \bar{z})x\| = 0$, while

$$\inf \|(T - z)x\| = \inf \|(N - z)x\| \geqslant \text{dist}(z, \text{sp}(T)) > 0,$$

and hence $\bar{z}$ belongs to the point spectrum of T^*. In particular, T surely has a nontrivial invariant subspace ($\neq 0, \mathfrak{H}$) if sp(N) is a proper subset of sp(T).

For any compact set X, let $C(X)$ denote the continuous functions on X, and $R(X)$ the functions on X which are uniformly approximable on X by rational functions with poles off X. It was shown by Clancey and Putnam [4] that if U is any open disk satisfying $\text{sp}(T) \cap U \neq \varnothing$ and if $R(\text{sp}(T) \cap U^-) = C(\text{sp}(T) \cap U^-)$ then T has a normal part, thus, $T = T_1 \oplus N$, where N ($\neq 0$) is normal. In view of the earlier

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remarks, it is clear that if $\text{sp}(N) \cap U \neq \varnothing$ and if $R(\text{sp}(N) \cap U^-) = C(\text{sp}(N) \cap U^-)$, then T has a nontrivial invariant subspace. In the special case in which $R(\text{sp}(N)) = C(\text{sp}(N))$, this result is essentially contained in Wermer [18]. (Actually, Wermer considers the case where $\text{sp}(N)$ has planar measure 0, so that $R(\text{sp}(N)) = C(\text{sp}(N))$ by the Hartogs–Rosenthal Theorem; see also Brennan [2] and Yoshino [19].)

A closed subset Q of a compact set X is said to be a peak set of $R(X)$ if there is a function $f \in R(X)$ for which $f(z) = 1$ for $z \in Q$ and $|f(z)| < 1$ for $z \in X - Q$ (see [6], p. 56). It has recently been shown by Lautzenheiser ([10], p. 84), that if T is subnormal, or even if $\text{sp}(T)$ is a spectral set of T, and if Q is a proper peak set of $R(\text{sp}(T))$ with the property that

$$Q \not\subset (\text{sp}(T) - Q)^-, \tag{1.1}$$

then T has a nontrivial reducing subspace. The condition (1.1) is used to ensure that a certain direct sum representation of T is nontrivial. It may be remarked that Lautzenheiser also extends certain results of Sarason [15] on the existence of reducing spaces of an operator T and that the methods of both authors involve the notion of Gleason parts of uniform algebras.

In this paper use will be made of the projection "operator" $P = P_C$ defined by

$$Px = P_C x = -(2\pi i)^{-1} \int_C (T^* - t)^{-1} x \, dt, \qquad \text{for} \quad x \in L, \tag{1.2}$$

where T is subnormal, L is the class of vectors defined in (1.7) below, and C is the rectifiable boundary of a bounded region (connected open set) S. It will be assumed that C consists of an outer rectifiable simple closed curve and possibly also a finite number of nonintersecting inner rectifiable simple closed curves having disjoint interiors and lying inside the outer curve, and that C is oriented positively with respect to S. It is not assumed however that C lies in the resolvent set of T^* (as is the usual case, cf. Riesz and Sz.-Nagy [12], p. 421). By $\text{int}(C)$ and $\text{ext}(C)$ will be meant the sets S and $\mathbf{C} - S^-$, respectively. The notion of a peak set will again play a role in establishing the existence of nontrivial invariant (though possibly not reducing) spaces of certain subnormals T, even though the condition (1.1) need not hold. First, some preliminaries will be set forth.

Since T is subnormal, it is also hyponormal, i.e.,

$$T^*T - TT^* = D \geqslant 0. \tag{1.3}$$

Suppose that

$$\text{the point spectrum of } T^* \text{ is empty,} \tag{1.4}$$

and hence, by (1.3), the point spectrum of T is also empty, and let D have the spectral resolution

$$D = \int_0^\infty t\, dF_t\,. \tag{1.5}$$

Since $T_z{}^*T_z = T_zT_z{}^* + D \geqslant D$, where $T_z = T - z$, it follows from Putnam [11] that if $x = F(s, \infty)x$, where $s > 0$, then, for any y in $\mathfrak{H}$,

$$F(z) = ((T^* - z)^{-1}\, x, y) \text{ is continuous in } \mathbf{C} \text{ and analytic for } z \notin \operatorname{sp}(T^*). \tag{1.6}$$

Note that such vectors x are dense in the range of D. Further, it is clear that (1.6) remains valid if x is replaced by $T^{*n}x$ for $n = 0, 1, 2, \ldots$, ($T^0 = I$). Hence, if L denotes the linear manifold of finite linear combinations of these vectors, so that

$$L = \left\{x = \sum_{n=1}^{N} T^{*n}F(s_n\ , \infty)\, x_n : s_n > 0, x_n \in \mathfrak{H}\right\}, \tag{1.7}$$

then (1.6) holds for any x in L. Next, let S denote a region with boundary C, as described above, and define $Px = P_Cx$ by (1.2). (Note that $(Px, y) = \int_C F(t)\, dt$ exists as a Riemann integral.) For later use, note that the linear manifold $P_C(L) = \{P_Cx\colon x \in L\}$ is invariant under T^*.

Incidentally, there exist subnormal operators which are not normal and for which (1.4) holds; examples have been given by Clancey and Morrel [3], using a result of Brennan [2].

There will be proved the following two lemmas.

LEMMA 1. *Let T be subnormal on $\mathfrak{H}$ with the minimal normal extension N on $\mathfrak{K} \supset \mathfrak{H}$ and suppose* (1.4). *Let N^* have the spectral resolution*

$$N^* = \int t\, dE_t\,, \tag{1.8}$$

let S, C and P_Cx be defined as above, and let

$$dm(t) = d\,\|\, E_t(P_Cx)\|^2, \qquad x \in L, \tag{1.9}$$

as a Stieltjes differential. Then

$$\int_K F(t)\, dm(t) = \int_C F(t) f(t)\, dt, \qquad F \in R(K_0), \tag{1.10}$$

where $f(t) = -((T^* - t)^{-1}x, P_C x)/2\pi i$ *(which is continuous in* $\mathbf{C}$*),* $K = \mathrm{sp}(T^*)$*, and* $K_0 = K \cup S^- \ (= K \cup (\mathrm{int}(C))^-)$.

LEMMA 2. *Under the conditions of Lemma* 1, *let* Q *be a peak set of* $R(K_0)$ *and suppose that*

$$\mathrm{meas}_1(Q \cap C) = 0, \tag{1.11}$$

the measure denoting arc length on C. *Then*

$$E(Q)\, P_C x = 0. \tag{1.12}$$

If, in addition,

$$E(Q) \neq 0 \qquad \textit{and} \qquad P_C(L) \neq 0, \tag{1.13}$$

where L *is defined in* (1.7), *then* $M_C = (P_C(L))^-$ *is a nontrivial invariant subspace of* T^* *satisfying*

$$M_C \perp R_{E(Q)}\,. \tag{1.14}$$

The proofs of the lemmas will be given in Sections 2 and 3. Various consequences concerning the existence of invariant spaces for certain subnormal operators will be obtained in the theorems of Sections 4–7.

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Proof of Lemma 1. The vector function

$$A_x(z) = -(2\pi i)^{-1} \int_C (t - z)^{-1} (T^* - t)^{-1} x\, dt$$

is clearly analytic for $z \notin C$, in particular, for $z \in \mathrm{ext}(C)$ $(= \mathbf{C} - S^-)$. Since $T^* - z = T^* - t + (t - z)$, one sees that $(T^* - z)\, A_x(z) = Px$ $(= P_C x)$ for $z \in \mathrm{ext}(C)$ and hence, in particular, that

$$G(z) = ((T^* - z)^{-1} Px, Px), \tag{2.1}$$

is analytic for $z \in \text{ext}(C)$. If $z \notin \text{sp}(T^*)$, then

$$G(z) = (Px, (T - \bar{z})^{-1} Px) = (Px, (N - \bar{z})^{-1} Px) = ((N^* - z)^{-1} Px, Px).$$

(Note that if $u = (T - \bar{z})^{-1} Px$, then $Px = (T - \bar{z})u = (N - \bar{z})u$ and hence $u = (N - \bar{z})^{-1} Px$.) Consequently, by (1.8),

$$\hat{m}(z) = G(z), \qquad z \notin \text{sp}(T^*), \tag{2.2}$$

where $\hat{m}(z)$ denotes the Cauchy transform

$$\hat{m}(z) = \int (t - z)^{-1}\, dm(t), \qquad dm(t) = d \| E_t Px \|^2. \tag{2.3}$$

(For properties of the Cauchy transform, see [6], p. 46, [20], p. 118, or, for a detailed exposition, [7], Chap. II.)

Next, note that $G(z)$ of (2.1) has the representation $G(z) = -(2\pi i)^{-1} \int_C (t - z)^{-1} ((T^* - t)^{-1}x, Px)\, dt$, for $z \in \text{ext}(C)$, so that if $\hat{n}(z) = \int_C (t - z)^{-1}\, dn(t)$, where n denotes the measure on C defined by $dn(t) = -((T^* - t)^{-1}x, Px)\, dt/2\pi i$, then $G(z) = \hat{n}(z)$ for $z \in \text{ext}(C)$. Consequently, if $dm_1(z) = dm(z) - dn(z)$, then, by (2.2),

$$\hat{m}_1(z) = \int (t - z)^{-1}\, dm_1(t) = 0 \qquad \text{for} \quad z \in \text{ext}(C) - \text{sp}(T^*). \tag{2.4}$$

Hence, relation (1.10) holds for rational functions F with poles off $K_0 = K \cup (\text{int}(C))^-$, while its validity for any $F \in R(K_0)$ follows by taking uniform limits.

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Proof of Lemma 2. Let $h \in R(K_0)$ be a peak function of Q, so that $h = 1$ on Q and $|h| < 1$ on $K_0 - Q$. If $F = F_n = h^n$ for $n = 1, 2,\ldots,$ in (1.10), then $F_n \in R(K_0)$, $|F_n| \leqslant 1$ on K_0 and $F_n(t) \to 1$ or 0 according as $t \in Q$ or $t \in K_0 - Q$. Hence, by (1.10), (1.11) and Lebesgue's uniform boundedness term by term integration theorem, $\int_{K \cap Q} dm(t) = 0$, that is, since $E(\mathbf{C} - K) = 0$, the relation (1.12).

Next, suppose that (1.13) holds. Then clearly $M_C \neq 0$ is an invariant space of T^* satisfying (1.14), and there remains only to show that $M_C \neq \mathfrak{H}$. But if $M_C = \mathfrak{H}$, then, by (1.14), $R_{E(Q)} \subset \mathfrak{K} \ominus \mathfrak{H}$ and hence $\mathfrak{H} \subset \mathfrak{K} \ominus R_{E(Q)}$. Since this last space clearly reduces N, it follows from the minimal property of N that $\mathfrak{R} \ominus R_{E(Q)} = \mathfrak{R}$, that is, $E(Q) = 0$, in contradiction to (1.13). This completes the proof of Lemma 2.

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THEOREM 1. *Let T be subnormal on $\mathfrak{H}$ with the minimal normal extension N. Suppose that T is not normal and that the point spectrum of T^* is empty. Let N^* have the spectral resolution* (1.8) *and let C be any rectifiable simple closed curve. Let $K_0 = K \cup (\text{int}(C))^-$ and $K_1 = K \cup (U^- - (\text{int}(C)))$, where $K = \text{sp}(T^*)$ and U is some open disk containing K and C. Suppose that, for $j = 0$ and 1, Q_j is a peak set of $R(K_j)$ and that $\text{meas}_1(Q_j \cap C) = 0$ and $E(Q_j) \neq 0$. Then T^* has a non-trivial invariant subspace M satisfying either $M \perp R_{E(Q_0)}$ or $M \perp R_{E(Q_1)}$.*

Proof. If the linear manifold $P_C(L) \neq 0$, the theorem is a consequence of Lemma 2 with Q corresponding to Q_0. On the other hand, if $P_C(L) = 0$, then let C_1 denote the (positively oriented) region $U - (\text{int}(C))^-$. The boundary of U surrounds K, and so $P_{C_1}x = x - P_C x = x$ for any x in L of (1.7). Since T is not normal, then $P_{C_1}(L) = L \neq 0$, hence also $M_{C_1} = (P_{C_1}(L))^- \neq 0$. Since $Q_j \cap C_1 = Q_j \cap C$ $(j = 0, 1)$, then, in particular, $\text{meas}_1(Q_1 \cap C_1) = 0$. Consequently, Lemma 2, with Q, K_0 and C replaced by Q_1, K_1 and C_1, can be again applied and this completes the proof of Theorem 1.

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THEOREM 2. *Let T be subnormal on $\mathfrak{H}$ with the minimal normal extension N. Suppose that T is not normal and that the point spectrum of T^* is empty. Let N^* have the spectral resolution* (1.8) *and let C be a rectifiable simple closed curve satisfying*

$$K \cap (\text{int}(C)) = \varnothing, \qquad K = \text{sp}(T^*). \tag{5.1}$$

Suppose that $Q \subset (K \cap C)$ and that Q is a peak set of $K_1 = U^- - \text{int}(C)$ $(\supset K)$, where U is some open disk containing K and C, and that also $E(Q) \neq 0$ and $\text{meas}_1(Q \cap C) = 0$. Then T^* has 'a nontrivial invariant subspace M satisfying $M \perp R_{E(Q)}$.

Proof. Let C' be a rectifiable simple closed curve, and consider the vector $P_{C'}x$ defined by (1.2) with $C = C'$, thus

$$P_{C'}x = -(2\pi i)^{-1} \int_{C'} (T^* - t)^{-1} x \, dt,$$

for any fixed $x \neq 0$, $x \in L$. If $C' \subset \operatorname{int}(C)$, then $P_{C'}x = 0$ by (5.1), (1.6), and Cauchy's Theorem. Since, by (1.6), $(T^* - t)^{-1}x$ is a weakly continuous function of t in $\mathbf{C}$, then clearly $P_{C'}x$ is also a weakly continuous function of C' under continuous deformations of C' in which also the arc length of C' varies continuously. It follows that $P_C(L) = 0$. If C_1 is defined as in Section 4, the remainder of the proof is like that of Theorem 1.

COROLLARY OF THEOREM 2. *Let T be subnormal on $\mathfrak{H}$ with the minimal normal extension N. Suppose that T is not normal and that the point spectrum of T^* is empty, and let N^* have the spectral resolution* (1.8). *Let C be a circle satisfying $K \cap (\operatorname{int}(C)) = \varnothing$ and $K \cap C = \{z_0\}$, where $K = \operatorname{sp}(T^*)$, and suppose that $E(z_0) \neq 0$, that is, z_0 is in the point spectrum of N^*. Then T^* has a nontrivial invariant subspace M satisfying $M \perp R_{E(z_0)}$.*

Proof. One need only note that z_0 is a peak set of $R(K_1)$, where $K_1 = U^- - \operatorname{int}(C)$ and U is any open disk containing K and C. In fact, if z_1 is any point in the interior of a segment joining z_0 to the center of C then $f(z) = (z_0 - z_1)(z - z_1)^{-1}$ is a peak function.

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THEOREM 3 (Lautzenheiser). *Let T be subnormal on $\mathfrak{H}$. Let C be a simple closed curve of class C^2 which separates $K = \operatorname{sp}(T^*)$, so that both $\operatorname{int}(C) \cap K$ and $\operatorname{ext}(C) \cap K$ are non-empty. Suppose that*

$$\operatorname{meas}_1(C \cap K) = 0, \tag{6.1}$$

where the measure denotes arc length on C. Then T has a non-trivial invariant subspace.

Remark. Actually, Theorem 3 is only a special case of what Lautzenheiser has proved. Among other things, he has shown that T even has a nontrivial reducing subspace. Further, although his proof involves the notion of Gleason parts and ours that of the projection operator, parts of the arguments overlap, in particular, those involving references to Davie and Øksendal, Rudin and Vitushkin.

For other results concerning invariant spaces of certain operators having "almost disconnected" spectra see Stampfli [16].

Proof. It will be supposed that T is not normal and that the point spectrum of T^* is empty (otherwise, the assertion of the Theorem is trivial). Further, it can be supposed that $P_C x$ of (1.2) satisfies

$$P_C x \neq 0, \qquad \text{for some} \quad x \in L. \tag{6.2}$$

(In case $P_C x = 0$ for all $x \in L$, one replaces C by $C \cup C_1$ where C_1 is a large circle surrounding $C \cup K$. This amounts to replacing $P_C x$ by $(I - P_C)x$ and entails minor modifications of the ensuing argument.) As in [10], one can use a result of Rudin as given in Hoffman [9], p. 81 (cf. also [13]) to conclude the existence of a function h defined and continuous on $(\text{int}(C))^-$, analytic inside C and such that $h = 1$ on $C \cap K$ (cf. (6.1)) and $|h| < 1$ otherwise. If F is defined on $K_1 = K \cup (\text{int}(C))^- = K' \cup K''$, where $K' = (\text{int}(C))^-$ and $K'' = (\text{ext}(C))^- \cap K$, by $F = h$ on K' and $F = 1$ on K'', then $F \in C(K_1)$, $f \mid K' \in R(K')$ (by Mergelyan's Theorem or its generalization, see, e.g., Rudin [14], pp. 386, 390) and $f \mid K'' \in R(K'')$. Since $K' \cap K''$ is a subset of a C^2 curve it is analytically negligible ([17]; cf. [6], p. 237). It then follows from a result of Davie and Øksendal [5] that $F \in R(K_1)$. Thus, $Q = K''$ is a peak set of $R(K_1)$ and $\text{meas}_1(Q \cap C) = 0$. The assertion of Theorem 3 now follows from Lemma 2.

7. Further Results on Almost Disconnected Spectra

Whether the C^2 hypothesis on C of Theorem 3 can be weakened to the requirement that C be rectifiable or even of class C^1 is not known. However, one can obtain a certain C^1 variation of Theorem 3 in Theorem 4 below. First, some preliminaries will be needed.

Consider a family C_t, where $0 \leqslant t \leqslant 1$, of simple closed curves, where $C_t \subset \text{int}(C_{t'})$ if $t < t'$, parametrized as follows.

$$C_t = \{(x, y) : x = x(t, u) \text{ and } y = y(t, u),\ 0 \leqslant u \leqslant 1\}, \tag{7.1}$$

where x, y are of class C^1 on $[0, 1] \times [0, 1]$. Suppose that the mapping is 1:1 from $\{(t, u): 0 \leqslant t, u \leqslant 1\}$ onto $M = \{(x, y): x = x(t, u),$

$y = y(t, u)\}$ and that $J = \partial(x, y)/\partial(t, u) \neq 0$. For each fixed $t \in [0, 1]$, let ds denote the element of arc length on C_t, so that $ds = (x_u^2 + y_u^2)^{1/2} du$. The element of area dA on M is then given by $dA = |J| \, dt \, du$ and hence

$$ds \, dt = (x_u^2 + y_u^2)^{1/2} \, |J|^{-1} \, dA \leqslant (\text{const.}) \, dA. \tag{7.2}$$

There will be proved the following theorem.

THEOREM 4. *Let T be subnormal on $\mathfrak{H}$. Let a rectifiable simple closed curve C and a vector x be chosen so that $P_C x$ of* (1.2) *satisfies $P_C x \neq 0$ for some $x \in L$. In addition, suppose that there exists a family, C_t, of C^1 curves, as described above, for which*

$$(\text{int}(C_t))^- \subset \text{ext}(C) \qquad \text{for} \quad 0 \leqslant t \leqslant 1; \tag{7.3}$$

for which, in addition, each C_t separates $\text{sp}(T^*)$, *so that* $(\text{int}(C_t)) \cap K \neq \varnothing$ *and* $\text{ext}(C_t) \cap K \neq \varnothing$, *for* $0 \leqslant t \leqslant 1$ *and* $K = \text{sp}(T^*)$; *and, finally, for which there exists some subset E of* $[0, 1]$ *satisfying*

$$\text{meas}_1(E) > 0, \tag{7.4}$$

and

$$\text{meas}_1(C_t \cap K) = 0 \qquad \textit{for} \quad t \in E, \qquad K = \text{sp}(T^*), \tag{7.5}$$

where the latter measure denotes arc length on C_t. Then T has a nontrivial invariant subspace.

Remark. The gist of the hypothesis of Theorem 4 as against that of Theorem 3 is that, rather than having a single curve separating $\text{sp}(T^*)$ as in Theorem 3, one now has many such curves C_t ($t \in E$, $\text{meas}_1(E) > 0$). However, the somewhat delicate concept of analytic negligibility can now be avoided and, instead, replaced by the more primitive notions involved in Fubini's theorem. In this way, the smoothness requirement is thus reduced from C^2 to C^1. Incidentally, in both Theorems 3 and 4, the full smoothness on C or on the C_t can be relaxed to some type of piecewise smoothness on the corresponding curves.

Proof. As before, it can be supposed that T is not normal and that the point spectrum of T^* is empty. If p denotes any (finite, complex) measure with compact support X, then

$$\hat{p}(z) = \int (t - z)^{-1} \, dp(t)$$

is analytic off X and

$$\int_Y \left(\int_X | t - z |^{-1} d | p | (t) \right) dA < \infty \qquad (dA = dx\, dy), \tag{7.6}$$

where, say, Y is any compact set in the plane (cf. [6], p. 46). If, as in Lemma 1, $dm(z)$ is defined by (1.9) for an x satisfying $P_C x \neq 0$, then $d | m | = dm$ and, by (7.6) with $p = m$, together with (7.2) and Fubini's Theorem, we have

$$\int_E \left(\int_{C_t} \left(\int_{K_1} | u - z |^{-1} dm(u) \right) | dz | \right) dt < \infty, \qquad K_1 = K \cup (\text{int}(C))^-, \tag{7.7}$$

where $| dz |$ $(= ds)$ is the element of arc length on C_t.

Since $G(z)$ of (2.1) is analytic outside C, it follows from (2.4), (7.7) and Fubini's Theorem that

$$0 = (2\pi i)^{-1} \int_E \left(\int_{C_t} \hat{m}(z)\, dz \right) dt = (2\pi i)^{-1} \int_E \left[\int_K \left(\int_{C_t} (u - z)^{-1} dz \right) dm(u) \right] dt$$

$$= \int_E \left[\int_{K \cap \text{int}(C_t)} dm(u) \right] dt \geqslant \int_E \left[\int_{K \cap \text{int}(C_0)} dm(u) \right] dt,$$

where C_0 denotes the innermost curve of the family C_t. (Note that $\int_{C_t} dm(u) = 0$ for almost all t in E, in view of (7.7).) It follows from (7.4) that the last integrand $[\cdots] = 0$ and hence, if $Q = K \cap \text{int}(C_0)$, that $E(Q) P_C x = 0$. Since C_0 separates K then $E(Q) \neq 0$, so that, since $P_C x \neq 0$, relation (1.13) holds. Although it is not now claimed that Q is a peak set of an algebra, nevertheless, the argument of the last paragraph of Section 3 shows that T has a nontrivial invariant subspace. This completes the proof of Theorem 4.

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