Pairs of positive periodic solutions of second order nonlinear equations with indefinite weight

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**Abstract**

We study the problem of the existence and multiplicity of positive periodic solutions to the scalar ODE

\[ u'' + \lambda a(t)g(u) = 0, \quad \lambda > 0, \]

where \( g(x) \) is a positive function on \( \mathbb{R}^+ \), superlinear at zero and sublinear at infinity, and \( a(t) \) is a \( T \)-periodic and sign indefinite weight with negative mean value. We first show the nonexistence of solutions for some classes of nonlinearities \( g(x) \) when \( \lambda \) is small. Then, using critical point theory, we prove the existence of at least two positive \( T \)-periodic solutions for \( \lambda \) large. Some examples are also provided.

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1. Introduction

In this paper we address our investigation to the existence and multiplicity of positive (i.e., \( u(t) > 0 \) for every \( t \in \mathbb{R} \)) \( T \)-periodic solutions of the second order nonlinear scalar ODE

\[ u'' + f(t, u) = 0, \quad (1.1) \]

where \( f = f(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) satisfies the Carathéodory assumptions [22], is \( T \)-periodic in the \( t \)-variable and such that \( f(t, 0) \equiv 0 \).

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1 Supported by the Project PRIN “Ordinary Differential Equations and Applications”.
Just to start our discussion, assume for a moment that $f(t, x) = f(x)$. In such a case, positive periodic solutions exist if and only if $f(\bar{x}) = 0$ for some $\bar{x} > 0$. Of course, any positive zero of $f(x)$ is a constant periodic solution and, in order to have the existence of nontrivial (i.e., non-constant) positive solutions, one has to look for a closed orbit in the right half plane $\{(u, u') \mid u > 0\}$ of the phase plane. This is possible only if $f(x)$ changes its sign (passing from negative to positive values) on $\mathbb{R}^+_0 := ]0, +\infty[$. Extending such elementary observations to the non-autonomous equation (1.1) and using the fact that $\int_0^T f(t, u(t))\,dt = 0$ for every $T$-periodic solution $u(t)$, one is led to assume some sign conditions on $f(t, x)$. For instance, splitting $f(t, x)$ as

$$f(t, x) = -V(t)x + h(t, x),$$

a possibility is that of assuming $h(t, x)/x \to 0$ for $x \to 0^+$ (so that (1.1) linearizes at zero as $u'' - V(t)u = 0$) and imposing some suitable sign and asymptotic conditions on $h(t, x)/x$ for $x \to +\infty$. Symmetrically, one can also assume that $h(t, x)/x \to 0$ for $x \to +\infty$ (so that (1.1) linearizes at infinity as $u'' - V(t)u = 0$) and require suitable sign conditions for $h(t, x)$ near zero. In this direction, results (at different levels of generality, that is, involving hypotheses on $h(t, x)$ or its potential $H(t, x) := \int_0^x h(t, \xi)\,d\xi$) have been obtained by various authors (see, for instance, [29,35] and the references therein).

In the present work we consider a case which appears rather new from the point of view of the existing literature. Indeed, we suppose that

$$f(t, x) = q(t)g(x)$$

with $g : \mathbb{R} \to \mathbb{R}$ of constant (positive) sign on $\mathbb{R}^+_0$ (and, of course, $g(0) = 0$). In such a situation and for $q \not\equiv 0$, a necessary condition for the existence of positive periodic solutions is that the weight function $q(t)$ changes its sign on $[0, T]$. Notice that any positive periodic solution (if it exists) will be non-constant.

Nonlinear boundary value problems with sign indefinite weights have been studied from different points of view in the past fifty years. In 1965 and 1967, Waltman [36] and Kiguradze [23] studied the oscillatory behavior of the solutions to the superlinear equation $u'' + q(t)u^{2n-1} = 0$ (see also [37] for extensions to more general nonlinearities and a rather exhaustive list of references till to the 2000 year). The periodic problem for

$$u'' + q(t)g(u) = 0$$

(1.2)

was considered by Butler in [14] and [15] for $g(x)$ having superlinear growth at infinity or sublinear growth at zero, respectively. In [14], infinitely many periodic oscillatory solutions are found. Further results in this direction have been obtained by Terracini and Verzini in [34] and by Papini and Zanolin in [30]. In these papers solutions with a large number of zeros in the intervals of positivity of the weight are produced. This, in turn, led Capietto, Dambrosio and Papini in [16] to study the existence of chaotic dynamics for (1.2) as well as for

$$u'' + cu' + q(t)g(u) = 0.$$ 

Both in [34] and in [16] the authors obtain solutions of (1.2) globally defined on $\mathbb{R}$ and, again, with a complex oscillatory behavior expressed in terms of their number of zeros.

On the other hand, starting with the nineties, many authors have investigated the existence and multiplicity of positive solutions to boundary value problems (typically, the Dirichlet or the Neumann one) associated to the nonlinear elliptic PDE

$$\Delta u - ku + q(x)g(u) = 0, \quad x \in \Omega \subset \mathbb{R}^N,$$

(1.3)
with a sign changing $q(x)$ (see [2,5–8] for some classical results in this direction). Clearly, (1.2) is the one-dimensional case of (1.3) for $k = 0$. Such kind of problems arise from the search of stationary solutions to some reaction diffusion systems (see [6]); a large (although incomplete) list of references on the subject is contained in [31]. Multiplicity results for positive solutions of Eq. (1.3), with Dirichlet boundary conditions on a bounded domain $\Omega$, have been obtained in [9,17,20,21,26] in the superlinear case. Such results deal with the case in which $q(x)$ splits as

$$q(x) = \varepsilon a^+(x) - \mu a^-(x),$$

with $a^+(x)$, $a^-(x)$ the positive and the negative part of a sign-changing weight $a(x)$, and hold for $\varepsilon > 0$ small or $\mu > 0$ large. When $k = 0$, however, the Neumann problem for (1.3), as well as the periodic problem for (1.2), exhibit some peculiar difficulties due to the presence of some necessary conditions for the existence of positive solutions (see [7] and the results in Section 2 below, particularly Proposition 2.1 and Remark 2.1).

There are interesting connections between the periodic and the Neumann boundary value problems for Eq. (1.2), besides the obvious fact that, in both cases, $k_0 = 0$ is the principal eigenvalue for the operator $-u''$, with the corresponding eigenspace made up by the constant functions. For instance, if the $T$-periodic weight function $q(t)$ is even-symmetric with respect to some $t_0 \in [0, T]$, in the sense that

$$q(t_0 + s) = q(t_0 - s), \quad \forall s \in \mathbb{R}, \quad (1.4)$$

then any (positive) solution $u(t)$ of (1.2) satisfying the Neumann boundary conditions

$$u'(t_0 - \frac{T}{2}) = u'(t_0) = 0$$

can be extended, by symmetry with respect to $t_0$ and by $T$-periodicity, to a (positive) $T$-periodic solution of (1.2). Note also that a function $q(t)$ satisfying the symmetry condition (1.4) changes its sign on $[0, T]$ if and only if it is of non-constant sign on $[t_0, t_0 + \frac{T}{2}]$. Using this remark, we can translate some theorems for the existence of positive solutions for the Neumann BVP associated to (1.3) (with $k = 0$) to corresponding periodicity results for (1.2). As an example in this direction, we can apply a recent theorem in [10] (which extends the result of [20] to the Neumann problem) in order to provide multiple positive $T$-periodic solutions to (1.2) when $g(x)$ is superlinear at infinity and $q(t) = a^+(t) - \mu a^-(t)$, with $\mu > 0$ large.

In the present work we study the existence of multiple positive $T$-periodic solutions to (1.2) under a different set of assumptions for $g(x)$; indeed, we impose some hypotheses which imply that $g(x)$ is superlinear at zero and sublinear at infinity, that is:

$$\lim_{x \to 0^+} \frac{g(x)}{x} = 0, \quad \lim_{x \to +\infty} \frac{g(x)}{x} = 0. \quad (1.5)$$

The choice of this terminology is due to the fact that a possible function $g(x)$ satisfying (1.5) is one which behaves like $x^\alpha$ with $\alpha > 1$ near zero and like $x^\beta$ with $0 \leq \beta < 1$ near infinity. It is also consistent with the analogous case for the Dirichlet problem where the term super-sublinear is referred to a function having slope less than the first eigenvalue at zero and at infinity [18, p. 361]. For the precise technical conditions on $g(x)$ assumed in this paper, see Proposition 2.2 and Theorem 3.1.

For what concerns the weight function $q(t)$, we write it as $q(t) = \lambda a(t)$, with $\lambda > 0$, so that we are finally led to consider the equation

$$u'' + \lambda a(t)g(u) = 0, \quad (1.6)$$

with $\lambda > 0$ playing the role of a parameter.
The Dirichlet (two-point) boundary value problem for equation

\[ u'' + \lambda a^+(t) g(u) = 0 \]

\((a^+ \neq 0)\) and with a nonlinearity \(g(x)\) satisfying (1.5), or related conditions, has been widely investigated in the literature mainly in the frame of the more general elliptic PDE

\[ \Delta u + \lambda f(x, u) = 0, \quad x \in \Omega. \]

Starting with the classical papers of Amann [3] and Rabinowitz [32] (see also [4,19,25]), typical results in this setting guarantee the existence of at least two positive solutions for \(\lambda > 0\) large enough. Our aim is to show that such a classical condition, paired with the hypothesis that \(a(t)\) has negative mean value, i.e.,

\[ \int_0^T a(t) \, dt < 0, \]

ensures the multiplicity of positive periodic solutions to (1.6). More precisely, in Section 3 we prove (see Theorem 3.1) the existence of at least two positive \(T\)-periodic solutions to (1.6) for every \(\lambda > \lambda^*\) (with, of course, \(\lambda^*\) depending on \(g(x)\) and \(a(t)\)). This goal is achieved using critical point techniques in a variational setting. It is worth noticing that the result holds under mild regularity assumptions on the weight (even continuity is not required provided that the solutions are meant in the generalized sense). Moreover, we stress again that the role of \(\lambda^*\) is similar to that played by the same coefficient in Rabinowitz’s paper [32] and the assumption \(\lambda > \lambda^*\) is justified by a nonexistence result for \(\lambda\) small (see Proposition 2.2). On the other hand, (1.7) is a sharp hypothesis since it is a necessary condition for the existence of positive periodic solutions in the case of (1.6) with \(g'(x) > 0\) (see Proposition 2.1).

As a possible corollary of Theorem 3.1 we have the following result (where, for simplicity, we assume \(a(t)\) continuous).

**Theorem 1.1.** Let \(g : \mathbb{R}^+ \to \mathbb{R}^+ := [0, +\infty[\) be a continuous function with \(g(0) = 0\) and \(g(x) > 0\) for \(x > 0\), such that

\[ g(x) \sim x^\alpha, \quad \text{for } x \to 0^+ \quad \text{and} \quad g(x) \sim x^\beta, \quad \text{for } x \to +\infty, \]

for some \(0 \leq \beta < 1 < \alpha\). Let \(a : \mathbb{R} \to \mathbb{R}\) be a continuous and \(T\)-periodic function with \(\int_0^T a(t) \, dt < 0\) and \(a^+ \neq 0\). Then there exists \(\lambda^* > 0\) such that, for each \(\lambda > \lambda^*\), Eq. (1.6) has at least two positive \(T\)-periodic solutions.

Examples in which our result applies are the equations

\[ u'' + \lambda (\rho + \cos t) \frac{u^\alpha}{1 + u^\gamma} = 0, \]

with \(-1 < \rho < 0\) and \(0 < \alpha - 1 \leq \gamma\), or

\[ u'' + \lambda \frac{\sin t}{c + \sin t} \arctan(u^\alpha + u^\gamma) = 0, \]

with \(c > 1\) and \(\gamma \geq \alpha > 1\). In both the cases the existence of at least two positive \(2\pi\)-periodic solutions is guaranteed for \(\lambda > 0\) sufficiently large. Estimates for \(\lambda^*\) can be provided too (see Remark 3.3).
On the other hand, for $\gamma = \alpha$, in both the above examples, we have that for every $m \geq 1$, there is a $\lambda_\alpha(m) > 0$ such that no positive $2m\pi$-periodic solutions exist when $0 < \lambda < \lambda_\alpha(m)$ (see Proposition 2.2).

We conclude this introduction with a list of notation and assumptions used throughout the paper.

We denote by $\mathbb{R}^+$ (resp., $\mathbb{Z}^+$) the set of nonnegative real (resp., integer) numbers and by $\mathbb{R}_0^+$ (resp., $\mathbb{Z}_0^+$) the set of positive real (resp., integer) numbers. The basic assumptions on the functions involved in Eq. (1.6) are the following:

$$(g_*) \quad g : \mathbb{R}^+ \to \mathbb{R} \text{ is continuous, with } g(0) = 0 \text{ and } g(x) > 0, \forall x > 0,$$

$$(a_*) \quad a : \mathbb{R} \to \mathbb{R} \text{ is } L^1_{\text{loc}} \text{ and } T\text{-periodic, with } \int_0^T a^+(t) \, dt, \int_0^T a^-(t) \, dt > 0.$$

Here,

$$a^+(t) := \max(a(t), 0), \quad a^-(t) := \max(-a(t), 0)$$

denote the positive and the negative part of $a(t)$, respectively.

We use standard notation for the Sobolev spaces $W^{k,p}$ ($k \in \mathbb{Z}^+, 1 \leq p < \infty$). As usual, we set $H^k := W^{k,2}$, while, by convention, $W^{0,p} = L^p$. Moreover, for every $m \in \mathbb{Z}_0^+$, we set

$$W^{k,p}_{mT} := \{ u \in W^{k,p}_{\text{loc}}(\mathbb{R}) \mid u \text{ is } mT\text{-periodic} \},$$

with norm

$$\|u\|_{W^{k,p}_{mT}} := \left( \sum_{j=0}^{k} mT \int_0^{mT} |u^{(j)}(t)|^p \, dt \right)^{\frac{1}{p}}.$$

We recall that, if $k \geq 1$ and $p > 1$,

$$W^{k,p}_{mT} \hookrightarrow C^{k-1}_{mT} := \{ u \in C^{k-1}(\mathbb{R}) \mid u \text{ is } mT\text{-periodic} \}$$

with compact embedding. For $u \in L^1_{mT}$, we set

$$\tilde{u} := \frac{1}{mT} \int_0^{mT} u(s) \, ds, \quad \tilde{u}(t) := u(t) - \tilde{u}$$

and we recall that, if $u \in H^1_{mT}$, the Sobolev and Wirtinger inequalities hold true:

$$\left( \sup_{t \in \mathbb{R}} |\tilde{u}(t)| \right)^2 \leq \frac{mT}{12} \int_0^{mT} u'(t)^2 \, dt, \quad (1.9)$$

$$\int_0^{mT} \tilde{u}(t)^2 \, dt \leq \left( \frac{mT}{2\pi} \right)^2 \int_0^{mT} u'(t)^2 \, dt. \quad (1.10)$$

As a consequence, the quantity $(\tilde{u}^2 + \int_0^{mT} (u')^2 \, dt)^{\frac{1}{2}}$ is an equivalent norm on $H^1_{mT}$.

Solutions to Eq. (1.6) will be considered in the generalized (Carathéodory) sense. Namely, by a (globally defined) solution of (1.6) (or of related equations) we mean a function in $W^{2,1}_{\text{loc}}(\mathbb{R})$ satisfying
the differential equation for almost every \( t \in \mathbb{R} \); in particular, an \( mT \)-periodic solution of (2.1), with \( m \in \mathbb{Z}_0^+ \), is a solution which belongs to \( W^{2,1}_{mT} \). Of course, if \( a(t) \in L^p_T \) (for some \( p > 1 \)) then every solution is in \( W^{2,p}_{loc} \), while if \( a(t) \) is continuous then every solution is of class \( C^2 \) and solves the equation for every \( t \in \mathbb{R} \) (that is, it is a classical solution).

By a positive solution, we mean a solution \( u(t) \) such that \( u(t) > 0 \) for every \( t \in \mathbb{R} \).

2. Nonexistence results

In this section, we give two nonexistence results (Proposition 2.1 and Proposition 2.2) for positive periodic solutions of equation

\[
 u'' + \lambda a(t) g(u) = 0. \tag{2.1}
\]

Our analysis for this section concerns both the harmonic solutions (i.e. \( T \)-periodic) as well as the subharmonics. As anticipated in the Introduction, \( \lambda > 0 \) plays the role of a parameter, while \( g(x) \) and \( a(t) \) satisfy the assumptions \((g_+)\) and \((a_+)\) listed above. Note that \((g_+)\) and \((a_+)\) imply that no positive constant solutions exist for (2.1).

Our first nonexistence result is an adaptation to the periodic case of similar results obtained in \([7,8]\) for the Neumann problem.

**Proposition 2.1.** Let us suppose \( g \in C^1(\mathbb{R}_0^+) \) with \( g'(x) > 0 \) for every \( x > 0 \).

If \( \int_0^T a(t) \, dt \geq 0 \), then Eq. (2.1) has no positive \( mT \)-periodic solutions for every \( m \in \mathbb{Z}_0^+ \) and every \( \lambda > 0 \).

**Proof.** If \( u(t) \) is a positive \( mT \)-periodic solution of (2.1), an integration by parts gives

\[
 \int_0^{mT} \frac{u''(t)}{g(u(t))} \, dt = \left[ \frac{u'(t)}{g(u(t))} \right]_0^{mT} + \int_0^{mT} g'(u(t)) \left( \frac{u'(t)}{g(u(t))} \right)^2 \, dt.
\]

\[
 = \int_0^{mT} g'(u(t)) \left( \frac{u'(t)}{g(u(t))} \right)^2 \, dt.
\]

Being, for a.e. \( t \in [0, mT] \),

\[
 \frac{u''(t)}{g(u(t))} = -\lambda a(t)
\]

and \( u' \not\equiv 0 \), we obtain

\[
 m\lambda \int_0^T a(t) \, dt = \lambda \int_0^{mT} a(t) \, dt = -\int_0^{mT} g'(u(t)) \left( \frac{u'(t)}{g(u(t))} \right)^2 \, dt < 0,
\]

a contradiction. \( \square \)

**Remark 2.1.** The above result extends to a wider class of equations. In particular, we propose the following general formulation which may have some independent interest.

Let \( J \subset \mathbb{R} \) be an open interval and \( g \in C^1(J) \) with \( g(x) > 0 \) and \( g'(x) > 0 \) for every \( x \in J \). Moreover, let \( q \in L^1([0, T]) \). Then, the condition
\[
\int_0^T q(t) \, dt < 0
\] 

(2.2)

is necessary for the existence of a solution \( u(t) \) with \( u(t) \in J \) for every \( t \in [0, T] \) for both the Neumann and the periodic problems

\[
\begin{aligned}
&u'' + q(t)g(u) = 0, \\
&u'(0) = u'(T) = 0,
\end{aligned}
\]

\[
\begin{aligned}
&u'' + q(t)g(u) = 0, \\
&u(0) = u(T), \quad u'(0) = u'(T).
\end{aligned}
\]

Obviously, Proposition 2.1 is just a particular case, with \( J = \mathbb{R}_0^+ \). Observe however, that, in its general formulation, the result allows also the case of singularities for \( g(x) \) like \( g(0^+) = -\infty \). This suggests the following question: when the condition (2.2) is a sufficient one? Positive answers have been given (with different assumptions on \( g(x) \)) for the Neumann problem with \( J = \mathbb{R}_0^+ \) in \([7,8,13]\) and for the Neumann and the periodic problems on \( J = \mathbb{R} \) in \([24]\). Recently, in \([12]\) for a stepwise weight \( q(t) \), the necessary condition has been proved to be sufficient for the periodic problem with \( J = \mathbb{R}_0^+ \) and \( g(x) \) singular in 0.

Our second result shows that, in the super-sublinear case, nonexistence can occur also in the case \( \int_0^T a(t) \, dt < 0 \). Indeed, in order to have positive periodic solutions, also \( \lambda \) must be not too small. This gives a (negative) answer, for the problem considered in the paper, to the general question raised in Remark 2.1. Of course, we cannot exclude that the mean value condition on the weight is a necessary and sufficient one for different classes of increasing nonlinearities.

**Proposition 2.2.** Let us suppose \( g \in C^1(\mathbb{R}_0^+) \); moreover assume that:

- there exist \( \alpha > 1, K > 0 \) such that
  \[
  \lim_{x \to 0^+} \frac{g(x)}{x^\alpha} = l_\alpha > 0
  \] 
  \( (2.3) \)
  and
  \[
  |g'(x)| \leq Kx^{\alpha-1}, \quad \text{for every } x > 0;
  \] 
  \( (2.4) \)

- there exists
  \[
  \lim_{x \to +\infty} g(x) =: g^+ \in [0, +\infty[.
  \] 
  \( (2.5) \)

Finally, let us assume \( a \in L^2_T \) with \( \int_0^T a(t) \, dt \neq 0 \).

Then for every \( m \in \mathbb{Z}_0^+ \) there exists \( \lambda_+(m) > 0 \) such that, for every \( 0 < \lambda < \lambda_+(m) \), Eq. (2.1) has no positive \( mT \)-periodic solutions.

Possible examples of a function \( g(x) \) satisfying the assumptions of Proposition 2.2 are given by

\[
 g(x) = \frac{x^\alpha}{x^\alpha + 1}
\]

or

\[
 g(x) = \arctan x^\alpha,
\]

with \( \alpha > 1 \).
Proof of Proposition 2.2. Let us assume by contradiction that \((u_k, \lambda_k)\) is a sequence of positive \(mT\)-periodic solutions of (2.1) with \(\lambda_k \neq 0\) and \(\lambda_k \to 0^+\). Since \(a \in L^2_T\), \(u_k \in H^2_{mT}\).

**Step 1.** We claim that \(\|u_k\|_{H^2_{mT}}\) is bounded. In fact, multiplying
\[
u_k''(t) + \lambda_k a(t) g(u(t)) = 0
\]
by \(\tilde{u}_k(t)\) and integrating by parts we get
\[
\int_0^{mT} \nu_k'(t)^2 \, dt = \lambda_k \int_0^{mT} a(t) g(u(t)) \tilde{u}_k(t) \, dt \leq C_1 \lambda_k \|a\|_{L^1_{mT}} \|\tilde{u}_k\|_{L^\infty}
\]
\[
\leq C_2 \lambda_k \|a\|_{L^1_{mT}} \left( \int_0^{mT} \nu_k'(t)^2 \, dt \right)^{1/2},
\]
where \(C_1, C_2 > 0\) are suitable positive constants which do not depend on \(u_k\). The above inequality implies that \(\int_0^{mT} \nu_k'(t)^2 \, dt\) is bounded (actually, it converges to 0).

Moreover, \(\tilde{u}_k\) is bounded too. In fact, let us assume by contradiction that, up to subsequences, \(\tilde{u}_k \to +\infty\); as \(\|\tilde{u}_k\|_{L^\infty}\) is bounded by Sobolev inequality (1.9), \(u_k(t) \to +\infty\) uniformly and the dominated convergence theorem and (2.5) imply that
\[
0 = \int_0^{mT} a(t) g(\tilde{u}_k + \tilde{u}_k(t)) \, dt \to g^+ \int_0^{mT} a(t) \, dt \neq 0,
\]
which is a contradiction. By the equivalence of \((\tilde{u}^2 + \int_0^{mT} (u')^2 \, dt)^{\frac{1}{2}}\) with the standard norm of \(H^1_{mT}\), we deduce that \(\|u_k\|_{H^1_{mT}}\) is bounded. Moreover, directly from the equation we get that
\[
\int_0^{mT} u_k'(t)^2 \, dt = \lambda_k \int_0^{mT} a(t)^2 g(u_k(t))^2 \, dt \leq C_3 \lambda_k^2 \|a\|_{L^2_{mT}}^2 \quad (2.6)
\]
(for suitable \(C_3 > 0\)), which implies the \(H^2_{mT}\) bound for \((u_k)_k\).

**Step 2.** We prove that \(u_k \to 0\) in \(H^2_{mT}\). In fact, by reflexivity, there exists \(u \in H^2_{mT}\), \(u \geq 0\) such that, up to subsequences, \(u_k \to u\) weakly in \(H^2_{mT}\) and strongly in \(C^1_{mT}\). Moreover (2.6) implies that \(u''(t) = 0\), that is \(u(t) \equiv c\) for some constant \(c \geq 0\). But
\[
0 = \int_0^{mT} a(t) g(u_k(t)) \, dt \to g(c) \int_0^{mT} a(t) \, dt,
\]
which implies \(g(c) = 0\) and hence \(0 = c = u\). By compactness, \(u_k \to 0\) in \(H^1_{mT}\); moreover (2.6) implies that \(u_k'' \to 0\) in \(L^2_{mT}\). Hence, \(u_k \to 0\) in \(H^2_{mT}\) as claimed.

**Step 3.** Set \(v_k(t) := \frac{u_k(t)}{\|u_k\|_{H^2_{mT}}^2};\) then \(\|v_k\|_{H^2_{mT}} = 1\) and \(v_k\) satisfies
\[
v'' + \lambda_k a(t) \frac{g(u_k)}{\|u_k\|_{H^{2}_{mT}}} = 0.
\] (2.7)

By reflexivity and up to subsequences, \(v_k \to v \geq 0\) weakly in \(H^{2}_{mT}\) and strongly in \(C^{1}_{mT}\). Moreover, (2.3) and (2.5) imply that, for some \(C_4 > 0\),

\[
g(x) \leq C_4 x, \quad \forall x \geq 0;
\]

hence, for every \(t \in [0, mT]\),

\[
\frac{g(u_k(t))}{\|u_k\|_{H^{2}_{mT}}} \leq C_4 \frac{u_k(t)}{\|u_k\|_{H^{2}_{mT}}} \leq C_4 \frac{\|u_k\|_{L^\infty}}{\|u_k\|_{H^{2}_{mT}}} \leq C_5
\]

(for a suitable constant \(C_5 > 0\)), which implies that

\[
\lambda_k \int_0^{mT} a(t) \left( \frac{g(u_k(t))}{\|u_k\|_{H^{2}_{mT}}} \right)^2 \, dt \leq C_5^2 \lambda_k \int_0^{mT} a(t)^2 \, dt. \quad (2.8)
\]

Hence passing to the limit into (2.7) we get that \(v''(t) = 0\), that is \(v(t) \equiv c\) for some constant \(c \geq 0\). Moreover, we have

\[
0 = \int_0^{mT} a(t) \frac{g(u_k(t))}{\|u_k\|_{H^{2}_{mT}}} \, dt = \int_0^{mT} a(t) \left( \frac{g(v_k(t)\|u_k\|_{H^{2}_{mT}}) - g(c\|u_k\|_{H^{2}_{mT}})}{\|u_k\|_{H^{2}_{mT}}} \right) \, dt + \int_0^{mT} a(t) \frac{g(c\|u_k\|_{H^{2}_{mT}})}{\|u_k\|_{H^{2}_{mT}}} \, dt.
\]

By the Lagrange theorem and in view of (2.4), for some \(0 \leq s_k = s(k, t) \leq 1\),

\[
\frac{|g(v_k(t)\|u_k\|_{H^{2}_{mT}}) - g(c\|u_k\|_{H^{2}_{mT}})|}{\|u_k\|_{H^{2}_{mT}}} = \frac{\int_{s_k}^{1} |g'(\|u_k\|_{H^{2}_{mT}} (s_k v_k(t) + (1-s_k)c))\|u_k\|_{H^{2}_{mT}} v_k(t) - c| \, ds_k}{\|u_k\|_{H^{2}_{mT}}} \leq K \frac{\|u_k\|_{H^{2}_{mT}}}{\|u_k\|_{H^{2}_{mT}}} \leq C_6 (\|v_k(t)\|_{L^\infty}^{\alpha-1} + c^{\alpha-1}) |v_k(t) - c| \\
\leq C_7 (\|v_k(t)\|_{H^{2}_{mT}}^{\alpha-1} + c^{\alpha-1}) |v_k(t) - c| = C_7 (1 + c^{\alpha-1}) |v_k(t) - c|
\]

(with \(K\) coming from (2.4) and \(C_6, C_7 > 0\) suitably chosen). Since \(v_k(t) \to c\) uniformly, the dominated convergence theorem implies that

\[
\int_0^{mT} a(t) \left( \frac{g(v_k(t)\|u_k\|_{H^{2}_{mT}}) - g(c\|u_k\|_{H^{2}_{mT}})}{\|u_k\|_{H^{2}_{mT}}} \right) \, dt \to 0.
\]
On the other hand, (2.3) implies that

\[
\int_0^{mT} a(t) \frac{g(c\|u_k\|_{H^{2}_{mT}})}{\|u_k\|_{H^{2}_{mT}}} dt \to l_\alpha c^\alpha \int_0^{mT} a(t) dt.
\]

Hence we deduce that \( c = 0. \) By compactness \( v_k \to 0 \) strongly in \( H^{1}_{mT}, \) while (2.7) and (2.8) imply that \( v''_k \to 0 \) strongly in \( L^2_{mT}. \) Hence \( v_k \to 0 \) strongly in \( H^{2}_{mT}, \) in contradiction with the fact that \( \|v_k\|_{H^{2}_{mT}} = 1. \)

**Remark 2.2.** It is worth noticing that \( \lambda_* \) cannot be taken independent of \( m. \) In fact, we will show (see the last part of Remark 3.2 in the next section) that, whenever \( \int_0^T a(t) dt < 0 \) and \( g(x) \) satisfies suitable assumptions at 0 and at infinity (see \((g_0)\) and \((g_\infty)\) of Theorem 3.1 below), compatible with (2.3)–(2.5), Eq. (2.1) always has two positive \( mT \)-periodic solutions for every \( m \) large enough.

### 3. Two positive \( T \)-periodic solutions via critical point theory

In this section, using critical point theory, we prove our main multiplicity result for positive \( T \)-periodic solutions to the equation

\[
u'' + \lambda a(t) g(u) = 0, \tag{3.1}
\]

with \( \lambda > 0 \) a parameter and \( g(x), a(t) \) satisfying assumptions \((g_*), (a_*)\) of the Introduction. More precisely, we are going to show that, under suitable assumptions on \( g(x) \) at zero and at infinity (see conditions \((g_0)\) and \((g_\infty)\) of Theorem 3.1 below) and under the mean value condition \( \int_0^T a(t) dt < 0, \) the existence of a pair of positive \( T \)-periodic solutions to Eq. (3.1) is guaranteed for \( \lambda \) large enough.

**Theorem 3.1.** Let us suppose that:

\((g_0)\) there exists \( \alpha > 1 \) such that

\[
\lim_{x \to 0^+} \frac{g(x)}{x^\alpha} = l_\alpha > 0;
\]

\((g_\infty)\) there exists \( 0 \leq \beta < 1 \) such that

\[
\limsup_{x \to +\infty} \frac{g(x)}{x^\beta} < +\infty \quad \text{and} \quad \lim_{x \to +\infty} \frac{G(x)}{x^{2\beta}} = +\infty,
\]

where we have set \( G(x) := \int_0^x g(\xi) d\xi. \)

Moreover, assume

\[
\int_0^T a(t) dt < 0 \tag{3.2}
\]

and also suppose that there exists an open interval \( I \subset [0, T] \) with
\[ a(t) \geq 0 \quad \text{for a.e. } t \in I, \quad \text{with} \quad \int_I a(t) \, dt > 0. \]  

(3.3)

Then there exists \( \lambda^* > 0 \) such that, for every \( \lambda > \lambda^* \), Eq. (3.1) has at least two positive \( T \)-periodic solutions.

**Remark 3.1.** As explained in the Introduction, conditions \((g_0)\) and \((g_\infty)\) of Theorem 3.1 imply that \( g(x) \) has a super-sublinear behavior, namely

\[ \lim_{x \to +\infty} \frac{g(x)}{x} = 0, \quad \lim_{x \to +\infty} \frac{g(x)}{x^\beta} = 0. \]

Notice that, in such a case, both the linearization at zero and the linearization at infinity of Eq. (3.1) are resonant with respect to the principal eigenvalue of \(-u''\) with \( T \)-periodic boundary conditions.

With this interpretation, \((g_\infty)\) may be seen as a generalized Ahmad–Lazer–Paul (nonresonance) condition at infinity for the potential \( F(t,x) := a(t)G(x) \), following Tang [33]. Recall that the classical Ahmad–Lazer–Paul condition [1], read in our context, requires that \( \int_0^1 F(t,x) \, dt \to +\infty \) for \( x \to \infty \) with \( f(t,x) := a(t)g(x) \) bounded, i.e., \((g_\infty)\) with \( \beta = 0 \). See also Mawhin and Willem [28, Theorems 1.5 and 4.8] for the periodic problem.

Observe also that \((g_\infty)\) is satisfied when \( g(x) \) has a precise (sublinear) power-growth at infinity, namely if, for some \( 0 \leq \beta < 1 \),

\[ (g_\infty') \quad \lim_{x \to +\infty} \frac{g(x)}{x^\beta} = l_\beta > 0. \]

Indeed, in this case, l'Hopital rule implies that \( \frac{G(x)}{x^{\beta+1}} \rightarrow l_\beta \frac{x^\beta}{\beta+1} \) for \( x \to +\infty \).

We finally notice that if \( a(t) \) is piecewise continuous then (3.3) just follows from \((a_*)\).

**Remark 3.2.** Since the assumptions \((g_0)\), \((g_\infty)\) are compatible with the hypotheses of both Propositions 2.1 and 2.2, then the lower bound \( \lambda > \lambda^* \) and the condition \((3.2)\) are in some sense “unavoidable” for the existence of positive \( T \)-periodic solutions of (3.1). In particular, if \( g(x) \) satisfies the assumptions of Proposition 2.2, then we enter in the setting of Theorem 3.1 with \((g_\infty)\) satisfied with \( \beta = 0 \). In this case, assuming further that \( a(t) \) is continuous with \( \max a(t) > 0 \), our results provide the existence of two constants \( 0 < \lambda_* \leq \lambda^* \) such that Eq. (3.1) has no positive \( T \)-periodic solution for \( 0 < \lambda < \lambda_* \) and at least two positive \( T \)-periodic solutions for \( \lambda > \lambda^* \). More general conditions on \( a(t) \) can be considered in view of Proposition 2.2 and Theorem 3.1.

We also point out that, via a time-dilatation, we can state the following.

Let \( g(x) \) and \( a(t) \) be as in Theorem 3.1. Then there exists an integer \( m^* > 0 \) such that, for every \( m > m^* \), the equation

\[ u'' + a(t)g(u) = 0 \]

has at least two positive \( mT \)-periodic solutions.

Of course, we have in general no information about the minimal period of such solutions; on the other hand, we have suppressed the parameter \( \lambda \) in the equation. Such a consideration can be of some interest in view of Proposition 2.2 and Remark 2.2. The existence of (positive) subharmonic solutions to (3.1) will be discussed in a forthcoming paper [11].

For the proof of Theorem 3.1, we will use a variational approach on the Hilbert space \( H^{1}_T = W^{1,2}_T \).

To this aim, we first extend \( g(x) \) to the whole real line and then define a corresponding action functional. More precisely, let us introduce the null extension of \( g(x) \),

\[ g^0(x) := g(x^+) = \begin{cases} g(x) & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases} \]

(3.4)

and observe that the following lemma holds (its standard proof is omitted).
Lemma 3.1. Let $u \in W^{2,1}_{\text{loc}}(\mathbb{R})$ be a $T$-periodic solution of the equation

$$u'' + \lambda a(t)g^0(u) = 0 \quad (3.5)$$

such that $u(\tilde{t}) \geq 0$ for some $\tilde{t} \in \mathbb{R}$. Then $u(t) \geq 0$ for every $t \in \mathbb{R}$.

Keeping in mind Lemma 3.1, from now on we identify $g(x)$ with its extension $g^0(x)$ and define also

$$G(x) := \int_0^x g(\xi) d\xi, \quad \forall x \in \mathbb{R}.$$  

Clearly, $G(x) = 0$ for $x \leq 0$. Then we define the functional $\mathcal{J}_\lambda : H^1_T \to \mathbb{R}$ by

$$\mathcal{J}_\lambda(u) := \frac{1}{2} \int_0^T u'(t)^2 \, dt - \lambda \int_0^T a(t)G(u(t)) \, dt.$$ 

It is well known (see for example [28, Corollary 1.1]) that $\mathcal{J}_\lambda$ is of class $C^1$ on $H^1_T$ and its critical points correspond to $W^{2,1}_T$ solutions of Eq. (3.1). Roughly speaking, we are going to show that a first solution can be characterized as a global minimum point of $\mathcal{J}_\lambda$ on $H^1_T$, while a second one is provided by a classical Mountain Pass procedure. By maximum-principle arguments (see Lemma 3.1 above and Lemma 3.7 at the end of the proof), both these solutions will be shown to be nontrivial and positive.

For technical reasons (see Remark 3.4), a second functional has to be introduced. Denoting by $g_0(x)$ the odd extension of $g$, we define $\mathcal{J}_0^\lambda : H^1_T \to \mathbb{R}$ by setting

$$\mathcal{J}_0^\lambda(u) := \frac{1}{2} \int_0^T u'(t)^2 \, dt - \lambda \int_0^T a(t)G_0(u(t)) \, dt,$$

where, as usual, $G_0(x) := \int_0^x g_0(\xi) d\xi$. Again, $\mathcal{J}_0^\lambda$ turns out to be of class $C^1$ on $H^1_T$ and its critical points correspond to $W^{2,1}_T$ solutions of the equation

$$u'' + \lambda a(t)g^0(u) = 0. \quad (3.6)$$

To summarize, $\mathcal{J}_\lambda$ and $\mathcal{J}_0^\lambda$ refer to the null extension and to the odd extension of $g$, respectively.

The proof will follow from some lemmas. The first one provides some useful estimates which will be used throughout the proof. From now on, we use the standard decomposition

$$u(t) = \bar{u} + \tilde{u}(t), \quad \forall u \in H^1_T$$

with $\bar{u}$ and $\tilde{u}(t)$ defined as in (1.8) for $m = 1$; moreover, for simplicity of notation we will set

$$a_\lambda(t) := \lambda a(t).$$
Lemma 3.2. Let $G := G$ or $G := G^0$. Then there exists $C > 0$ such that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that, for every $u \in H^1_T$,

$$
\left| \int_0^T a_\lambda(t)(G(u(t)) - G(\tilde{u})) \, dt \right| \leq \epsilon |\tilde{u}|^{\alpha+1} + C_\epsilon \|u'\|_{L^1_T}^{\alpha+1},
$$

(3.7)

$$
\left| \int_0^T a_\lambda(t)(G(u(t)) - G(\tilde{u})) \, dt \right| \leq \epsilon \|u'\|_{L^2_T}^2 + C_\epsilon |\tilde{u}|^{2\beta} + C \|u'\|_{L^2_T}^{\beta+1}.
$$

(3.8)

Proof. We prove the result for $G = G^0$; the proof for $G$ is analogous. We begin with relation (3.7). Using assumptions $(g_0)$ and $(g_\infty)$, it is easy to see that there exists a constant $C_1 > 0$ such that

$$
|g^0(x)| \leq C_1 |x|^\alpha, \quad \forall x \in \mathbb{R}.
$$

Hence, if $0 \leq s \leq 1$,

$$
|g^0(\tilde{u} + s\tilde{u}(t))\tilde{u}(t)| \leq C_1 |\tilde{u} + s\tilde{u}(t)|^{\alpha}|\tilde{u}(t)| \leq C_2 (|\tilde{u}|^{\alpha} + |\tilde{u}(t)|^{\alpha})|\tilde{u}(t)|
$$

$$
= C_2 (|\tilde{u}|^{\alpha} |\tilde{u}(t)| + |\tilde{u}(t)|^{\alpha+1})
$$

(for $C_2 > 0$ another constant). By Young's inequality, for every $\eta > 0$,

$$
|\tilde{u}|^{\alpha} |\tilde{u}(t)| = \left( \frac{\alpha + 1}{\alpha} \frac{\eta}{C_2} \right)^{\frac{\alpha}{\alpha+1}} |\tilde{u}|^{\alpha} \left( \frac{\alpha + 1}{\alpha} \frac{\eta}{C_2} \right)^{-\frac{\alpha}{\alpha+1}} |\tilde{u}(t)|
$$

$$
\leq \frac{\eta}{C_2} |\tilde{u}|^{\alpha+1} + \frac{1}{\alpha + 1} \left( \frac{\alpha + 1}{\alpha} \frac{\eta}{C_2} \right)^{-\frac{\alpha}{\alpha+1}} |\tilde{u}(t)|^{\alpha+1};
$$

hence we obtain

$$
|g^0(\tilde{u} + s\tilde{u}(t))\tilde{u}(t)| \leq \eta |\tilde{u}|^{\alpha+1} + K_\eta |\tilde{u}(t)|^{\alpha+1} \leq \eta |\tilde{u}|^{\alpha+1} + K_\eta \|u'\|_{L^1_T}^{\alpha+1}
$$

$$
\leq \eta |\tilde{u}|^{\alpha+1} + K_{\eta}' \|u'\|_{L^1_T}^{\alpha+1},
$$

where $K_\eta := (C_2/(\alpha + 1))(\alpha C_2/\eta(\alpha + 1))^{\alpha} + C_2$ and $K_{\eta}' := (T/12)(\alpha+1)/2K_\eta$ (cf. (1.9)). In conclusion, we find

$$
\left| \int_0^T a_\lambda(t)(G^0(u(t)) - G^0(\tilde{u})) \, dt \right| = \left| \int_0^T \int_0^1 a_\lambda(t)g^0(\tilde{u} + s\tilde{u}(t))\tilde{u}(t) \, ds \, dt \right|
$$

$$
\leq (\eta |\tilde{u}|^{\alpha+1} + K_{\eta}' \|u'\|_{L^1_T}^{\alpha+1}) \|a_\lambda\|_{L^1_T}.
$$

Thus, taking $C_\epsilon := K_{\eta}' \|a_\lambda\|_{L^1_T}$ for $\eta = \frac{\epsilon}{\|a_\lambda\|_{L^1_T}}$, we are done.

We pass to prove relation (3.8). Similarly as before, $(g_0)$ and $(g_\infty)$ imply that there exists a constant $C_2 > 0$ such that
\[ |g^\alpha(x)| \leq C_3 |x|^\beta, \quad \forall x \in \mathbb{R}. \]

Hence, if \( 0 \leq s \leq 1 \) and using Sobolev inequality (1.9),
\[
|g^\alpha(\bar{u} + s\bar{u}(t))\bar{u}(t)| \leq C_3 |\bar{u} + s\bar{u}(t)|^{\beta} |\bar{u}(t)| \leq C_4 (|\bar{u}|^{\beta} + |\bar{u}(t)|^{\beta}) |\bar{u}(t)|
\]
\[
= C_4 (|\bar{u}|^{\beta} |\bar{u}(t)| + |\bar{u}(t)|^{\beta+1}) \leq C_4 (|\bar{u}|^{\beta} \|u\|_{L^\infty} + \|\bar{u}\|_{L^\infty}^{\beta+1})
\]
\[
\leq C_5 (|\bar{u}|^{\beta} \|u\|_{L^1_T}^2 + \|u\|_{L^2_T}^{\beta+1})
\]

(for \( C_4, C_5 \) further positive constants). By the elementary inequality \( 2ab \leq a^2 + b^2 \), we find that for every \( \eta > 0 \),
\[
|\bar{u}|^{\beta} \|u\|_{L^1_T}^2 = \left( \frac{2\eta}{C_5} \right)^{-\frac{1}{\beta}} |\bar{u}|^{\beta} \left( \frac{2\eta}{C_5} \right)^{\frac{1}{\beta}} \|u\|_{L^2_T}^{\beta}
\]
\[
\leq C_5 \frac{1}{4\eta} |\bar{u}|^{2\beta} + \eta C_5 \|u\|_{L^1_T}^2.
\]

Hence, we obtain
\[
|g^\alpha(\bar{u} + s\bar{u}(t))\bar{u}(t)| \leq \eta \|u\|_{L^2_T}^2 + D_\eta |\bar{u}|^{2\beta} + C_5 \|u\|_{L^2_T}^{\beta+1},
\]

with \( D_\eta := C_5^2 / 4\eta \). In conclusion, we have
\[
\left| \int_0^T a_\lambda(t)(G^\alpha(u(t)) - C^\alpha(\bar{u})) \right| = \left| \int_0^T \int_0^1 a_\lambda(t)g^\alpha(\bar{u} + s\bar{u}(t))\bar{u}(t) ds dt \right|
\]
\[
\leq (\eta \|u\|_{L^2_T}^2 + C_\eta |\bar{u}|^{2\beta} + C \|u\|_{L^2_T}^{\beta+1}) \|a_\lambda\|_{L^1_T}
\]

and, taking \( C := C_5 \|a_\lambda\|_{L^1_T} \) and \( C_\epsilon := D_\eta \|a_\lambda\|_{L^1_T} \) for \( \eta = \frac{\epsilon}{\|a_\lambda\|_{L^1_T}} \), we are done. \( \square \)

Notice that the constants \( C, C_\epsilon \) of (3.7), (3.8) surely depend on \( \lambda \), which, however, does not play any special role at this step.

The next lemma concerns the possibility of minimizing the functional \( \mathcal{S}_\alpha \). Indeed, as a consequence of estimate (3.7) and of the mean value assumption (3.2), we can prove that \( \mathcal{S}_\lambda \) is bounded from below.

**Lemma 3.3.** For every \( \lambda > 0 \), the functional \( \mathcal{S}_\lambda \) is bounded from below on \( H^1_T \).

**Proof.** Let us suppose by contradiction that there exists \( (u_k) \subset H^1_T \) such that \( \mathcal{S}_\lambda(u_k) \to -\infty \). Since, in view of relation (3.8), we have
\[
\mathcal{S}_\lambda(u_k) \geq \frac{1}{2} \int_0^T u_k(t)^2 dt - \int_0^T a_\lambda(t)G(u_k(t)) dt
\]
\[
\frac{1}{2} \int_0^T u_k'(t)^2 \, dt - \int_0^T a_\lambda(t) \left( G(u_k(t)) - G(\bar{u}_k) \right) \, dt - G(\bar{u}_k) \int_0^T a_\lambda(t) \, dt
\]

\[
\geq \left( \frac{1}{2} - \epsilon \right) \int_0^T u_k'(t)^2 \, dt - C \left( \int_0^T u_k'(t)^2 \, dt \right)^{\frac{\beta+1}{2}} - |\bar{u}_k|^{2\beta} \left( \frac{G(\bar{u}_k)}{|\bar{u}_k|^{2\beta}} \int_0^T a_\lambda(t) \, dt + C \epsilon \right),
\]

and taking into account hypothesis \((g_\infty)\), together with the fact that \(\int_0^T a_\lambda(t) \, dt < 0\), we have that \(\bar{u}_k\) is not lower bounded. So, up to subsequences, we can suppose \(\bar{u}_k \to -\infty\). We now distinguish two cases. If \(\|\tilde{u}_k\|_{L^\infty} |\bar{u}_k| \to 0\), then for every \(k\) large enough,

\[
u_k(t) = \bar{u}_k \left( 1 + \frac{\tilde{u}_k(t)}{\bar{u}_k} \right) \leq 0, \quad \forall t \in \mathbb{R},
\]

and hence

\[
\mathcal{J}_\lambda(u_k) = \frac{1}{2} \int_0^T u_k'(t)^2 \, dt - \int_0^T a_\lambda(t) G(u_k(t)) \, dt = \frac{1}{2} \int_0^T u_k'(t)^2 \, dt \geq 0,
\]

in contradiction with the fact that \(\mathcal{J}_\lambda(u_k) \to -\infty\). On the other hand, if, for a subsequence \(u_{kj}\),

\[
\frac{\|\tilde{u}_{kj}\|_{L^\infty}}{|\bar{u}_{kj}|} \geq \delta > 0,
\]

then using Sobolev inequality (1.9) we get that

\[
\int_0^T u_{kj}'(t)^2 \, dt \geq (12/T)\|\tilde{u}_{kj}\|_{L^\infty}^2 \geq \delta^2 (12/T) |\tilde{u}_{kj}|^2
\]

and hence

\[
\mathcal{J}_\lambda(u_{kj}) \geq \left( \frac{1}{2} - \epsilon \right) \int_0^T u_{kj}'(t)^2 \, dt - C \left( \int_0^T u_{kj}'(t)^2 \, dt \right)^{\frac{\beta+1}{2}} - C \epsilon |\tilde{u}_{kj}|^{2\beta}
\]

\[
\geq \left( \frac{1}{4} - \frac{\epsilon}{2} \right) \int_0^T u_{kj}'(t)^2 \, dt - C \left( \int_0^T u_{kj}'(t)^2 \, dt \right)^{\frac{\beta+1}{2}}
\]

\[
+ \left( \frac{1}{4} - \frac{\epsilon}{2} \right) (12/T) \delta^2 |\tilde{u}_{kj}|^2 - C \epsilon |\tilde{u}_{kj}|^{2\beta}.
\]

Since \(2\beta < 2\), we have \(\liminf_{j \to +\infty} \mathcal{J}_\lambda(u_{kj}) > -\infty\), a contradiction again. Thus the claim is proved. \(\square\)

Since \(\mathcal{J}_\lambda(0) = 0\), we have \(\inf_{H^1_T} \mathcal{J}_\lambda \leq 0\). The next lemma shows that the strict inequality actually occurs if \(\lambda\) is large enough.
Lemma 3.4. There exist $\lambda^* > 0$ and $e \in H^1_T$ with $e(t) \geq 0$ such that, for every $\lambda > \lambda^*$,

$$\mathcal{J}_\lambda(e) = \mathcal{J}_\lambda^0(e) < 0.$$ 

Proof. We claim that there exists $e \in H^1_0(I)$, with $e(t) \geq 0$ and such that

$$\int_I a(t)G(e(t)) \, dt = \int_I a^+(t)G(e(t)) \, dt > 0.$$ 

In fact, defining

$$e_k(t) := \begin{cases} k(t - \inf I) & \text{for } \inf I \leq t \leq \inf I + \frac{1}{k}, \\ 1 & \text{for } \inf I + \frac{1}{k} \leq t \leq \sup I - \frac{1}{k}, \\ -k(t - \sup I) & \text{for } \sup I - \frac{1}{k} \leq t \leq \sup I, \end{cases}$$

we have that $e_k \in H^1_0(I)$ and $e_k(t) \to 1$ almost everywhere; hence the dominated convergence theorem implies that

$$\int_I a(t)G(e_k(t)) \, dt \to G(1) \int_I a(t) \, dt = G(1) \int_I a^+(t) \, dt > 0.$$

Hence we can choose $e(t) := e_k(t)$ for $k$ large enough. Defining $e(t)$ to be 0 on $[0, T] \setminus I$ and extending by $T$-periodicity, it is easily seen that $e \in H^1_T$ and

$$\mathcal{J}_\lambda(e) = \mathcal{J}_\lambda^0(e) = \frac{1}{2} \int_0^T e'(t)^2 \, dt - \lambda \int_0^T a^+(t)G(e(t)) \, dt < 0 \quad (3.9)$$

for every $\lambda$ large enough. $\square$

As a consequence of Lemmas 3.3 and 3.4, there exists $\lambda^* > 0$ such that

$$-\infty < \inf_{H^1_T} \mathcal{J}_\lambda < 0, \quad \forall \lambda > \lambda^*. \quad (3.10)$$

Remark 3.3. Condition (3.10), together with a suitable Mountain Pass geometry that we study in a subsequent lemma, is the key point to obtain our multiplicity result. The constant $\lambda^*$ can be found as a value of $\lambda$ for which (3.9) holds. For a given choice of $a(t)$ and $g(x)$ such a constant can be computed explicitly, by choosing a suitable function $e(t)$.

Now we prove that a Mountain Pass geometry occurs for the functional $\mathcal{J}_\lambda^0$. Precisely, we have the following result.

Lemma 3.5. For every $\lambda > \lambda^*$, the functional $\mathcal{J}_\lambda^0$ has the Mountain Pass geometry, namely:

(i) there exist $r, \rho > 0$ such that

$$\mathcal{J}_\lambda^0(u) \geq \rho, \quad \text{for every } u \in H^1_T \text{ with } \|u\|_{H^1_T} = r;$$

(ii) there exists $e \in H^1_T$, with $\|e\|_{H^1_T} > r$, such that $\mathcal{J}_\lambda^0(e) < 0$. 
Proof. By Lemma 3.4, there exists $e \in H^1_I$ such that $J^0_\lambda(e) < 0$. It remains to show that there exist $\rho > 0$ and $0 < r < \|e\|_{H^1_I}$ such that, if $u \in H^1_I$ with $\|u\|_{H^1_I} = r$, then

$$\mathcal{J}_\lambda(u) \geq \rho.$$ 

We follow an argument similar to that of [2, Lemma 1.4]. We set

$$m = -\int_0^T a_\lambda(t) dt > 0;$$

and

$$I^0_\lambda(u) = \frac{1}{2} \int_0^T u'(t)^2 dt - \int_0^T a_\lambda(t) G^0(u(t)) dt$$

$$= \frac{1}{2} \int_0^T u'(t)^2 dt + \frac{l_\alpha}{\alpha + 1} m \bar{u}^{\alpha + 1}$$

$$+ \left( \frac{l_\alpha}{\alpha + 1} \bar{u}^{\alpha + 1} - G^0(\bar{u}) \right) \int_0^1 a_\lambda(t) dt + \int_0^T a_\lambda(t)(G^0(\bar{u}) - G^0(u(t))) dt.$$ 

Being, by l'Hopital rule,

$$\lim_{x \to 0} \frac{G^0(x)}{|x|^{\alpha + 1}} = \frac{l_\alpha}{\alpha + 1} > 0,$$

it is easily seen that

$$\left( \frac{l_\alpha}{\alpha + 1} \bar{u}^{\alpha + 1} - G^0(\bar{u}) \right) \int_0^1 a_\lambda(t) dt = o(|\bar{u}|^{\alpha + 1}), \quad |\bar{u}| \to 0.$$ 

Hence using relation (3.7), we obtain

$$\mathcal{J}^0_\lambda(u) \geq \frac{1}{2} \int_0^T u'(t)^2 dt - C_\epsilon \left( \int_0^1 u'(t)^2 dt \right)^{\frac{\alpha + 1}{2}} + \left( \frac{l_\alpha}{\alpha + 1} m - \epsilon \right) |\bar{u}|^{\alpha + 1} + o(|\bar{u}|^{\alpha + 1}).$$ 

Being $\alpha + 1 > 2$ and recalling the equivalence of $(\bar{u}^2 + \int_0^T u'(t)^2 dt)^{\frac{1}{2}}$ with the standard norm of $H^1_I$, we achieve the conclusion for $r, \rho > 0$ small enough. 

Remark 3.4. It is worth noticing that the functional $\mathcal{J}_\lambda$ does not have a Mountain Pass geometry. In fact, if $u \in H^1_I$ is a negative constant, then $\mathcal{J}_\lambda(u) = 0$; hence, condition (i) cannot be true. It is necessary, indeed, to use the functional $\mathcal{J}^0_\lambda$ at this point.

Finally, we prove standard compactness properties for the functionals $\mathcal{J}_\lambda, \mathcal{J}^0_\lambda$. Being $\mathcal{J} = \mathcal{J}_\lambda$ or $\mathcal{J} = \mathcal{J}^0_\lambda$, we recall that:

- $\mathcal{J}$ is said to satisfy the $(PS)_c$-condition if from $\mathcal{J}(u_k) \to c$ and $\mathcal{J}'(u_k) \to 0$, it follows that $u_k$ has a convergent subsequence;
- $\mathcal{J}$ is said to satisfy the $(PS)$-condition if from sup_k $|\mathcal{J}(u_k)| < +\infty$ and $\mathcal{J}'(u_k) \to 0$, it follows that $u_k$ has a convergent subsequence.
Lemma 3.6. Let $\lambda > 0$. Then:

- the functional $\mathcal{F}_\lambda$ satisfies the (PS)$_c$-condition for every $c < 0$;
- the functional $\mathcal{F}_\lambda^0$ satisfies the (PS)-condition.

Proof. We first deal with the functional $\mathcal{F}_\lambda$.

Let $(u_k) \subset H^1_T$ be a (PS)$_c$ sequence for some $c < 0$; by standard arguments, it is enough to prove that $\|u_k\|_{H^1_T}$ is bounded. Suppose by contradiction that, up to subsequences, $\|u_k\|_{H^1_T} \to +\infty$.

Step 1. We claim that $|\tilde{u}_k| \to +\infty$.

To this aim, we preliminarily verify that there exist $A, B > 0$ such that

$$\int_0^T u'_k(t)^2 \, dt \leq A |\tilde{u}_k|^{2\beta} + B. \quad (3.11)$$

In fact, arguing as in the proof of Lemma 3.2 we get that

$$\left| \int_0^T a_\lambda(t) g(u_k(t)) \tilde{u}_k(t) \, dt \right| \leq \epsilon \|u'_k\|_{L^2_T}^2 + C \epsilon |\tilde{u}_k|^{2\beta} + C \|u'_k\|^{\beta+1}_{L^2_T}. \quad (3.12)$$

This estimate, together with the Wirtinger inequality (1.10), implies that for $k$ large enough it holds that

$$\left( \frac{T^2}{4\pi^2} + 1 \right)^{1/2} \left( \int_0^T u'_k(t)^2 \, dt \right)^{1/2} \geq \|\tilde{u}_k\|_{H^1_T} \geq \mathcal{F}'_\lambda(u_k)[\tilde{u}_k]$$

$$= \int_0^T u'_k(t)^2 \, dt - \int_0^T a_\lambda(t) g(u_k(t)) \tilde{u}_k(t) \, dt$$

$$\geq (1 - \epsilon) \int_0^T u'_k(t)^2 \, dt - C \epsilon |\tilde{u}_k|^{2\beta} - C \left( \int_0^T u'_k(t)^2 \, dt \right)^{\frac{\beta+1}{2}},$$

that is, for $\epsilon = \frac{1}{2}$ and a suitable $C_1$

$$\frac{1}{2} \int_0^T u'_k(t)^2 \, dt \leq C_1 \left( |\tilde{u}_k|^{2\beta} + \left( \int_0^T u'_k(t)^2 \, dt \right)^{1/2} + \left( \int_0^T u'_k(t)^2 \, dt \right)^{\frac{\beta+1}{2}} \right). \quad (3.12)$$

On the other hand, for every $\delta > 0$, there exists $D_\delta > 0$ such that

$$C_1 \left( \int_0^T u'_k(t)^2 \, dt \right)^{1/2} \leq \delta \int_0^T u'_k(t)^2 \, dt + D_\delta.$$
Combining these relations with (3.12), we immediately get (3.11), proving the claim.

Now we can show that \( \| \tilde{u}_k \| \rightarrow +\infty \). In fact, if a subsequence \( \tilde{u}_{k_j} \) is bounded, relation (3.11) implies that \( \int_0^T u_k'(t)^2 \, dt \) is bounded too. This gives the boundedness of \( \| u_{k_j} \|_{H^2} \), in contradiction with the fact that \( \| u_{k_j} \|_{H^2} \rightarrow +\infty \).

**Step 2.** We claim that \( \tilde{u}_k \rightarrow +\infty \).

In fact, let us suppose that, for some subsequences, \( \tilde{u}_{k_j} \rightarrow -\infty \). By the Sobolev inequality (1.9) and (3.11) again we deduce that

\[
\frac{\| \tilde{u}_k \|_{L^\infty}}{\| \tilde{u}_k \|} \leq \frac{(T/12)^{1/2} (\int_0^T u_k'(t)^2 \, dt)^{1/2}}{\| \tilde{u}_k \|} \leq (T/12)^{1/2} (A|\tilde{u}_k|^{2\beta} + B)^{1/2} \rightarrow 0.
\]

Then, for every \( j \) large enough,

\[
u_k(t) = \tilde{u}_k \left( 1 + \frac{\tilde{u}_{k_j}(t)}{\tilde{u}_k} \right) \leq 0, \quad \forall t \in \mathbb{R}.
\]

We deduce that

\[
\mathcal{J}_\lambda(u_k) = \frac{1}{2} \int_0^T u_k'(t)^2 \, dt - \int_0^T a_\lambda(t) G(u_k(t)) \, dt = \frac{1}{2} \int_0^T u_k'(t)^2 \, dt \geq 0,
\]

in contradiction with the fact that \( \mathcal{J}_\lambda(u_k) \rightarrow c < 0 \).

**Step 3.** We are in position to conclude. Using relations (3.8) and (3.11), we obtain, for suitable constants \( A', B' > 0 \),

\[
\mathcal{J}_\lambda(u_k) = \frac{1}{2} \int_0^T u_k'(t)^2 \, dt - \int_0^T a_\lambda(t) G(u_k(t)) \, dt
\]

\[
= \frac{1}{2} \int_0^T u_k'(t)^2 \, dt - \int_0^T a_\lambda(t) \left( G(u_k(t)) - G(\tilde{u}_k) \right) \, dt - G(\tilde{u}_k) \int_0^T a_\lambda(t) \, dt
\]

\[
\geq - \int_0^T a_\lambda(t) \left( G(u_k(t)) - G(\tilde{u}_k) \right) \, dt - G(\tilde{u}_k) \int_0^T a_\lambda(t) \, dt
\]

\[
\geq -\epsilon \int_0^T u_k'(t)^2 \, dt - C_\epsilon (\tilde{u}_k)^{2\beta} - C \left( \int_0^T u_k'(t)^2 \, dt \right)^{\beta+1} - G(\tilde{u}_k) \int_0^T a_\lambda(t) \, dt
\]

\[
\geq -\epsilon A(\tilde{u}_k)^{2\beta} - C_\epsilon (\tilde{u}_k)^{2\beta} - A' (\tilde{u}_k)^{(\beta+1)} - G(\tilde{u}_k) \int_0^T a_\lambda(t) \, dt - \epsilon B - B'
\]

\[
\geq -(\tilde{u}_k)^{2\beta} \left( \epsilon A + C_\epsilon + \frac{G(\tilde{u}_k)}{(\tilde{u}_k)^{2\beta}} \int_0^T a_\lambda(t) \, dt \right) - A' (\tilde{u}_k)^{(\beta+1)} - \epsilon B - B'.
\]
Being

\[
\lim_{k \to +\infty} \left( \frac{G(\bar{u}_k)}{|\bar{u}_k|^{2\beta}} \int_0^T a_\lambda(t) \, dt \right) = -\infty,
\]

and \(\beta(\beta + 1) < 2\beta\), we get a contradiction with the fact that \(\mathcal{J}_\lambda(u_k)\) is upper bounded.

For the verification of the (PS)-property for the functional \(\mathcal{J}_\lambda^o\) (see also [33]), it is sufficient to argue as in Step 1, Step 3 and use the fact that, if \(|\bar{u}_k| \to +\infty\), then

\[
\lim_{k \to +\infty} \left( \frac{G^o(\bar{u}_k)}{|\bar{u}_k|^{2\beta}} \int_0^T a_\lambda(t) \, dt \right) = -\infty. \quad \square
\]

Finally, we need a strong maximum principle for solutions to (3.1). It only relies on the behavior of \(g(x)\) near the origin.

**Lemma 3.7.** Let \(u \in W^{2,1}_T\) be a nontrivial solution of (3.1) with \(u(t) \geq 0\) for every \(t \in \mathbb{R}\). Then \(u(t) > 0\) for every \(t \in \mathbb{R}\).

**Proof.** Let us assume that for some \(\bar{t} \in \mathbb{R}\), \(u(\bar{t}) = 0\); then, necessarily, \(u'(\bar{t}) = 0\). Being \(\lim_{x \to 0^+} \frac{g(x)}{x} = 0\), [27, Lemma 2.1] implies that \(u(t) \equiv 0\), a contradiction. \(\square\)

We are now in a position to collect the previous results in order to prove Theorem 3.1.

**Proof of Theorem 3.1.** Fix \(\lambda > \lambda^*\). From Lemmas 3.3 and 3.4, we get that \(\mathcal{J}_\lambda\) is bounded from below with \(\inf_{H^1_T} \mathcal{J}_\lambda < 0\). Moreover, by Lemma 3.6, the Palais–Smale condition is satisfied at every strictly negative level. Hence, from the minimization theorem [28, Theorem 4.4] we get that \(\mathcal{J}_\lambda\) attains its minimum at a point \(u_1 \in H^1_T\) with

\[
\mathcal{J}_\lambda(u_1) = \inf_{H^1_T} \mathcal{J}_\lambda < 0.
\]

Being \(G(x) = 0\) for \(x \leq 0\), the function \(u_1(t)\) is non-constant and, by Lemmas 3.1 and 3.7, it is a positive \(T\)-periodic solution of (3.1).

On the other hand, in view of Lemmas 3.5 and 3.6, the Mountain Pass Theorem can be applied to the functional \(\mathcal{J}_\lambda^o\), yielding the existence of a critical point \(u_2 \in H^1_T\) of \(\mathcal{J}_\lambda^o\) such that

\[
\mathcal{J}_\lambda^o(u_2) = \inf_{\gamma \in \Gamma} \max_{\sigma \in [0,1]} \mathcal{J}_\lambda^o(\gamma(\sigma)) > 0,
\]

where \(\Gamma = \{\gamma \in C([0,1], H^1_T) \mid \gamma(0) = 0, \gamma(1) = e\}\). Such \(u_2\) is a nontrivial solution of (3.6). Moreover, being \(\mathcal{J}_\lambda^o(u) = \mathcal{J}_\lambda^o(|u|)\) and \(e \geq 0\), it is a general fact (see [8, Theorem 10], or argue as in the proof of [2, Theorem 1.6]) that \(u_2(t) \geq 0\). Hence \(u_2(t)\) is a solution of (3.1) and, being \(u_2 \not\equiv 0\), Lemma 3.7 implies that \(u_2(t) > 0\) for every \(t \in \mathbb{R}\).

The fact that \(u_1 \neq u_2\) follows since \(\mathcal{J}_\lambda(u_1) < 0 < \mathcal{J}_\lambda(u_2) = \mathcal{J}_\lambda^o(u_2)\). \(\square\)

**Acknowledgment**

The authors thank the referee for his/her remarks which led to an enhanced version of the paper.
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